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Restrictions of Certain Degenerate Principal Series of the Universal Covering of the Symplectic Group

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Abstract. Let $\widetilde{Sp}(n,\mathbb{R})$ be the universal covering of the symplectic group. In this paper, we study the restrictions of the degenerate unitary principal series $I(\epsilon,t)$ of $\widetilde{Sp}(n,\mathbb{R})$ onto $\widetilde{Sp}(p,\mathbb{R})\widetilde{Sp}(n-p,\mathbb{R})$. We prove that if $n \geq 2p$, $I(\epsilon,t)|_{\widetilde{Sp}(p,\mathbb{R})\widetilde{Sp}(n-p,\mathbb{R})}$ is unitarily equivalent to an L^2-space of sections of a homogeneous line bundle $L^2(\tilde{S}p(n-p,\mathbb{R}) \times_{\widetilde{GL}(n-2p)N} \mathbb{C}_{\epsilon,t+\rho})$ (see Theorem 1.1). We further study the restriction of complementary series $C(\epsilon,t)$ onto $\tilde{U}(n-p)\widetilde{Sp}(p,\mathbb{R})$. We prove that this restriction is unitarily equivalent to $I(\epsilon,t)|_{\tilde{U}(n-p)\widetilde{Sp}(p,\mathbb{R})}$ for $t \in i\mathbb{R}$. Our results suggest that the direct integral decomposition of $C(\epsilon,t)|_{\widetilde{Sp}(p,\mathbb{R})\widetilde{Sp}(n-p,\mathbb{R})}$ will produce certain complementary series for $\widetilde{Sp}(n-p,\mathbb{R})$ ([He09]).
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1. Introduction

Let $\widetilde{Sp}(n,\mathbb{R})$ be the universal covering of $Sp(n,\mathbb{R})$. $\widetilde{Sp}(n,\mathbb{R})$ is a central extension of $Sp(n,\mathbb{R})$:

$$1 \to C \to \widetilde{Sp}(n,\mathbb{R}) \to Sp(n,\mathbb{R}) \to 1,$$

where $C \cong \mathbb{Z}$. The unitary dual of C is parametrized by a torus $\mathbb{T}$. For each $\kappa \in \mathbb{T}$, denote the corresponding unitary character of C by χ^κ. We say that a representation π of $\widetilde{Sp}(n,\mathbb{R})$ is of class κ if $\pi|_C = \chi^\kappa$. Since C commutes with $\widetilde{Sp}(n,\mathbb{R})$, for any irreducible representation π of $\widetilde{Sp}(n,\mathbb{R})$, $\pi|_C = \chi^\kappa$ for some κ.

Denote the projection $\widetilde{Sp}(n,\mathbb{R}) \to Sp(n,\mathbb{R})$ by p. For any subgroup H of $Sp(n,\mathbb{R})$, denote the full inverse image $p^{-1}(H)$ by $\tilde{H}$. We adopt the notation from [Sa]. Let P be the Siegel parabolic subgroup of $Sp(n,\mathbb{R})$. One dimensional characters of $\tilde{P}$ can be parametrized by (ϵ,t) where $\epsilon \in \mathbb{T}$ and $t \in \mathbb{C}$. Let $I(\epsilon,t)$ be the representation of $\widetilde{Sp}(n,\mathbb{R})$ induced from the one dimensional character $\mathbb{C}_{\epsilon,t}$ parametrized by (ϵ,t) of $\tilde{P}$. If $t \in i\mathbb{R}$ and $t \neq 0$, $I(\epsilon,t)$ is unitary and irreducible. $I(\epsilon,t)$ is called *unitary degenerate principal series.* If t is real, then $I(\epsilon,t)$ has a

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nontrivial invariant Hermitian form. Sahi gives a classification of all irreducible unitarizable $I(\epsilon,t)$. If $I(\epsilon,0)$ is irreducible, there are *complementary series* $C(\epsilon,t)$ for t in a suitable interval ([Sa]). Some of these complementary series are obtained by Kudla-Rallis [KR], Østed-Zhang [OZ], Branson-Østed-Olafsson [BOO], Lee [Lee]. Strictly speaking $C(\epsilon,t)$ should be called *degenerate complementary series* because there are complementary series associated with the principal series, which should be called complementary series ([Kos], [ABPTV]). Throughout this paper, complementary series will mean $C(\epsilon,t)$.

Let $(Sp(p,\mathbb{R}), Sp(n-p,\mathbb{R}))$ be a pair of symplectic groups diagonally embeded in $Sp(n,\mathbb{R})$ (see Definition 5.1). Let $U(n)$ be a maximal compact subgroup such that $Sp(n-p,\mathbb{R})\cap U(n)$ and $Sp(p,\mathbb{R})\cap U(n)$ are maximal compact subgroups of $Sp(n-p,\mathbb{R})$ and $Sp(p,\mathbb{R})$ respectively. Denote $Sp(n-p,\mathbb{R})\cap U(n)$ by $U(n-p)$ and $Sp(p,\mathbb{R})\cap U(n)$ by $U(p)$. The main results of this paper can be stated as follows.

