Restrictions of Certain Degenerate Principal Series of the Universal Covering of the Symplectic Group

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Abstract. Let $\widetilde{Sp}(n; \mathbb{R})$ be the universal covering of the symplectic group. In this paper, we study the restrictions of the degenerate unitary principal series $I(\epsilon, t)$ of $\widetilde{Sp}(n; \mathbb{R})$ onto $\widetilde{Sp}(p; \mathbb{R})\widetilde{Sp}(n-p; \mathbb{R})$. We prove that if $n \geq 2p$, $I(\epsilon, t)|_{\widetilde{Sp}(p; \mathbb{R})\widetilde{Sp}(n-p; \mathbb{R})}$ is unitarily equivalent to an $L^2$-space of sections of a homogeneous line bundle $L^2(\widetilde{Sp}(n-p; \mathbb{R}) \times \mathcal{L}(n-2p; N)\mathbb{C}_{\epsilon, t+p})$ (see Theorem 1.1). We further study the restriction of complementary series $C(\epsilon, t)$ onto $\widetilde{U}(n-p)\widetilde{Sp}(p; \mathbb{R})$. We prove that this restriction is unitarily equivalent to $I(\epsilon, t)|_{\widetilde{U}(n-p)\widetilde{Sp}(p; \mathbb{R})}$ for $t \in i\mathbb{R}$. Our results suggest that the direct integral decomposition of $C(\epsilon, t)|_{\widetilde{Sp}(p; \mathbb{R})\widetilde{Sp}(n-p; \mathbb{R})}$ will produce certain complementary series for $\widetilde{Sp}(n-p; \mathbb{R})$ ([He09]).

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1. Introduction

Let $\widetilde{Sp}(n; \mathbb{R})$ be the universal covering of $Sp(n; \mathbb{R})$. $\widetilde{Sp}(n; \mathbb{R})$ is a central extension of $Sp(n; \mathbb{R})$:

$$1 \to C \to \widetilde{Sp}(n; \mathbb{R}) \to Sp(n; \mathbb{R}) \to 1,$$

where $C \cong \mathbb{Z}$. The unitary dual of $C$ is parametrized by a torus $T$. For each $\kappa \in T$, denote the corresponding unitary character of $C$ by $\chi^\kappa$. We say that a representation $\pi$ of $\widetilde{Sp}(n; \mathbb{R})$ is of class $\kappa$ if $\pi|_{C} = \chi^\kappa$. Since $C$ commutes with $\widetilde{Sp}(n; \mathbb{R})$, for any irreducible representation $\pi$ of $\widetilde{Sp}(n; \mathbb{R})$, $\pi|_{C} = \chi^\kappa$ for some $\kappa$.

Denote the projection $\widetilde{Sp}(n; \mathbb{R}) \to Sp(n; \mathbb{R})$ by $p$. For any subgroup $H$ of $Sp(n; \mathbb{R})$, denote the full inverse image $p^{-1}(H)$ by $\tilde{H}$. We adopt the notation from [Sa]. Let $P$ be the Siegel parabolic subgroup of $Sp(n; \mathbb{R})$. One dimensional characters of $P$ can be parametrized by $(\epsilon, t)$ where $\epsilon \in T$ and $t \in \mathbb{C}$. Let $I(\epsilon, t)$ be the representation of $\widetilde{Sp}(n; \mathbb{R})$ induced from the one dimensional character $C_{\epsilon, t}$ parametrized by $(\epsilon, t)$ of $\tilde{P}$. If $t \in i\mathbb{R}$ and $t \neq 0$, $I(\epsilon, t)$ is unitary and irreducible. $I(\epsilon, t)$ is called unitary degenerate principal series. If $t$ is real, then $I(\epsilon, t)$ has a

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nontrivial invariant Hermitian form. Sahi gives a classification of all irreducible unitarizable $I(\epsilon,t)$. If $I(\epsilon,0)$ is irreducible, there are complementary series $C(\epsilon,t)$ for $t$ in a suitable interval ([Sa]). Some of these complementary series are obtained by Kudla-Rallis [KR], Østed-Zhang [OZ], Branson-Østed-Olafsson [BOO], Lee [Lee]. Strictly speaking $C(\epsilon,t)$ should be called degenerate complementary series because there are complementary series associated with the principal series, which should be called complementary series ([Kos], [ABPTV]). Throughout this paper, complementary series will mean $C(\epsilon,t)$.

Let $(Sp(p,\mathbb{R}),Sp(n-p,\mathbb{R}))$ be a pair of symplectic groups diagonally embedded in $Sp(n,\mathbb{R})$ (see Definition 5.1). Let $U(n)$ be a maximal compact subgroup such that $Sp(n-p,\mathbb{R}) \cap U(n)$ and $Sp(p,\mathbb{R})$ are maximal compact subgroups of $Sp(n-p,\mathbb{R})$ and $Sp(p,\mathbb{R})$ respectively. Denote $Sp(n-p,\mathbb{R}) \cap U(n)$ by $U(n-p)$ and $Sp(p,\mathbb{R}) \cap U(n)$ by $U(p)$. The main results of this paper can be stated as follows.

**Theorem 1.1.** Suppose $p \leq n-p$ and $t \in i\mathbb{R}$. Let $P_{p,n-2p}$ be a maximal parabolic subgroup of $Sp(n-p,\mathbb{R})$ with Langlands decomposition $Sp(p,\mathbb{R})GL(n-2p)N_{p,n-2p}$. Let $M_{\epsilon,t}$ be the homogeneous line bundle

$$
\tilde{Sp}(n-p,\mathbb{R}) \times_{GL(n-2p)N_{p,n-2p}} C_{\epsilon,t+p} \to Sp(n-p,\mathbb{R})/GL(n-2p)N_{p,n-2p} \quad (\cong Sp(p,\mathbb{R})U(n-p)/U(p)O(n-2p)),
$$

where $\rho = \frac{n+1}{2}$. Let $dg_{1d[k_2]}$ be an $Sp(p,\mathbb{R})U(n-p)$-invariant measure. Then

$$
I(\epsilon,t)|_{\tilde{Sp}(p,\mathbb{R})\tilde{Sp}(n-p,\mathbb{R})} \cong L^2(M_{\epsilon,t},dg_{1d[k_2]}),
$$
on which $\tilde{Sp}(n-p,\mathbb{R})$ acts from the left and $\tilde{Sp}(p,\mathbb{R})$ acts from the right.

**Theorem 1.2.** Let $C(\epsilon,t)$ be a complementary series representation. Suppose that $p \leq n-p$. Then

$$
C(\epsilon,t)|_{\tilde{U}(n-p)\tilde{Sp}(p,\mathbb{R})} \cong I(\epsilon,0)|_{\tilde{U}(n-p)\tilde{Sp}(p,\mathbb{R})} \cong I(\epsilon,i\lambda)|_{\tilde{U}(n-p)\tilde{Sp}(p,\mathbb{R})} \quad (\lambda \in \mathbb{R}).
$$

$p = \lceil \frac{n}{2} \rceil$ is the best possible value for such a statement. In particular, for $\tilde{Sp}(2m+1,\mathbb{R})$

$$
I(\epsilon,0)|_{\tilde{Sp}(m+1,\mathbb{R})} \not\cong C(\epsilon,t)|_{\tilde{Sp}(m+1,\mathbb{R})}.
$$

To see this, let $L^2(\tilde{Sp}(n,\mathbb{R}))_\kappa$ be the set of functions with

$$
f(zg) = \chi^{\kappa}(z)f(g) \quad (z \in C, g \in \tilde{Sp}(n,\mathbb{R}));
$$

$$
\|f\|^2 = \int_{\tilde{Sp}(n,\mathbb{R})} |f(g)|^2 d[g] < \infty \quad (g \in \tilde{Sp}(n,\mathbb{R}), [g] \in Sp(n,\mathbb{R})).
$$