Theorem 1.1. *Suppose $p\leq n-p$ and $t\in i\mathbb{R}$. Let $P_{p,n-2p}$ be a maximal parabolic subgroup of $Sp(n-p,\mathbb{R})$ with Langlands decomposition $Sp(p,\mathbb{R})GL(n-2p)N_{p,n-2p}$. Let $\mathcal{M}_{\epsilon,t}$ be the homogeneous line bundle*

$$\widetilde{Sp}(n-p,\mathbb{R})\times_{\widetilde{GL}(n-2p)N_{p,n-2p}}\mathbb{C}_{\epsilon,t+\rho}\rightarrow Sp(n-p,\mathbb{R})/GL(n-2p)N_{p,n-2p}$$
$$(\cong Sp(p,\mathbb{R})U(n-p)/U(p)O(n-2p)), \tag{1}$$

where $\rho=\frac{n+1}{2}$. Let $dg_1d[k_2]$ be an $Sp(p,\mathbb{R})U(n-p)$-invariant measure. Then

$$I(\epsilon,t)|_{\widetilde{Sp}(p,\mathbb{R})\widetilde{Sp}(n-p,\mathbb{R})}\cong L^2(\mathcal{M}_{\epsilon,t}, dg_1d[k_2]),$$

on which $\widetilde{Sp}(n-p,\mathbb{R})$ acts from the left and $\widetilde{Sp}(p,\mathbb{R})$ acts from the right.

Theorem 1.2. *Let $C(\epsilon,t)$ be a complementary series representation. Suppose that $p\leq n-p$. Then*

$$C(\epsilon,t)|_{\tilde{U}(n-p)\tilde{S}p(p,\mathbb{R})}\cong I(\epsilon,0)|_{\tilde{U}(n-p)\tilde{S}p(p,\mathbb{R})}\cong I(\epsilon,i\lambda)|_{\tilde{U}(n-p)\tilde{S}p(p,\mathbb{R})}\qquad(\lambda\in\mathbb{R}).$$

$p=[\frac{n}{2}]$ is the best possible value for such a statement. In particular, for $\widetilde{Sp}(2m+1,\mathbb{R})$

$$I(\epsilon,0)|_{\widetilde{Sp}(m+1,\mathbb{R})}\ncong C(\epsilon,t)|_{\widetilde{Sp}(m+1,\mathbb{R})}.$$

To see this, let $L^2(\widetilde{Sp}(n,\mathbb{R}))_\kappa$ be the set of functions with

$$f(zg)=\chi^\kappa(z)f(g)\qquad(z\in C, g\in\widetilde{Sp}(n,\mathbb{R}));$$

$$\|f\|^2=\int_{Sp(n,\mathbb{R})}|f(g)|^2d[g]<\infty\qquad(g\in\widetilde{Sp}(n,\mathbb{R}), [g]\in Sp(n,\mathbb{R})).$$

We say that a representation of class κ is tempered if it is weakly contained in $L^2(\widetilde{Sp}(n,\mathbb{R}))_\kappa$. By studying the leading exponents of $I(\epsilon,0)$ and $C(\epsilon,t)$, it can be shown that $I(\epsilon,0)|_{\widetilde{Sp}(m+1,\mathbb{R})}$ is tempered and $C(\epsilon,t)|_{\widetilde{Sp}(m+1,\mathbb{R})}$ is not tempered. Therefore

$$I(\epsilon,0)|_{\widetilde{Sp}(m+1,\mathbb{R})} \not\cong C(\epsilon,t)|_{\widetilde{Sp}(m+1,\mathbb{R})}.$$

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2. A Lemma on Friedrichs Extension

Let S be a semibounded densely defined symmetric operator on a Hilbert space H. S is said to be *positive* if $(Su,u)>0$ for every nonzero $u\in\mathcal{D}(S)$. Suppose that S is positive. For $u,v\in\mathcal{D}(S)$, define

$$(u,v)_S=(u,Sv),$$

$$\|u\|_S^2=(u,Su).$$

Let H_S be the completion of $\mathcal{D}(S)$ under the norm $\|\ \|_S$. Clearly $H_{S+I}\subseteq H$ and $H_{S+I}\subseteq H_S$.

The operator $S+I$ has a unique self-adjoint extension $(S+I)_0$ in H, the Friedrichs extension. $(S+I)_0$ has the following properties

- $\mathcal{D}(S)\subseteq\mathcal{D}((S+I)_0)\subseteq H_{S+I}\subseteq H$;
- $(u,v)_{S+I}=(u,(S+I)_0v)$ for all $u\in H_{S+I}$ and $v\in\mathcal{D}((S+I)_0)$

(see Theorem in Page 335 [RS]). Now consider $(S+I)_0-I$. It is an self-adjoint extension of S. It is nonnegative. By the spectral decomposition and functional calculus, $(S+I)_0-I$ has a unique *square root* T (See Pg. 127. 128. [RS]).