We say that a representation of class $\kappa$ is tempered if it is weakly contained in $L^2(\tilde{Sp}(n,\mathbb{R}))_\kappa$. By studying the leading exponents of $I(\epsilon,0)$ and $C(\epsilon,t)$, it can
be shown that \( I(\epsilon, 0)|_{\tilde{Sp}(m+1, \mathbb{R})} \) is tempered and \( C(\epsilon, t)|_{\tilde{Sp}(m+1, \mathbb{R})} \) is not tempered. Therefore

\[
I(\epsilon, 0)|_{\tilde{Sp}(m+1, \mathbb{R})} \not\subset C(\epsilon, t)|_{\tilde{Sp}(m+1, \mathbb{R})};
\]

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2. A Lemma on Friedrichs Extension

Let \( S \) be a semibounded densely defined symmetric operator on a Hilbert space \( H \). \( S \) is said to be positive if \( (Su, u) > 0 \) for every nonzero \( u \in D(S) \). Suppose that \( S \) is positive. For \( u, v \in D(S) \), define

\[
(u, v)_S = (u, Sv),
\]

\[
\|u\|_S^2 = (u, Su).
\]

Let \( H_S \) be the completion of \( D(S) \) under the norm \( \| \cdot \|_S \). Clearly \( H_{S+I} \subseteq H \) and \( H_{S+I} \subseteq H_S \).

The operator \( S+I \) has a unique self-adjoint extension \( (S+I)_0 \) in \( H \), the Friedrichs extension. \( (S+I)_0 \) has the following properties

- \( D(S) \subseteq D((S + I)_0) \subseteq H_{S+I} \subseteq H \);
- \( (u, v)_{S+I} = (u, (S + I)_0v) \) for all \( u \in H_{S+I} \) and \( v \in D((S + I)_0) \)

(see Theorem in Page 335 [RS]). Now consider \( (S + I)_0 - I \). It is an self-adjoint extension of \( S \). It is nonnegative. By the spectral decomposition and functional calculus, \( (S + I)_0 - I \) has a unique square root \( T \) (See Pg. 127, 128 [RS]).

**Lemma 2.1.** Let \( S \) be a positive densely defined symmetric operator. Then the square root of \( (S + I)_0 - I \) extends to an isometry from \( H_S \) into \( H \).

**Proof.** Clearly, the spectrum of \( T \) is contained in the nonnegative part of the real line. By spectral decomposition \( D((S + I)_0 - I) = D((S + I)_0) \subseteq D(T) \) and \( TT = (S + I)_0 - I \). In addition for any \( u, v \in D(S) \subseteq D((S + I)_0) \),

\[
(Tu, Tv) = (u, TTv) = (u, (S + I)_0v - v) = (u, Sv) = (u, v)_S
\]

So \( T \) is an isometry from \( D(S) \) into \( H \). Since \( D(S) \) is dense in \( H_S \), \( T \) extends to an isometry from \( H_S \) into \( H \).

3. Degenerate Principal Series of \( \tilde{Sp}(n, \mathbb{R}) \)

Fix the Lie algebra:

\[
\mathfrak{sp}(n, \mathbb{R}) = \left\{ \begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} \right| Y^t = Y, Z^t = Z \right\}
\]
and the Siegel parabolic algebra:

$$p = \left\{ \begin{pmatrix} X & Y \\ 0 & -X' \end{pmatrix} \mid Y' = Y \right\}.$$ 

Fix the Levi decomposition $$p = l \oplus n$$ with

$$l = \left\{ \begin{pmatrix} X \\ 0 & -X' \end{pmatrix} \mid X \in \mathfrak{gl}(n, \mathbb{R}) \right\}, \quad n = \left\{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \mid Y' = Y \right\}.$$ 

Fix a Cartan subalgebra

$$a = \{ \text{diag}(H_1, H_2, \ldots, H_n, -H_1, -H_2, \ldots, -H_n) \mid H_i \in \mathbb{R} \}.$$ 

Let $$\text{Sp}(n, \mathbb{R})$$ be the symplectic group and $$P$$ be the Siegel parabolic subgroup. Set $$U(n) = \text{Sp}(n, \mathbb{R}) \cap O(2n)$$ where $$O(2n)$$ is the standard orthogonal group. Let $$LN$$ be the Levi decomposition of $$P$$ and $$A$$ be the analytic group generated by the Lie algebra $$a$$. Clearly, $$L \cong GL(n, \mathbb{R})$$ and $$L \cap U(n) \cong O(n)$$. On the covering group, we have $$\tilde{L} \cap \tilde{U}(n) = \tilde{O}(n)$$. Recall that

$$\tilde{U}(n) = \{ (x, g) \mid g \in U(n), \exp 2\pi ix = \det g, x \in \mathbb{R} \}.$$ 

Therefore

$$\tilde{O}(n) = \{ (x, g) \mid g \in O(n), \exp 2\pi ix = \det g, x \in \mathbb{R} \}.$$ 

Notice that for $$g \in O(n)$$, $$\det g = \pm 1$$ and $$x \in \frac{1}{2} \mathbb{Z}$$. We have the following exact sequence

$$1 \rightarrow SO(n) \rightarrow \tilde{O}(n) \rightarrow \frac{1}{2} \mathbb{Z} \rightarrow 1.$$ 

Consequently, we have

$$1 \rightarrow GL_0(n, \mathbb{R}) \rightarrow \tilde{L} \rightarrow \frac{1}{2} \mathbb{Z} \rightarrow 1.$$ 

In fact,

$$\tilde{L} = \{ (x, g) \mid g \in L, \exp 2\pi ix = \frac{\det g}{|\det g|}, x \in \mathbb{R} \}.$$ 

The one dimensional unitary characters of $$\frac{1}{2} \mathbb{Z}$$ are parametrized by the one dimensional torus $$T$$. Identify $$T$$ with $$[0, 1)$$. Let $$\mu^\epsilon$$ be the character of $$\frac{1}{2} \mathbb{Z}$$ corresponding to $$\epsilon \in [0, 1)$$. Now each character $$\mu^\epsilon$$ yields a character of $$\tilde{L}$$, which in turn, yields a character of $$\tilde{P}$$. For simplicity, we retain $$\mu^\epsilon$$ to denote the character on $$\tilde{L}$$ and $$\tilde{P}$$. Let $$\nu$$ be the det-character on $$\tilde{L}_0$$, i.e.,

$$\nu(x, g) = |\det g| \quad (x, g) \in \tilde{L}. \quad (2)$$

Let

$$I(\epsilon, t) = \text{Ind}_{\tilde{P}}^{\tilde{G}_{\text{Sp}(n, \mathbb{R})}} \mu^\epsilon \otimes \nu^t$$

be the normalized induced representation with $$\epsilon \in [0, 1)$$ and $$t \in \mathbb{C}$$. This is Sahi’s notation in the case of the universal covering of the symplectic group ([Sa]). $$I(\epsilon, t)$$ is a degenerate principal series representation. Clearly, $$I(\epsilon, t)$$ is unitary.
When $t \in i\mathbb{R}$.

When $t$ is real and $I(\epsilon, t)$ is unitarizable, the unitary representation, often denoted by $C(\epsilon, t)$, is called a complementary series representation. Various complementary series of $Sp(n, \mathbb{R})$ and its metaplectic covering was determined explicitly or implicitly by Kudla-Rallis, Ørsted-Zhang, Brason-Olafsson-Ørsted and others. See [KR], [BOO], [OZ] and the references therein. The complete classification of the complementary series of the universal covering is due to Sahi.

**Theorem 3.1 (Thm A, [Sa]).** Suppose that $t$ is real. For $n$ even, $I(\epsilon, t)$ is irreducible and unitarizable if and only if $0 < |t| < |\frac{1}{2} - |2\epsilon - 1||$. For $n$ odd and $n > 1$, $I(\epsilon, t)$ is irreducible and unitarizable if and only if $0 < |t| < \frac{1}{2} - |\frac{1}{2} - |2\epsilon - 1||$.

One can easily check that the complementary series exist if $\epsilon \neq 0, \frac{1}{2}$ for $n$ odd and $n > 1$; if $\epsilon \neq \frac{1}{4}, \frac{3}{4}$ for $n$ even. It is interesting to note that complementary series always exist unless $I(\epsilon, t)$ descends into a representation of the metaplectic group. For the metaplectic group $Mp(2n + 1, \mathbb{R})$, there are two complementary series $I(\frac{1}{2}, t)(0 < t < \frac{1}{2})$ and $I(\frac{3}{2}, t)(0 < t < \frac{1}{2})$. For the metaplectic group $Mp(2n, \mathbb{R})$, there are two complementary series $I(0, t)(0 < t < \frac{1}{2})$ and $I(\frac{1}{2}, t)(0 < t < \frac{1}{2})$. These four complementary series are the “longest”.

For $n = 1$, the situation is quite different. The difference was pointed out in [KR]. For example, there are Bargmann’s complementary series representation for $I(0, t)$ ($t \in (0, \frac{1}{2})$). The classification of the complementary series of $\tilde{Sp}(1, \mathbb{R})$ can be found in [Bar], [Puk], [Howe].

Since our restriction theorem only makes sense for $n \geq 2$, we will assume $n \geq 2$ from now on. The parameters for the complementary series of $\tilde{Sp}(n, \mathbb{R})$ are illustrated in Fig. 1.

### 4. The generalized compact model and The Intertwining Operator

Recall that

$$I^\infty(\epsilon, t) = \{ f \in C^\infty(\tilde{Sp}(n, \mathbb{R})) \mid f(gln) = (\mu^t \otimes \nu^{t+\rho})(l^{-1})f(g), \quad (g \in \tilde{Sp}(n, \mathbb{R}), t \in \tilde{L}, n \in \mathbb{N}) \}$$

where $\rho = \frac{n+1}{2}$. Let $X = \tilde{Sp}(n, \mathbb{R})/\tilde{P}$. Then $I^\infty(\epsilon, t)$ consists of smooth sections of the homogeneous line bundle $\mathcal{L}_{\epsilon, t}$

$$\tilde{Sp}(n, \mathbb{R}) \times _{\tilde{\rho}} \mathbb{C}_{\mu^t \otimes \nu^{t+\rho}} \to X.$$
naturally. Notice that the homogeneous line bundle $\mathcal{K}_{\epsilon,t}$ does not depend on the parameter $t$. We denote this line bundle by $\mathcal{K}_{\epsilon}$. The representation $I^\infty(\epsilon, t)$ can then be modeled on smooth sections of $\mathcal{K}_{\epsilon}$. This model will be called the generalized compact model.

Let $d[k]$ be the normalized $\tilde{U}(n)$-invariant measure on $X$. The generalized compact model equips the smooth sections of $\mathcal{K}_{\epsilon,t}$ with a natural pre-Hilbert structure

$$ (f_1, f_2)_X = \int_{[k] \in X} f_1(k) \overline{f_2(k)} d[k], $$

where $k \in \tilde{U}(n)$ and $[k] \in X$. It is easy to verify that $f_1(k) \overline{f_2(k)}$ is a function of $[k]$ and it does not depend on any particular choice of $k$. Notice that our situation is different from the compact model since $\tilde{U}(n)$ is not compact. We denote the completion of $I^\infty$ with respect to $(\cdot, \cdot)_X$ by $I_X(\epsilon, t)$.

Secondly, the action of $\tilde{U}(n)$ on $\mathcal{K}_{\epsilon}$ induces an orthogonal decomposition of $I_X(\epsilon, t)$:

$$ I_X(\epsilon, t) = \bigoplus_{\alpha \in 2\mathbb{Z}^n} V(\alpha + \epsilon(2, 2, \ldots, 2)),$$

where $V(\alpha + \epsilon(2, 2, \ldots, 2))$ is an irreducible finite dimensional representation of $\tilde{U}(n)$ with highest weight $\alpha + \epsilon(2, 2, \ldots, 2)$ and $\alpha$ satisfies

$$ \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n. $$

This is essentially a consequence of Helgason’s theorem. Let

$$ V(\epsilon, t) = \bigoplus_{\alpha \in 2\mathbb{Z}^n} V(\alpha + \epsilon(2, 2, \ldots, 2)). $$

\[\text{Figure 1: Complementary Parameters } (E, t)\]
$V(\epsilon,t)$ possesses an action of the Lie algebra $\mathfrak{sp}(n,\mathbb{R})$. It is the Harish-Chandra module of $I(\epsilon,t)$. Clearly, $V(\epsilon,t) \subset I^\infty(\epsilon,t) \subset I_X(\epsilon,t)$.

For each $t$, there is an $\widetilde{Sp}(n,\mathbb{R})$-invariant sesquilinear pairing of $I_X(\epsilon,t)$ and $I_X(\epsilon,-\overline{t})$, namely,

$$(f_1,f_2) = \int_X f_1(k)\overline{f_2(k)}d[k],$$

where $f_1 \in I_X(\epsilon,t)$ and $f_2 \in I_X(\epsilon,-\overline{t})$. If $t \in i\mathbb{R}$, we obtain a $\widetilde{Sp}(n,\mathbb{R})$-invariant Hermitian form which is exactly $(\ ,\ )_X$. Since $(\ ,\ )_X$ is positive definite, $I_X(\epsilon,t)$ is a unitary representation of $\widetilde{Sp}(n,\mathbb{R})$.

For each real $t$, the form $(\ ,\ )$ gives an $\mathfrak{sp}(n,\mathbb{R})$-invariant sesquilinear pairing of $V(\epsilon,t)$ and $V(\epsilon,-t)$. In addition, there is an intertwining operator

$$A(\epsilon,t): V(\epsilon,t) \rightarrow V(\epsilon,-t)$$

which preserves the action of $\mathfrak{sp}(n,\mathbb{R})$ (see for example [BOO]). Define a Hermitian structure $(\ ,\ )_{\epsilon,t}$ on $V(\epsilon,t)$ by

$$(u,v)_{\epsilon,t} = (A(\epsilon,t)u,v), \quad (u,v \in V(\epsilon,t)).$$

Clearly, $(\ ,\ )_{\epsilon,t}$ is $\mathfrak{sp}(n,\mathbb{R})$-invariant. So $A(\epsilon,t)$ induces an invariant Hermitian form on $V(\epsilon,t)$.

Now $A(\epsilon,t)$ can also be realized as an unbounded operator on $I_X(\epsilon,t)$ as follows. For each $f \in V(\epsilon,t)$, define $A_X(\epsilon,t)f$ to be the unique section of $\mathcal{L}_{\epsilon,t}$ such that

$$(A_X(\epsilon,t)f)|_{\tilde{U}(n)} = (A(\epsilon,t)f)|_{\tilde{U}(n)}.$$  

Notice that $A_X(\epsilon,t)f \in I(\epsilon,t)$ and $A(\epsilon,t)f \in I(\epsilon,-t)$. They differ by a multiplier.

Now $A_X(\epsilon,t)$ is an unbounded operator on the Hilbert space $I_X(\epsilon,t)$. The following fact is well-known in many different forms. I state it in a way that is convenient for later use.