Lemma 2.1. *Let S be a positive densely defined symmetric operator. Then the square root of $(S+I)_0-I$ extends to an isometry from H_S into H.*

Proof. Clearly, the spectrum of T is contained in the nonnegative part of the real line. By spectral decomposition $\mathcal{D}((S+I)_0-I)=\mathcal{D}((S+I)_0)\subseteq\mathcal{D}(T)$ and $TT=(S+I)_0-I$. In addition for any $u,v\in\mathcal{D}(S)\subseteq D((S+I)_0)$,

$$(Tu,Tv)=(u,TTv)=(u,(S+I)_0v-v)=(u,Sv)=(u,v)_S.$$

So T is an isometry from $\mathcal{D}(S)$ into H. Since $\mathcal{D}(S)$ is dense in H_S, T extends to an isometry from H_S into H. ■

3. Degenerate Principal Series of $\widetilde{Sp}(n,\mathbb{R})$

Fix the Lie algebra:

$$\mathfrak{sp}(n,\mathbb{R})=\{\begin{pmatrix} X & Y\\ Z & -X^t\end{pmatrix}\mid Y^t=Y, Z^t=Z\}$$

and the Siegel parabolic algebra:

$$\mathfrak{p} = \{ \begin{pmatrix} X & Y \\ 0 & -X^t \end{pmatrix} \mid Y^t = Y \}.$$

Fix the Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ with

$$\mathfrak{l} = \{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} \mid X \in \mathfrak{gl}(n, \mathbb{R}) \}, \qquad \mathfrak{n} = \{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \mid Y^t = Y \}.$$

Fix a Cartan subalgebra

$$\mathfrak{a} = \{ \operatorname{diag}(H_1, H_2, \ldots, H_n, -H_1, -H_2 \ldots, -H_n) \mid H_i \in \mathbb{R} \}.$$

Let $Sp(n, \mathbb{R})$ be the symplectic group and P be the Siegel parabolic subgroup. Set $U(n) = Sp(n, \mathbb{R}) \cap O(2n)$ where $O(2n)$ is the standard orthogonal group. Let LN be the Levi decomposition of P and A be the analytic group generated by the Lie algebra $\mathfrak{a}$. Clearly, $L \cong GL(n, \mathbb{R})$ and $L \cap U(n) \cong O(n)$. On the covering group, we have $\tilde{L} \cap \tilde{U}(n) = \tilde{O}(n)$. Recall that

$$\tilde{U}(n) = \{ (x, g) \mid g \in U(n), \exp 2\pi i x = \det g, x \in \mathbb{R} \}.$$

Therefore

$$\tilde{O}(n) = \{ (x, g) \mid g \in O(n), \exp 2\pi i x = \det g, x \in \mathbb{R} \}.$$

Notice that for $g \in O(n)$, $\det g = \pm 1$ and $x \in \frac{1}{2}\mathbb{Z}$. We have the following exact sequence

$$1 \to SO(n) \to \tilde{O}(n) \to \frac{1}{2}\mathbb{Z} \to 1.$$

Consequently, we have

$$1 \to GL_0(n, \mathbb{R}) \to \tilde{L} \to \frac{1}{2}\mathbb{Z} \to 1.$$

In fact,

$$\tilde{L} = \{ (x, g) \mid g \in L, \exp 2\pi i x = \frac{\det g}{|\det g|}, x \in \mathbb{R} \}.$$

The one dimensional unitary characters of $\frac{1}{2}\mathbb{Z}$ are parametrized by the one dimensional torus T. Identify T with $[0, 1)$. Let μ^ϵ be the character of $\frac{1}{2}\mathbb{Z}$ corresponding to $\epsilon \in [0, 1)$ Now each character μ^ϵ yields a character of $\tilde{L}$, which in turn, yields a character of $\tilde{P}$. For simplicity, we retain μ^ϵ to denote the character on $\tilde{L}$ and $\tilde{P}$. Let ν be the det-character on $\tilde{L}_0$, i.e.,

$$\nu(x, g) = |\det g| \qquad (x, g) \in \tilde{L}. \tag{2}$$

Let

$$I(\epsilon, t) = Ind_{\tilde{P}}^{\widetilde{Sp}(n,\mathbb{R})} \mu^\epsilon \otimes \nu^t$$

be the normalized induced representation with $\epsilon \in [0, 1)$ and $t \in \mathbb{C}$. This is Sahi's notation in the case of the universal covering of the symplectic group ([Sa]). $I(\epsilon, t)$ is a degenerate principal series representation. Clearly, $I(\epsilon, t)$ is unitary when $t \in i\mathbb{R}$.

When t is real and $I(\epsilon, t)$ is unitarizable, the unitary representation, often denoted by $C(\epsilon, t)$, is called a *complementary series representation.* Various complementary series of $Sp(n, \mathbb{R})$ and its metaplectic covering was determined explicitly or implicitly by Kudla-Rallis, Ørsted-Zhang, Brason-Olafsson-Ørsted and others. See [KR], [BOO], [OZ] and the references therein. The complete classification of the complementary series of the universal covering is due to Sahi.

Theorem 3.1 (Thm A, [Sa]). *Suppose that t is real. For n even, $I(\epsilon, t)$ is irreducible and unitarizable if and only if $0 < |t| < |\frac{1}{2} - |2\epsilon - 1||$. For n odd and $n > 1$, $I(\epsilon, t)$ is irreducible and unitarizable if and only if $0 < |t| < \frac{1}{2} - |\frac{1}{2} - |2\epsilon - 1||$.*

One can easily check that the complementary series exist if $\epsilon \neq 0, \frac{1}{2}$ for n odd and $n > 1$; if $\epsilon \neq \frac{1}{4}, \frac{3}{4}$ for n even. It is interesting to note that complementary series always exist unless $I(\epsilon, t)$ descends into a representation of the metaplectic group. For the metaplectic group $Mp(2n + 1, \mathbb{R})$, there are two complementary series $I(\frac{1}{4}, t)(0 < t < \frac{1}{2})$ and $I(\frac{3}{4}, t)(0 < t < \frac{1}{2})$. For the metaplectic group $Mp(2n, \mathbb{R})$, there are two complementary series $I(0, t)(0 < t < \frac{1}{2})$ and $I(\frac{1}{2}, t)(0 < t < \frac{1}{2})$. These four complementary series are the "longest".

For $n = 1$, the situation is quite different. The difference was pointed out in [KR]. For example, there are Bargmann's complementary series representation for $I(0, t)$ $(t \in (0, \frac{1}{2}))$. The classification of the complementary series of $\widetilde{Sp}(1, \mathbb{R})$ can be found in [Bar], [Puk], [Howe].

Since our restriction theorem only makes sense for $n \geq 2$, we will **assume** $n \geq 2$ **from now on**. The parameters for the complementary series of $\widetilde{Sp}(n, \mathbb{R})$ are illustrated in Fig. 1.

4. The generalized compact model and The Intertwining Operator

Recall that

$$\begin{aligned} I^\infty(\epsilon, t) = &\{f \in C^\infty(\widetilde{Sp}(n, \mathbb{R})) \mid f(gln) = (\mu^\epsilon \otimes \nu^{t+\rho})(l^{-1}) f(g), \\ &(g \in \widetilde{Sp}(n, \mathbb{R}), l \in \tilde{L}, n \in N)\} \end{aligned} \tag{3}$$

where $\rho = \frac{n+1}{2}$. Let $X = \widetilde{Sp}(n, \mathbb{R})/\tilde{P}$. Then $I^\infty(\epsilon, t)$ consists of smooth sections of the homogeneous line bundle $\mathcal{L}_{\epsilon,t}$

$$\widetilde{Sp}(n, \mathbb{R}) \times_{\tilde{P}} \mathbb{C}_{\mu^\epsilon \otimes \nu^{t+\rho}} \to X.$$

Since $X \cong \tilde{U}(n)/\tilde{O}(n)$, $\tilde{U}(n)$ acts transitively on X. The function $f \in I^\infty(\epsilon, t)$ is uniquely determined by $f|_{\tilde{U}(n)}$ and vice versa. Moreover, the homogeneous vector bundle $\mathcal{L}_{\epsilon,t}$ can be identified with $\mathcal{K}_{\epsilon,t}$

$$\tilde{U}(n) \times_{\tilde{O}(n)} \mathbb{C}_{\mu^\epsilon \otimes \nu^{t+\rho}}|_{\tilde{O}(n)} \to X$$

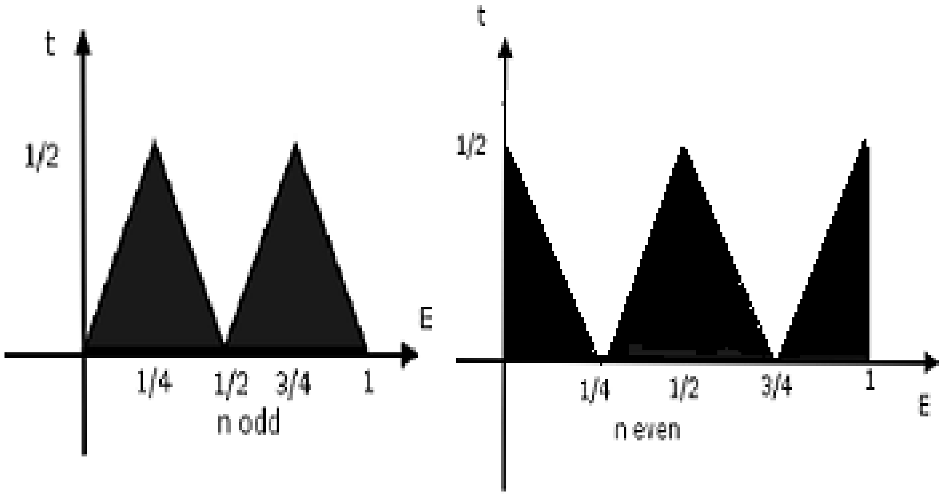

Figure 1: Complementary Parameters (E, t)

naturally. Notice that the homogeneous line bundle $\mathcal{K}_{\epsilon,t}$ does not depend on the parameter t. We denote this line bundle by $\mathcal{K}_{\epsilon}$. The representation $I^{\infty}(\epsilon, t)$ can then be modeled on smooth sections of $\mathcal{K}_{\epsilon}$. This model will be called *the generalized compact model.*

Let $d[k]$ be the normalized $\tilde{U}(n)$-invariant measure on X. The generalized compact model equips the smooth sections of $\mathcal{K}_{\epsilon,t}$ with a natural pre-Hilbert structure

$$(f_1, f_2)_X = \int_{[k]\in X} f_1(k)\overline{f_2(k)}d[k],$$

where $k \in \tilde{U}(n)$ and $[k] \in X$. It is easy to verify that $f_1(k)\overline{f_2(k)}$ is a function of $[k]$ and it does not depend on any particular choice of k. Notice that our situation is different from the compact model since $\tilde{U}(n)$ is not compact. We *denote the completion of I^{∞} with respect to $(\,,\,)_X$ by $I_X(\epsilon, t)$.*