**Lemma 4.1.** Let $t \in \mathbb{R}$. $I(\epsilon,t)$ is unitarizable if and only if $A_X(\epsilon,t)$ extends to a self-adjoint operator on $I_X(\epsilon,t)$ with spectrum on the nonnegative part of the real axis.

The spectrum of $A_X(\epsilon,t)$ was computed in [BOO] and [OZ] explicitly for special cases and in [Sa] implicitly. In particular, $A_X(\epsilon,t)$ restricted onto each $\tilde{U}(n)$-type is a scalar multiplication and the scalar is bounded by a polynomial on the highest weight. We obtain

**Lemma 4.2 ([WV]).** $A_X(\epsilon,t)$ extends to an unbounded operator from $I^\infty(\epsilon,t)$ to $I^\infty(\epsilon,t)$.  

This lemma follows from a standard argument that the norm of each $\tilde{U}(n)$-component in the Peter-Weyl expansion of any smooth section of $K_{\epsilon}$ decays rapidly with respect to the highest weight. It is true in general (see [WV]).

5. **Actions of $\tilde{Sp}(p, \mathbb{R})\tilde{Sp}(q, \mathbb{R})$**

Suppose that $p + q = n$ and $p \leq q$. Fix a standard basis
\[
\{e_1, e_2, \ldots, e_p; e_1^*, e_2^*, \ldots e_p^*\}
\]
for the symplectic form $\Omega_p$ on $\mathbb{R}^{2p}$. Fix a standard basis
\[
\{f_1, f_2, \ldots, f_q; f_1^*, f_2^*, \ldots f_q^*\}
\]
for the symplectic form $\Omega_q$ on $\mathbb{R}^{2q}$.

**Definition 5.1.** Let $Sp(p, \mathbb{R})$ be the symplectic group preserving $\Omega_p$ and $Sp(q, \mathbb{R})$ be the symplectic group preserving $\Omega_q$. Let
\[
\Omega = \Omega_p - \Omega_q
\]
and $Sp(n, \mathbb{R})$ be the symplectic group preserving $\Omega$. We say that $(Sp(p, \mathbb{R}), Sp(q, \mathbb{R}))$ is diagonally embedded in $Sp(n, \mathbb{R})$.

We shall make a remark here. In [Henu], $\Omega = \Omega_p + \Omega_q$. $Sp(p, \mathbb{R})Sp(q, \mathbb{R})$ is embedded differently there. The effect of this difference is an involution $\tau$ on the representation level.

Let $P_{p,q-p}$ be the subgroup of $Sp(q, \mathbb{R})$ that preserves the linear span of $\{f_{p+1}, \ldots, f_q\}$. Choose the Levi factor $GL(q-p)Sp(p, \mathbb{R})$ to be the subgroup of $P_{p,q-p}$ that preserves the span of $\{f_{p+1}^*, \ldots, f_q^*\}$. In particular the $Sp(p, \mathbb{R})$ factor can be identified with the symplectic group of
\[
\text{span}\{f_1, \ldots, f_p; f_1^*, \ldots f_p^*\},
\]
which will be identified with the standard $Sp(p, \mathbb{R})$. More precisely, for $x \in Sp(p, \mathbb{R})$, by identify $e_i$ with $f_i$ and $e_i^*$ with $f_i^*$ and extending $x$ trivially on $f_{p+1}, \ldots f_q; f_{p+1}^*, \ldots f_q^*$, we obtain the identification
\[
x \in Sp(p, \mathbb{R}) \rightarrow \dot{x} \in Sp(q, \mathbb{R}).
\]

Now fix a Lagrangian Grassmanian
\[
x_0 = \text{span}\{e_1 + f_1, \ldots, e_p + f_p; e_1^* + f_1^*, \ldots e_p^* + f_p^*, f_{p+1}, \ldots f_q\}.
\]
Then the stabilizer $Sp(q, \mathbb{R})_{x_0} = GL(q-p)N_{p,q-p}$ where $N_{p,q-p}$ is the nilradical of $P_{p,q-p}$. Put
\[
\Delta(Sp(p, \mathbb{R})) = \{(u, \dot{u}) \mid u \in Sp(p, \mathbb{R})\} \subseteq Sp(p, \mathbb{R})Sp(q, \mathbb{R})
\]
and
\[
H = \Delta(Sp(p, \mathbb{R}))GL(q-p)N_{p,q-p}.
\]
Lemma 5.2 ([Henu]). Let \( p \leq q \) and \( p + q = n \). Let \( X_0 \) be the \( \text{Sp}(p, \mathbb{R}) \times \text{Sp}(q, \mathbb{R}) \)-orbit generated by \( x_0 \). Then \( X_0 \) is open and dense in \( X \) and \( [\text{Sp}(p, \mathbb{R}) \text{Sp}(q, \mathbb{R})]_{x_0} = H \).

Notice here that \( \Delta(\epsilon, t) \) consists of smooth sections of \( L_{\epsilon, t} : \nabla \rightarrow X \).

Consider the subgroup \( \tilde{\text{Sp}}(p, \mathbb{R}) \tilde{\text{Sp}}(q, \mathbb{R}) \) in \( \tilde{\text{Sp}}(n, \mathbb{R}) \). Notice that \( \tilde{\text{Sp}}(p, \mathbb{R}) \cap \tilde{\text{Sp}}(q, \mathbb{R}) \cong \mathbb{Z} \). So \( \tilde{\text{Sp}}(p, \mathbb{R}) \tilde{\text{Sp}}(q, \mathbb{R}) \) is not a direct product, but rather the product of the two groups as sets.

Definition 5.3. For any \( f \in I_X(\epsilon, t) \), define

\[
\tilde{f}_{X_0} = f|_{\tilde{\text{Sp}}(p, \mathbb{R}) \tilde{\text{Sp}}(q, \mathbb{R})},
\]

Let \( \tilde{I}_{\epsilon, X_0}(\epsilon, t) \) be the set of smooth sections of \( L_{\epsilon, t} \) that are compactly supported in \( X_0 \).

Clearly \( \tilde{f}_{X_0} \) is a smooth section of

\[
\tilde{\text{Sp}}(p, \mathbb{R}) \tilde{\text{Sp}}(q, \mathbb{R}) \times \mathbb{R} C_{\mu^t \otimes \nu^t + \rho} \rightarrow X_0.
\]

Notice that \( \Delta(\text{Sp}(p, \mathbb{R})) \) sits inside of \( SL(n, \mathbb{R}) \subseteq GL(n, \mathbb{R}) \subseteq P \). The universal covering of \( \text{Sp}(n, \mathbb{R}) \) splits over \( SL(n, \mathbb{R}) \subseteq P \). Similarly the universal covering of \( \text{Sp}(q, \mathbb{R}) \) also splits over \( N_{p, q-p} \). So we have

\[
\tilde{H} \cong \Delta(\text{Sp}(p, \mathbb{R})) \tilde{GL}(q - p) N_{p, q-p},
\]

where \( \tilde{GL}(q - p) N_{p, q-p} \subseteq \tilde{\text{Sp}}(q, \mathbb{R}) \). In particular, \( \mu^t \otimes \nu^t + \rho|_{\Delta(\text{Sp}(p, \mathbb{R})) N_{p, q-p}} \) is trivial and \( \mu^t \otimes \nu^t + \rho|_{\tilde{GL}(q-p)} \) is essentially the restriction from \( \tilde{GL}(p + q) \) to \( \tilde{GL}(q) \).