Secondly, the action of $\tilde{U}(n)$ on $\mathcal{K}_{\epsilon}$ induces an orthogonal decomposition of $I_X(\epsilon, t)$:

$$I_X(\epsilon, t) = \hat{\oplus}_{\alpha\in 2\mathbb{Z}^n} V(\alpha + \epsilon(2, 2, \ldots, 2)),$$

where $V(\alpha + \epsilon(2, 2, \ldots, 2))$ is an irreducible finite dimensional representation of $\tilde{U}(n)$ with highest weight $\alpha + \epsilon(2, 2, \ldots, 2)$ and α satisfies

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n.$$

This is essentially a consequence of Helgason's theorem. Let

$$V(\epsilon, t) = \oplus_{\alpha\in 2\mathbb{Z}^n} V(\alpha + \epsilon(2, 2, \ldots, 2)).$$

$V(\epsilon,t)$ possesses an action of the Lie algebra $\mathfrak{sp}(n,\mathbb{R})$. It is the Harish-Chandra module of $I(\epsilon,t)$. Clearly, $V(\epsilon,t) \subset I^{\infty}(\epsilon,t) \subset I_X(\epsilon,t)$.

For each t, there is an $\widetilde{Sp}(n,\mathbb{R})$-invariant sesquilinear pairing of $I_X(\epsilon,t)$ and $I_X(\epsilon,-\overline{t})$, namely,

$$(f_1,f_2) = \int_X f_1(k)\overline{f_2(k)}d[k],$$

where $f_1 \in I_X(\epsilon,t)$ and $f_2 \in I_X(\epsilon,-\overline{t})$. If $t \in i\mathbb{R}$, we obtain a $\widetilde{Sp}(n,\mathbb{R})$-invariant Hermitian form which is exactly $(\,,\,)_X$. Since $(\,,\,)_X$ is positive definite, $I_X(\epsilon,t)$ is a unitary representation of $\widetilde{Sp}(n,\mathbb{R})$.

For each real t, the form $(\,,\,)$ gives an $\mathfrak{sp}(n,\mathbb{R})$-invariant sesquilinear pairing of $V(\epsilon,t)$ and $V(\epsilon,-t)$. In addition, there is an intertwining operator

$$A(\epsilon,t): V(\epsilon,t) \rightarrow V(\epsilon,-t)$$

which preserves the action of $\mathfrak{sp}(n,\mathbb{R})$ (see for example [BOO]). Define a Hermitian structure $(\,,\,)_{\epsilon,t}$ on $V(\epsilon,t)$ by

$$(u,v)_{\epsilon,t} = (A(\epsilon,t)u,v), \qquad (u,v \in V(\epsilon,t)).$$

Clearly, $(\,,\,)_{\epsilon,t}$ is $\mathfrak{sp}(n,\mathbb{R})$-invariant. So $A(\epsilon,t)$ induces an invariant Hermitian form on $V(\epsilon,t)$.

Now $A(\epsilon,t)$ can also be realized as an unbounded operator on $I_X(\epsilon,t)$ as follows. For each $f \in V(\epsilon,t)$, define $A_X(\epsilon,t)f$ to be the unique section of $\mathcal{L}_{\epsilon,t}$ such that

$$(A_X(\epsilon,t)f)|_{\tilde{U}(n)} = (A(\epsilon,t)f)|_{\tilde{U}(n)}.$$

Notice that $A_X(\epsilon,t)f \in I(\epsilon,t)$ and $A(\epsilon,t)f \in I(\epsilon,-t)$. They differ by a multiplier.

Now $A_X(\epsilon,t)$ is an unbounded operator on the Hilbert space $I_X(\epsilon,t)$. The following fact is well-known in many different forms. I state it in a way that is convenient for later use.

Lemma 4.1. *Let $t \in \mathbb{R}$. $I(\epsilon,t)$ is unitarizable if and only if $A_X(\epsilon,t)$ extends to a self-adjoint operator on $I_X(\epsilon,t)$ with spectrum on the nonnegative part of the real axis.*

The spectrum of $A_X(\epsilon,t)$ was computed in [BOO] and [OZ] explicitly for special cases and in [Sa] implicitly. In particular, $A_X(\epsilon,t)$ restricted onto each $\tilde{U}(n)$-type is a scalar multiplication and the scalar is bounded by a polynomial on the highest weight. We obtain

Lemma 4.2 ([WV]). *$A_X(\epsilon,t)$ extends to an unbounded operator from $I^{\infty}(\epsilon,t)$ to $I^{\infty}(\epsilon,t)$.*

This lemma follows from a standard argument that the norm of each $\tilde{U}(n)$-component in the Peter-Weyl expansion of any smooth section of $\mathcal{K}_\epsilon$ decays rapidly with respect to the highest weight. It is true in general (see [WV]).

5. Actions of $\widetilde{Sp}(p,\mathbb{R})\widetilde{Sp}(q,\mathbb{R})$

Suppose that $p+q=n$ and $p \leq q$. Fix a standard basis

$$\{e_1, e_2, \ldots, e_p; e_1^*, e_2^*, \ldots e_p^*\}$$

for the symplectic form Ω_p on $\mathbb{R}^{2p}$. Fix a standard basis

$$\{f_1, f_2, \ldots, f_q; f_1^*, f_2^*, \ldots, f_q^*\}$$

for the symplect form Ω_q on $\mathbb{R}^{2q}$.

Definition 5.1. Let $Sp(p,\mathbb{R})$ be the symplectic group preserving Ω_p and $Sp(q,\mathbb{R})$ be the symplectic group preserving Ω_q. Let

$$\Omega = \Omega_p - \Omega_q$$

and $Sp(n,\mathbb{R})$ be the symplectic group preserving Ω. We say that $(Sp(p,\mathbb{R}), Sp(q,\mathbb{R}))$ is *diagonally embedded* in $Sp(n,\mathbb{R})$.

We shall make a remark here. In [Henu], $\Omega = \Omega_p + \Omega_q$. $Sp(p,\mathbb{R})Sp(q,\mathbb{R})$ is embedded differently there. The effect of this difference is an involution τ on the representation level.

Let $P_{p,q-p}$ be the subgroup of $Sp(q,\mathbb{R})$ that preserves the linear span of $\{f_{p+1}, \ldots, f_q\}$. Choose the Levi factor $GL(q-p)Sp(p,\mathbb{R})$ to be the subgroup of $P_{p,q-p}$ that preserves the span of $\{f_{p+1}^*, \ldots, f_q^*\}$. In particular the $Sp(p,\mathbb{R})$ factor can be identified with the symplectic group of

$$\mathrm{span}\{f_1, \ldots, f_p; f_1^*, \ldots f_p^*\},$$

which will be identified with the standard $Sp(p,\mathbb{R})$. More precisely, for $x \in Sp(p,\mathbb{R})$, by identify e_i with f_i and e_i^* with f_i^* and extending x trivially on $f_{p+1}, \ldots f_q; f_{p+1}^*, \ldots, f_q^*$, we obtain the identification

$$x \in Sp(p,\mathbb{R}) \to \dot{x} \in Sp(q,\mathbb{R}). \tag{4}$$

Now fix a Lagrangian Grassmanian

$$x_0 = \mathrm{span}\{e_1 + f_1, \ldots, e_p + f_p, e_1^* + f_1^*, \ldots, e_p^* + f_p^*, f_{p+1}, \ldots f_q\}.$$

Then the stabilizer $Sp(q,\mathbb{R})_{x_0} = GL(q-p)N_{p,q-p}$ where $N_{p,q-p}$ is the nilradical of $P_{p,q-p}$. Put

$$\Delta(Sp(p,\mathbb{R})) = \{(u, \dot{u}) \mid u \in Sp(p,\mathbb{R})\} \subseteq Sp(p,\mathbb{R})Sp(q,\mathbb{R})\}$$

and

$$H = \Delta(Sp(p,\mathbb{R}))GL(q-p)N_{p,q-p}.$$

Lemma 5.2 ([Henu]). *Let $p \leq q$ and $p+q=n$. Let X_0 be the $Sp(p,\mathbb{R}) \times Sp(q,\mathbb{R})$-orbit generated by x_0. Then X_0 is open and dense in X and $[Sp(p,\mathbb{R})Sp(q,\mathbb{R})]_{x_0} = H$.*

Notice here that X_0 depends on (p,q). Let $P = Sp(n,\mathbb{R})_{x_0}$. The smooth representation $I^{\infty}(\epsilon,t)$ consists of smooth sections of $\mathcal{L}_{\epsilon,t}$:

$$\widetilde{Sp}(n,\mathbb{R}) \times_{\tilde{P}} \mathbb{C}_{\mu^{\epsilon}\otimes\nu^{t+\rho}} \rightarrow X.$$

Consider the subgroup $\widetilde{Sp}(p,\mathbb{R})\widetilde{Sp}(q,\mathbb{R})$ in $\widetilde{Sp}(n,\mathbb{R})$. Notice that $\widetilde{Sp}(p,\mathbb{R}) \cap \widetilde{Sp}(q,\mathbb{R}) \cong \mathbb{Z}$. So $\widetilde{Sp}(p,\mathbb{R})\widetilde{Sp}(q,\mathbb{R})$ is not a direct product, but rather the product of the two groups as sets.

Definition 5.3. For any $f \in I_X(\epsilon,t)$, define

$$f_{X_0} = f|_{\widetilde{Sp}(p,\mathbb{R})\widetilde{Sp}(q,\mathbb{R})}.$$

Let $I^{\infty}_{c,X_0}(\epsilon,t)$ be the set of smooth sections of $\mathcal{L}_{\epsilon,t}$ that are compactly supported in X_0.