If \( p = q \), then \( GL(0) \) will be the identity element. So \( \tilde{GL}(0) \) is just \( C \). We have

Lemma 5.4. The identification (4)

\[
x \in \text{Sp}(p, \mathbb{R}) \rightarrow \iota \text{Sp}(q, \mathbb{R})
\]

lifts naturally to \( \tilde{\text{Sp}}(p, \mathbb{R}) \rightarrow \tilde{\text{Sp}}(q, \mathbb{R}) \). Let \( \phi \in I^\infty(\epsilon, t) \). Then

\[
\phi(g_1, g_2) = \phi(1, g_2 g_1^{-1}) \quad (g_1 \in \tilde{\text{Sp}}(p, \mathbb{R}), g_2 \in \tilde{\text{Sp}}(q, \mathbb{R})).
\]

In addition

\[
\phi(1, g_2 h) = \mu^t \otimes \nu^t + \rho(h^{-1}) \phi(1, g_2) \quad (h \in \tilde{GL}(q - p) N_{p, q-p}).
\]

Now let us consider the action of \( \tilde{\text{Sp}}(p, \mathbb{R}) \) and \( \tilde{\text{Sp}}(q, \mathbb{R}) \) on \( I(\epsilon, t) \). By Lemma 5.4, we obtain
Lemma 5.5. Let \( \phi \in I^\infty(\epsilon, t) \) and \( h_1 \in \tilde{Sp}(p, \mathbb{R}) \) and \( g_2 \in \tilde{Sp}(q, \mathbb{R}) \). Then
\[
[I(\epsilon, t)(h_1)\phi](1, g_2) = f(1, g_2 h_1).
\]
In particular the restriction map
\[
\phi \in I^\infty(\epsilon, t) \rightarrow \phi |_{\tilde{Sp}(q, \mathbb{R})} \in C^\infty(\tilde{Sp}(q, \mathbb{R}) \times GL(q-p)N_{p,q-p} \mathbb{C}^{\mu \otimes \nu^t})
\]
intertwines the left regular action of \( \tilde{Sp}(p, \mathbb{R}) \) on \( I^\infty(\epsilon, t) \) with the right regular action of \( \tilde{Sp}(p, \mathbb{R}) \) on \( C^\infty(\tilde{Sp}(q, \mathbb{R}) \times GL(q-p)N_{p,q-p} \mathbb{C}^{\mu \otimes \nu^t}) \).

Obviously, the restriction map also intertwines the left regular actions of \( \tilde{Sp}(q, \mathbb{R}) \).

6. Mixed Model

Now fix complex structures on \( \mathbb{R}^{2p} \) and \( \mathbb{R}^{2q} \) and inner products \((,)_p\), \((,)_q\) such that
\[
\Omega_p = \mathfrak{H}(,)_p, \quad \Omega_q = -\mathfrak{H}(,)_q.
\]
Let \( U(p) \) and \( U(q) \) be the unitary groups preserving \((,)_p\) and \((,)_q\) respectively. \( U(p) \) and \( U(q) \) are maximal compact subgroups of \( Sp(p, \mathbb{R}) \) and \( Sp(q, \mathbb{R}) \). Let \( U(n) \) be the unitary group preserving \((,)_p + (,)_q\). Then \( U(n) \) is a maximal compact subgroup of \( Sp(n, \mathbb{R}) \). In addition,
\[
U(p) = Sp(p, \mathbb{R}) \cap U(n) \quad U(q) = Sp(q, \mathbb{R}) \cap U(n).
\]
Identify \( U(q) \cap P_{p,q-p} \) with \( O(q-p)U(p) \). Recall that \( X_0 \cong Sp(q, \mathbb{R})/GL(q-p)N_{p,q-p} \). The group \( Sp(p, \mathbb{R}) \) acts on \( X_0 \) freely from the right. We obtain a principal fibration
\[
Sp(p, \mathbb{R}) \rightarrow X_0 \rightarrow Sp(q, \mathbb{R})/P_{p,q-p} \cong U(q)/O(q-p)U(p).
\]
This fibration allows us to visualize the action of \( \tilde{Sp}(p, \mathbb{R})\tilde{Sp}(q, \mathbb{R}) \) on \( I^\infty(\epsilon, t) \). Let \( dg_1 \) be a Haar measure on \( Sp(p, \mathbb{R}) \) and \( d[k_2] \) be an invariant probability measure on \( U(q)/O(q-p)U(p) \). Then \( dg_1 d[k_2] \) defines an \( U(q)Sp(p, \mathbb{R}) \) invariant measure on \( X_0 \).

Definition 6.1. Let \( M = Sp(p, \mathbb{R})U(q) \subset Sp(p, \mathbb{R})Sp(q, \mathbb{R}) \subset Sp(n, \mathbb{R}) \). Elements in \( X_0 \) are parametrized by a pair \( (g_1, [k_2]) \) for \( (g_1, k_2) \in M \). For each \( g \in \tilde{Sp}(n, \mathbb{R}) \), write \( g = \tilde{u}(g)p(g) \) where \( \tilde{u}(g) \in \tilde{U}(n) \) and \( p(g) \in P_0 \), the identity component of \( \tilde{P} \). For each \( (g_1, k_2) \in (\tilde{Sp}(p, \mathbb{R}), \tilde{U}(q)) \), we have
\[
g_1k_2 = \tilde{u}(g_1k_2)p(g_1k_2) = k_2\tilde{u}(g_1)p(g_1).
\]
The component \( \tilde{u} \) defines a map from \( \tilde{M} \) to \( \tilde{U}(n) \). In particular, \( \tilde{u} \) induces a map from \( \tilde{M}/O(q-p)\tilde{U}(p) \) to \( \tilde{U}(n)/\tilde{O}(n) \) which will be denoted by \( j \). The map \( j \) parametrizes the open dense subset \( X_0 \) in \( X \) by
\[
([g_1], [k_2]) \in \tilde{Sp}(p, \mathbb{R})/C \times \tilde{U}(q)/\tilde{O}(q-p)\tilde{U}(p).
\]
Change the variables on $X_0$ from $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$ to $\tilde{U}(n)/\tilde{O}(n)$. Let $J([g_1],[k_2])$ be the Jacobian:

$$
\frac{dj([g_1],[k_2])}{d[g_1][d[k_2]]}.
$$

$J$ can be regarded as a function on $Sp(p,\mathbb{R})U(q)$ or $Sp(p,\mathbb{R})U(q)/U(p)O(q-p)$, even though it is defined as a function on the covering. Denote the line bundle

$$
\tilde{Sp}(q,\mathbb{R})\times_{\tilde{GL}(q-p)} N_{p,q-p} \mathbb{C}^{\mu^*} \otimes \mu^+ \rightarrow X_0.
$$

by $\mathcal{M}_{\epsilon,t}$. Denote the line bundle

$$
\tilde{M} \times \tilde{O}(q-p)\tilde{U}(p) \mathcal{C}^{\mu^*} \rightarrow \tilde{M}/\tilde{O}(q-p)\tilde{U}(p) \cong X_0.
$$

by $\mathcal{M}_{\epsilon}$.

Clearly, $I_{c,X_0}(\epsilon,t) \subset I^\infty(\epsilon,t)$. Consider the restriction of $( , )_X$ onto $I_{c,X_0}(\epsilon,t)$. We are interested in expressing $( , )_X$ as an integral on $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$. This boils down to a change of variables from $\tilde{U}(n)/\tilde{O}(n)$ to $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$. We have

**Lemma 6.2.** Let $\Delta_t(g_1,k_2) = \nu(p(g_1))^{t+2p}J([g_1],[k_2])$ (see Equ.(2)). Then for every $f_1,f_2 \in I^\infty(\epsilon,t)$ we have

$$(f_1,f_2)_X = \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(g_1k_2)f_2(g_1k_2)\Delta_t(g_1,k_2)d[g_1][d[k_2]]$$

where $g_1 \in \tilde{Sp}(p,\mathbb{R})$, $k_2 \in \tilde{U}(q)$, $[g_1] \in Sp(p,\mathbb{R})$ and $[k_2] \in \tilde{U}(q)/\tilde{O}(q-p)\tilde{U}(p)$. Furthermore, $\Delta_t(g_1,k_2)$ is a nonnegative right $\tilde{O}(q-p)\tilde{U}(p)$-invariant function on $\tilde{M}$.