Clearly f_{X_0} is a smooth section of

$$\widetilde{Sp}(p,\mathbb{R})\widetilde{Sp}(q,\mathbb{R}) \times_{\tilde{H}} \mathbb{C}_{\mu^{\epsilon}\otimes\nu^{t+\rho}} \rightarrow X_0.$$

Notice that $\Delta(Sp(p,\mathbb{R}))$ sits inside of $SL(n,\mathbb{R}) \subseteq GL(n,\mathbb{R}) \subseteq P$. The universal covering of $Sp(n,\mathbb{R})$ splits over $SL(n,\mathbb{R}) \subseteq P$. Similarly the universal covering of $Sp(q,\mathbb{R})$ also splits over $N_{p,q-p}$. So we have

$$\tilde{H} \cong \Delta(Sp(p,\mathbb{R}))\widetilde{GL}(q-p)N_{p,q-p},$$

where $\widetilde{GL}(q-p)N_{p,q-p} \subseteq \widetilde{Sp}(q,\mathbb{R})$. In particular, $\mu^{\epsilon}\otimes\nu^{t+\rho}|_{\Delta(Sp(p,\mathbb{R}))N_{p,q-p}}$ is trivial and $\mu^{\epsilon}\otimes\nu^{t+\rho}|_{\widetilde{GL}(q-p)}$ is essentially the restriction from $\widetilde{GL}(p+q)$ to $\widetilde{GL}(q-p)$. If $p=q$, then $GL(0)$ will be the identity element. So $\widetilde{GL}(0)$ is just C. We have

Lemma 5.4. *The identification (4)*

$$x \in Sp(p,\mathbb{R}) \rightarrow \dot{x}Sp(q,\mathbb{R})$$

lifts naturally to $\widetilde{Sp}(p,\mathbb{R}) \rightarrow \widetilde{Sp}(q,\mathbb{R})$. Let $\phi \in I^{\infty}(\epsilon,t)$. Then

$$\phi(g_1,g_2) = \phi(1, g_2\dot{g_1}^{-1}) \qquad (g_1 \in \widetilde{Sp}(p,\mathbb{R}), g_2 \in \widetilde{Sp}(q,\mathbb{R})).$$

In addition

$$\phi(1,g_2h) = \mu^{\epsilon}\otimes\nu^{t+\rho}(h^{-1})\phi(1,g_2) \qquad (h \in \widetilde{GL}(q-p)N_{p,q-p}).$$

Now let us consider the action of $\widetilde{Sp}(p,\mathbb{R})$ and $\widetilde{Sp}(q,\mathbb{R})$ on $I(\epsilon,t)$. By Lemma 5.4, we obtain

Lemma 5.5. *Let $\phi \in I^{\infty}(\epsilon, t)$ and $h_1 \in \widetilde{Sp}(p,\mathbb{R})$ and $g_2 \in \widetilde{Sp}(q,\mathbb{R})$. Then*

$$[I(\epsilon,t)(h_1)\phi](1,g_2) = f(1, g_2\dot{h_1}).$$

In particular the restriction map

$$\phi \in I^{\infty}(\epsilon,t) \to \phi|_{\widetilde{Sp}(q,\mathbb{R})} \in C^{\infty}(\widetilde{Sp}(q,\mathbb{R}) \times_{\widetilde{GL}(q-p)N_{p,q-p}} \mathbb{C}_{\mu^{\epsilon}\otimes\nu^{t+\rho}})$$

intertwines the left regular action of $\widetilde{Sp}(p,\mathbb{R})$ on $I^{\infty}(\epsilon,t)$ with the right regular action of $\widetilde{Sp}(p,\mathbb{R})$ on $C^{\infty}(\widetilde{Sp}(q,\mathbb{R}) \times_{\widetilde{GL}(q-p)N_{p,q-p}} \mathbb{C}_{\mu^{\epsilon}\otimes\nu^{t+\rho}})$.

Obviously, the restriction map also intertwines the left regular actions of $\widetilde{Sp}(q,\mathbb{R})$.

6. Mixed Model

Now fix complex structures on $\mathbb{R}^{2p}$ and $\mathbb{R}^{2q}$ and inner products $(\,,\,)_p$, $(\,,\,)_q$ such that

$$\Omega_p = \Im(\,,\,)_p, \qquad \Omega_q = -\Im(\,,\,)_q.$$

Let $U(p)$ and $U(q)$ be the unitary groups preserving $(\,,\,)_p$ and $(\,,\,)_q$ respectively. $U(p)$ and $U(q)$ are maximal compact subgroups of $Sp(p,\mathbb{R})$ and $Sp(q,\mathbb{R})$. Let $U(n)$ be the unitary group preserving $(\,,\,)_p + (\,,\,)_q$. Then $U(n)$ is a maximal compact subgroup of $Sp(n,\mathbb{R})$. In addition,

$$U(p) = Sp(p,\mathbb{R}) \cap U(n) \qquad U(q) = Sp(q,\mathbb{R}) \cap U(n).$$

Identify $U(q) \cap P_{p,q-p}$ with $O(q-p)U(p)$. Recall that $X_0 \cong Sp(q,\mathbb{R})/GL(q-p)N_{p,q-p}$. The group $Sp(p,\mathbb{R})$ acts on X_0 freely from the right. We obtain a principal fibration

$$Sp(p,\mathbb{R}) \to X_0 \to Sp(q,\mathbb{R})/P_{p,q-p} \cong U(q)/O(q-p)U(p).$$

This fibration allows us to visualize the action of $\widetilde{Sp}(p,\mathbb{R})\widetilde{Sp}(q,\mathbb{R})$ on $I^{\infty}(\epsilon,t)$. Let dg_1 be a Haar measure on $Sp(p,\mathbb{R})$ and $d[k_2]$ be an invariant probability measure on $U(q)/\,O(q-p)U(p)$. Then $dg_1 d[k_2]$ defines an $U(q)Sp(p,\mathbb{R})$ invariant measure on X_0.