**Proof.** We compute

$$
\int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(g_1k_2)f_2(g_1k_2)\Delta_t(g_1,k_2)d[g_1][d[k_2]]
$$

$$
= \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(\hat{u}(g_1k_2))f_2(\hat{u}(g_1k_2))\nu(p(g_1))^{-t-2p}\Delta_t(g_1,k_2)d[g_1][d[k_2]]
$$

$$
= \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(\hat{u}(g_1k_2))f_2(\hat{u}(g_1k_2))\nu(p(g_1))^{-t-2p}\Delta_t(g_1,k_2)J^{-1}(g_1,k_2)dj([g_1],[k_2])
$$

$$
= \int_{X_0} f_1(\hat{u})f_2(\hat{u})d[\hat{u}] = (f_1,f_2)_X.
$$

(5)

Since $\nu(p(g_1)$ and $J([g_1],[k_2])$ remain the same when we multiply $k_2$ on the right by $\tilde{O}(q-p)\tilde{U}(p)$, $\Delta_t(g_1,k_2)$ is a nonnegative right $\tilde{O}(q-p)\tilde{U}(p)$-invariant function.

**Corollary 6.3.** As representations of $\tilde{Sp}(p,\mathbb{R})\tilde{Sp}(q,\mathbb{R})$,

$$
I_X(\epsilon,t) \cong L^2(\mathcal{M}_{\epsilon,t}, \Delta_t d[g_1][d[k_2]]).
$$
For each $f_1, f_2 \in I_{c,X_0}^\infty(\epsilon, t)$, define

$$ (f_1, f_2)_{M,t} = \int_{\tilde{\mathcal{M}}/\tilde{\mathcal{O}}(q-p)\tilde{\mathcal{U}}(p)} \frac{f_1(g_1k_2)f_2(g_1k_2)}{f_2(g_1k_2)} \Delta_t(g_1k_2)d|g_1|d|k_2|, $$

$$ (f_1, f_2)_M = \int_{\tilde{\mathcal{M}}/\tilde{\mathcal{O}}(q-p)\tilde{\mathcal{U}}(p)} f_1(g_1, k_2)f_2(g_1k_2)d|g_1|d|k_2|. $$

The completion of $I_{c,X_0}^\infty(\epsilon, t)$ under $(\cdot, \cdot)_{M,t}$ is $L^2(\mathcal{M}_{c,t}, \Delta_t d|g_1|d|k_2|)$. We call $L^2(\mathcal{M}_{c,t}, \Delta_t d|g_1|d|k_2|)$, the mixed model. We denote it by $I_M(\epsilon, t)$. On $I_M(\epsilon, t)$, the actions of $\tilde{Sp}(p, \mathbb{R})$ and $\tilde{Sp}(q, \mathbb{R})$ are easy to manipulate.

7. Mixed Model for Unitary Principal Series

**Lemma 7.1.** If $t \in i\mathbb{R}$, then $\Delta_t(g_1, k_2)$ is a constant and $(\cdot, \cdot)_{M,t}$ is a constant multiple of $(\cdot, \cdot)_M$.

**Proof.** Let $t \in i\mathbb{R}$. Let $f_1, f_2 \in I^\infty(\epsilon, t)$ and $h \in \tilde{Sp}(p, \mathbb{R})$. Recall that $X_0$ is parametrized by a pair $[g_1] \in \tilde{Sp}(p, \mathbb{R})/C$ and $[k_2] \in \tilde{U}(q)/\tilde{O}(q-p)\tilde{U}(p)$. By Lemma 6.2, we have

$$ (I(\epsilon, t)(h)f_1, I(\epsilon, t)(h)f_2)_X = \int_{X_0} f_1(h^{-1}g_1k_2)f_2(h^{-1}g_1k_2)\Delta_t(g_1, k_2)d|g_1|d|k_2| $$

$$ = \int_{X_0} f_1(g_1k_2)f_2(g_1k_2)\Delta_t(hg_1, k_2)d|g_1|d|k_2| $$

(6)

Since $I(\epsilon, t)$ is unitary, $(I(\epsilon, t)(h)f_1, I(\epsilon, t)(h)f_2)_X = (f_1, f_2)_X$. We have

$$ \int_{X_0} f_1(g_1k_2)f_2(g_1k_2)\Delta_t(hg_1, k_2)d|g_1|d|k_2| = \int_{X_0} f_1(g_1k_2)f_2(g_1k_2)\Delta_t(g_1, k_2)d|g_1|d|k_2|. $$

It follows that $\Delta_t(hg_1, k_2) = \Delta_t(g_1, k_2)$ for any $h \in \tilde{Sp}(p, \mathbb{R})$. Similarly, we obtain $\Delta_t(g_1, k_2) = \Delta_t(g_1, k_2)$ for any $k \in \tilde{U}(q)$. Hence, $\Delta_t(g_1, k_2)$ is a constant for purely imaginary $t$.

Combining with Corollary 6.3, we obtain

**Theorem 7.2.** Let $t \in i\mathbb{R}$. The restriction map $f \rightarrow f_{X_0}$ induces an isometry between $I(\epsilon, t)$ and $L^2(\mathcal{M}_{c,t}, d|g_1|d|k_2|)$. In addition, this isometry intertwines the actions of $\tilde{Sp}(p, \mathbb{R})\tilde{Sp}(q, \mathbb{R})$. So as $\tilde{Sp}(p, \mathbb{R})\tilde{Sp}(q, \mathbb{R})$ representations,

$$ I(\epsilon, t) \cong L^2(\mathcal{M}_{c,t}, d|g_1|d|k_2|); $$

and as $\tilde{Sp}(p, \mathbb{R})\tilde{U}(q)$ representations,

$$ I(\epsilon, t) \cong L^2(\mathcal{M}_c, d|g_1|d|k_2|). $$
Notice that $L^2(\mathcal{M}_c, d[g_1]d[k_2])$ does not depend on the parameter $t$. The following corollary is automatical.

**Corollary 7.3.** Suppose that $p + q = n$ and $p \leq q$. For $t$ real,

$$I(\epsilon, t)|_{\tilde{Sp}(p, \mathbb{R})U(q)} \cong I(\epsilon, 0)|_{\tilde{Sp}(p, \mathbb{R})U(q)} \cong L^2(\mathcal{M}_c, d[g_1]d[k_2]).$$

For $t$ a nonzero real number, $\Delta_t(g, k)$ is not a constant. So $C(\epsilon, t)$ cannot be modeled naturally on $L^2(\mathcal{M}_{c, t}, d[g_1]d[k_2])$. Nevertheless, we have

**Theorem 7.4 (Main Theorem).** Suppose that $p + q = n$ and $p \leq q$. Given a complementary series representation $C(\epsilon, t)$,

$$C(\epsilon, t)|_{\tilde{Sp}(p, \mathbb{R})U(q)} \cong I(\epsilon, 0)|_{\tilde{Sp}(p, \mathbb{R})U(q)} \cong L^2(\mathcal{M}_c, d[g_1]d[k_2]).$$

In other words, there is an isometry between $C(\epsilon, t)$ and $I(\epsilon, 0)$ that intertwines the actions of $U(q)$ and of $\tilde{Sp}(p, \mathbb{R})$.

We shall postpone the proof of this theorem to the next section. We will first derive some corollaries from Lemma 7.1 concerning $\Delta$ and $\nu(g_1)$.

**Corollary 7.5.** $J([g_1], [k_2]) = cv(p(g_1))^{-2p}$ for a constant $c$ and $\Delta_t(g_1, k_2) = cv(p(g_1))^{t+\gamma}$. So both $\Delta_t$ and $J([g_1], [k_2])$ do not depend on $k_2$. Furthermore,

$$I(\epsilon, t) \cong L^2(\mathcal{M}_{c, t}, \nu(p(g_1))^{t+\gamma}d[g_1]d[k_2]) = I_M(\epsilon, t). \quad (7)$$

$\nu(p(g_1))$ is a function on $\tilde{Sp}(p, \mathbb{R})/C$. So it can be regarded as a function on $Sp(p, \mathbb{R})$.

**Corollary 7.6.** $\nu(p(g_1))^{-\rho} \in L^2(\tilde{Sp}(p, \mathbb{R}))$ and $\nu(p(g_1))^{-1}$ is a bounded positive function.

**Proof.** Since $X$ is compact,

$$\int_{\tilde{Sp}(p, \mathbb{R})} \nu(p(g_1))^{-2\rho} dg_1 = C \int_{\tilde{M}/\tilde{O}(q-p)/\tilde{O}(p)} J([g_1], [k_2])d[g_1]d[k_2] = C \int_{\tilde{O}(n)/\tilde{O}(n)} 1d[k] < \infty.$$ 

So $\nu(p(g_1))^{-\rho} \in L^2(\tilde{Sp}(p, \mathbb{R}))$. Now we need to compute $\nu(g_1)$. Recall that $P$ is defined to be the stabilizer of

$$x_0 = \text{span}\{e_1 + f_1, \ldots, e_p + f_p, e_1^*, f_1^*, \ldots, e_p^*, f_p^*, f_{p+1}, \ldots, f_q\}.$$ 

So $j(g_1, 1)$ is the following Lagrangian

$$\text{span}\{g_1e_1 + f_1, \ldots, g_1e_p + f_p, g_1e_1^* + f_1^*, \ldots, g_1e_p^* + f_p^*, f_{p+1}, \ldots, f_q\}.$$ 

The action of $\tilde{U}(n)$ will not change the volume of the $n$-dimensional cube spanned by the basis above. So $\nu(p(g_1))$, as the determinant character, is equal to the volume of the $n$-dimensional cube, up to a constant. Hence

$$\nu(p(g_1)) = \left[2^{-n} \det(g_1g_1^* + I)\right]^\frac{1}{2}.$$ 

Clearly, $\nu(p(g_1))^{-1}$ is bounded and positive. ■
This corollary is easy to understand in terms of compactification. Notice that the map \( j \), without the covering,

\[
Sp(p, \mathbb{R})U(q)/U(p)O(q - p) \to U(n)/O(n)
\]
is an analytic compactification. Hence the Jacobian \( J(g_1, [k_2]) \) should be positive and bounded above. Since \( J(g_1, [k_2]) = cv(p(g_1))^{-2p} \), \( \nu(p(g_1))^{-1} \) must also be positive and bounded above. The situation here is similar to [He02] (see Appendix) and [He06] (Theorem 2.3). It is not clear that \( j(g_1, 1) \) gets mapped onto \( U(2p)/O(2p) \) though.

If \( f \in I_M(\epsilon, t_1) \) and \( h > 0 \), by Cor. 7.6 and Equation (7), we have \( \| f \|_{M, h - h} \leq C\| f \|_{M, A_1} \). So \( I_M(\epsilon, t_1) \subset I_M(\epsilon, t_1 - h) \).

**Corollary 7.7.** Suppose that \( h > 0 \). Then \( I_M(\epsilon, t_1) \subset I_M(\epsilon, t_1 - h) \).

## 8. “Square Root” of the Intertwining Operator

Suppose from now on \( t \in \mathbb{R} \). For \( f \in I^\infty(\epsilon, t)|\tilde{M} \), define a function on \( \tilde{M} \),

\[
(A_M(\epsilon, t)f)(g_1k_2) = A(\epsilon, t)f(g_1k_2) \quad (g_1 \in \tilde{Sp}(p, \mathbb{R}), k_2 \in \tilde{U}(q)).
\]

So \( A_M(\epsilon, t) \) is the “restriction” of \( A(\epsilon, t) \) onto \( \tilde{M} \). \( A_M(\epsilon, t) \) is not yet an unbounded operator on \( I_M(\epsilon, t) \). In fact, for \( t > 0 \), \( A_M(\epsilon, t) \) does not behave well and it is not clear whether \( A_M(\epsilon, t) \) can be realized as an unbounded operator on \( I_M(\epsilon, t) \). The function \( A_M(\epsilon, t)f \) differs from \( A_X(\epsilon, t)f \).

**Lemma 8.1.** For \( t \in \mathbb{R} \) and \( f \in I^\infty(\epsilon, t) \),

\[
(A_M(\epsilon, t)f|\tilde{M})(g_1k_2) = (A_X(\epsilon, t)f)(g_1k_2)\nu(p(g_1))^{2t} = (A_X(\epsilon, t)f)(g_1k_2)\Delta_t(g_1, k_2).
\]

This Lemma is due to the fact that \( A_X(\epsilon, t)f \in I(\epsilon, t) \) but \( A(\epsilon, t)f \in I(\epsilon, -t) \).

Let \( f \in I^\infty(\epsilon, t) \). In terms of the mixed model, the invariant Hermitian form \( (, )_{\epsilon,t} \) can be written as follows:

\[
(f, f)_{\epsilon,t} = (A_X(\epsilon, t)f, f)_X = \int_{\tilde{M}/\tilde{\partial}(q-p)U(p)} A_M(\epsilon, t)f|\tilde{M} \overline{f}|\tilde{M} d[g_1]d[k_2].
\]

This follows from Lemma 8.1 and Lemma 6.2. We obtain

**Lemma 8.2.** For \( f_1, f_2 \in I^\infty(\epsilon, t) \), \( (f_1, f_2)_{\epsilon,t} = (A_M(\epsilon, t)f_1|\tilde{M}, f_2|\tilde{M})_M \).

**Theorem 8.3.** If \( t < 0 \) and \( C(\epsilon, \cdot) \) is a complementary series representation, then \( A_M(\epsilon, t) \) is a positive and densely defined symmetric operator. Its self-adjoint-extension \((A_M(\epsilon, t) + I)|_{L^2(\mathcal{M}, d[g_1]d[k_2])} \to I(\epsilon) \) has a unique square root which extends to an isometry from \( L^2(\mathcal{M}, d[g_1]d[k_2]) \).
Proof. Let \( t < 0 \). Put
\[
\mathcal{H} = L^2(\mathcal{M}_t, d[g_1]d[k_2]).
\]
Let \( f \in I^\infty(\epsilon, t) \). Then \( A_M(\epsilon, t)(f|_{\tilde{M}})(g_1k_2) = \nu(p(g_1))^{2t}A_X(\epsilon, t)f(g_1k_2) \). By Lemma 8.1, Cor. 7.6 and Lemma 6.2, we have
\[
\begin{align*}
&= \int_{M/\tilde{O}(q-p)/(p)} A_M(\epsilon, t)(f|_{\tilde{M}})A_M(\epsilon, t)(f|_{\tilde{M}})d[g_1]d[k_2] \\
&= \int_{M/\tilde{O}(q-p)/(p)} \nu(p(g_1))^{2t}|(A_X(\epsilon, t)f)(g_1k_2)|^2 \nu(p(g_1))^{2t}d[g_1]d[k_2] \\
&= \int_{M/\tilde{O}(q-p)/(p)} \nu(p(g_1))^{2t}|(A_X(\epsilon, t)f)(g_1k_2)|^2 \Delta_t(g_1, k_2)d[g_1]d[k_2] \\
&\leq C \int_{M/\tilde{O}(q-p)/(p)} |A_X(\epsilon, t)f(g_1k_2)|^2 \Delta_t(g_1, k_2)d[g_1]d[k_2] \\
&= C(A_X(\epsilon, t)f, A_X(\epsilon, t)f)_X < \infty.
\end{align*}
\]
Therefore, \( A_M(\epsilon, t)(f|_{\tilde{M}}) \in \mathcal{H} \). Let \( \mathcal{D} = I^\infty(\epsilon, t)|_{\tilde{M}} \). Clearly, \( \mathcal{D} \) is dense in \( \mathcal{H} \). So \( A_M(\epsilon, t) \) is a densely defined unbounded operator. It is positive and symmetric by Lemma 8.2.

**Definition 8.4.** Define \( \mathcal{U}(\epsilon, t) = ((A_M(\epsilon, t) + I)_0 - I)^{\frac{1}{2}} \).

Now \((f, g)_{\mathcal{U}} = (A_M(\epsilon, t)f|_{\tilde{M}}, g|_{\tilde{M}})_M\) for any \( f, g \in I^\infty(\epsilon, t) \). So \( C(\epsilon, t) = \mathcal{H}_{A_M(\epsilon, t)} \). By Lemma 2.1, \( \mathcal{U}(\epsilon, t) \), mapping from \( C(\epsilon, t) \) into \( \mathcal{H} \), is an isometry.

Suppose that \( \mathcal{U}(\epsilon, t) \) is not onto. Let \( f \in \mathcal{H} \) such that for any \( u \in \mathcal{D}(\mathcal{U}(\epsilon, t)) \),
\[
(f, \mathcal{U}(\epsilon, t)u)_M = 0.
\]
Notice that
\[
I^\infty(\epsilon, t)|_{\tilde{M}} \subset \mathcal{D}((A_M(\epsilon, t) + I)_0 - I) \subset \mathcal{D}(\mathcal{U}(\epsilon, t)),
\]
and
\[
\mathcal{U}(\epsilon, t)\mathcal{U}(\epsilon, t) = (A_M(\epsilon, t) + I)_0 - I.
\]
In particular,
\[
\mathcal{U}(\epsilon, t)I^\infty(\epsilon, t)|_{\tilde{M}} \subset \mathcal{D}(\mathcal{U}(\epsilon, t)).
\]
It follows that
\[
\begin{align*}
&= (f, (A_M(\epsilon, t) + I)_0 - I)I^\infty(\epsilon, t)|_{\tilde{M}})_M \\
&= (f, (A_M(\epsilon, t) + I)_0 - I)I^\infty(\epsilon, t)|_{\tilde{M}})_M \\
&= (f, \mathcal{U}(\epsilon, t)\mathcal{U}(\epsilon, t)I^\infty(\epsilon, t)|_{\tilde{M}})_M \\
&= 0.
\end{align*}
\]
Let \( f_{\epsilon, t} \) be a function such that \( f_{\epsilon, t}|_{\tilde{M}} = f \) and
\[
f_{\epsilon, t}(gln) = (\mu^t \otimes \nu^{t+\rho})(l^{-1})f_{\epsilon, t}(g) \quad (l \in \tilde{L}, n \in N).
\]
By Lemma 8.2, \( \forall u \in V(\epsilon, t) \),
\[
0 = (f, A_M(\epsilon, t)(u|_M))_M = (f_{\epsilon,t}, A_X(\epsilon, t)u)_X = (f_{\epsilon,t}, u)_{\epsilon,t}.
\]
This equality is to be interpreted as an equality of integrals according to the definitions of \( ( , )_M \) and \( ( , )_X \). Since \( A_X(\epsilon, t) \) acts on \( U(n) \)-types in \( V(\epsilon, t) \) as scalars, \( A_X(\epsilon, t)V(\epsilon, t) = V(\epsilon, t) \). We now have
\[
(f_{\epsilon,t}, V(\epsilon, t))_X = 0.
\]
In particular, \( f_{\epsilon,t}|_{U(n)} \in L^1(X) \). By Peter-Weyl Theorem, \( f_{\epsilon,t} = 0 \). We see that \( U(\epsilon, t) \) is an isometry from \( C(\epsilon, t) \) onto \( L^2(M_\epsilon, d[g_1]d[k_2]) \).

The Hilbert space \( L^2(M_\epsilon, d[g_1]d[k_2]) \) is the mixed model for \( I(\epsilon, 0) \) restricted to \( \tilde{M} \). We now obtain an isometry from \( C(\epsilon, t) \) onto \( I(\epsilon, 0) \). Within the mixed model, the action of \( I(\epsilon, t)(g_1k_2) \) is simply the left regular action and it is independent of \( t \). We obtain

**Lemma 8.5.** Suppose \( t < 0 \). Let \( g \in \tilde{U}(q) \). Let \( L(g) \) be the left regular action on \( L^2(M_\epsilon, d[g_1]d[k_2]) \). As an operator on \( I_\infty(\epsilon, t)|_{\tilde{M}} \), \( L(g) \) commutes with \( A_M(\epsilon, t) \). Furthermore, \( L(g) \) commutes with \( (A_M(\epsilon, t) + I)_{0}^{-1} \). Similar statement holds for \( g \in Sp(p, \mathbb{R}) \).

**Proof.** Let \( g \in \tilde{M} \). Both \( A_M(\epsilon, t) \) and \( L(g) \) are well-defined operator on \( I_\infty(\epsilon, t)|_{\tilde{M}} \). Regarding \( A(\epsilon, t)I(\epsilon, t)(g) = I(\epsilon, -t)(g)A(\epsilon, t) \) as operators on the mixed model \( L^2(M_\epsilon, d[g_1]d[k_2]) \), we have
\[
A_M(\epsilon, t)L(g) = L(g)A_M(\epsilon, t).
\]
It follows that
\[
L(g)^{-1}(A_M(\epsilon, t) + I)L(g) = (A_M(\epsilon, t) + I).
\]
Since \( L(g) \) is unitary, \( L(g)^{-1}(A_M(\epsilon, t) + I)_{0}L(g) = (A_M(\epsilon, t) + I)_{0} \). In fact, \( (A_M(\epsilon, t) + I)_{0} \) can be defined as the inverse of \( (A_M(\epsilon, t) + I)^{-1} \), which exists and is bounded. So \( L(g) \) commutes with both \( (A_M(\epsilon, t) + I)^{-1} \) and \( (A_M(\epsilon, t) + I)_{0} \).

**Lemma 8.6.** We have, for \( g \in \tilde{M} \), \( U(\epsilon, t)I(\epsilon, t)(g) = I(\epsilon, 0)(g)U(\epsilon, t) \).

**Proof.** Recall from Theorem 7.2 that the action of \( \tilde{M} \) on the mixed model is independent of \( t \). It suffices to show that on the mixed model, \( U(\epsilon, t) \) commutes with \( L(g) \) for any \( g \in \tilde{M} \). By Lemma 8.5,
\[
L(g)^{-1}[(A_M(\epsilon, t) + I)_{0} - I)L(g) = (A_M(\epsilon, t) + I)_{0} - I.
\]
Since \( L(g) \) is unitary on \( L^2(M_\epsilon, d[g_1]d[k_2]) \), both sides are positive self-adjoint operators. Taking square roots, we obtain \( L(g)^{-1}U(\epsilon, t)L(g) = U(\epsilon, t) \). Theorem 7.2 is proved.
References


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