Definition 6.1. Let $M = Sp(p,\mathbb{R})U(q) \subset Sp(p,\mathbb{R})Sp(q,\mathbb{R}) \subset Sp(n,\mathbb{R})$. Elements in X_0 are parametrized by a pair $(g_1,[k_2])$ for $(g_1,k_2) \in M$. For each $g \in \widetilde{Sp}(n,\mathbb{R})$, write $g = \tilde{u}(g)p(g)$ where $\tilde{u}(g) \in \tilde{U}(n)$ and $p(g) \in P_0$, the identity component of $\tilde{P}$. For each $(g_1,k_2) \in (\widetilde{Sp}(p,\mathbb{R}), \tilde{U}(q))$, we have

$$g_1k_2 = \tilde{u}(g_1k_2)p(g_1k_2) = k_2\tilde{u}(g_1)p(g_1).$$

The component $\tilde{u}$ defines a map from $\tilde{M}$ to $\tilde{U}(n)$. In particular, $\tilde{u}$ induces a map from $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$ to $\tilde{U}(n)/\tilde{O}(n)$ which will be denoted by j. The map j parametrizes the open dense subset X_0 in X by

$$([g_1],[k_2]) \in \widetilde{Sp}(p,\mathbb{R})/C \times \tilde{U}(q)/\tilde{O}(q-p)\tilde{U}(p).$$

Change the variables on X_0 from $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$ to $\tilde{U}(n)/\tilde{O}(n)$. Let $J([g_1],[k_2])$ be the Jacobian:

$$\frac{dj([g_1],[k_2])}{d[g_1]d[k_2]}.$$

J can be regarded as a function on $Sp(p,\mathbb{R})U(q)$ or $Sp(p,\mathbb{R})U(q)/U(p)O(q-p)$, even though it is defined as a function on the covering. Denote the line bundle

$$\widetilde{Sp}(q,\mathbb{R}) \times_{\widetilde{GL}(q-p)N_{p,q-p}} \mathbb{C}_{\mu^\epsilon \otimes \nu^{t+\rho}} \to X_0.$$

by $\mathcal{M}_{\epsilon,t}$. Denote the line bundle

$$\tilde{M} \times_{\tilde{O}(q-p)\tilde{U}(p)} \mathbb{C}_{\mu^\epsilon} \to \tilde{M}/\tilde{O}(q-p)\tilde{U}(p) \cong X_0.$$

by $\mathcal{M}_\epsilon$.

Clearly, $I^\infty_{c,X_0}(\epsilon,t) \subset I^\infty(\epsilon,t)$. Consider the restriction of $(\,,\,)_X$ onto $I^\infty_{c,X_0}(\epsilon,t)$. We are interested in expressing $(\,,\,)_X$ as an integral on $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$. This boils down to a change of variables from $\tilde{U}(n)/\tilde{O}(n)$ to $\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)$. We have

Lemma 6.2. *Let $\Delta_t(g_1,k_2) = \nu(p(g_1))^{t+\bar{t}+2\rho}J([g_1],[k_2])$ (see Equ.(2)). Then for every $f_1, f_2 \in I^\infty(\epsilon,t)$ we have*

$$(f_1,f_2)_X = \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(g_1k_2)\overline{f_2(g_1k_2)}\Delta_t(g_1,k_2)d[g_1]d[k_2]$$

where $g_1 \in \widetilde{Sp}(p,\mathbb{R})$, $k_2 \in \tilde{U}(q)$, $[g_1] \in Sp(p,\mathbb{R})$ and $[k_2] \in \tilde{U}(q)/\tilde{O}(q-p)\tilde{U}(p)$. Furthermore, $\Delta_t(g_1,k_2)$ is a nonnegative right $\tilde{O}(q-p)\tilde{U}(p)$-invariant function on $\widetilde{M}$.

Proof. We compute

$$\begin{aligned}
&\int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(g_1k_2)\overline{f_2(g_1k_2)}\Delta_t(g_1,k_2)d[g_1]d[k_2] \\
=&\int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(\tilde{u}(g_1k_2))\overline{f_2(\tilde{u}(g_1k_2))}\nu(p(g_1))^{-t-\bar{t}-2\rho}\Delta_t(g_1,k_2)d[g_1]d[k_2] \\
=&\int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(\tilde{u}(g_1k_2))\overline{f_2(\tilde{u}(g_1k_2))}\nu(p(g_1))^{-t-\bar{t}-2\rho}\Delta_t(g_1,k_2)J^{-1}(g_1,k_2)dj([g_1],[k_2]) \\
=&\int_{X_0} f_1(\tilde{u})\overline{f_2(\tilde{u})}d[\tilde{u}] = (f_1,f_2)_X.
\end{aligned} \tag{5}$$

Since $\nu(p(g_1)$ and $J([g_1],[k_2])$ remain the same when we multiply k_2 on the right by $\tilde{O}(q-p)\tilde{U}(p)$, $\Delta_t(g_1,k_2)$ is a nonnegative right $\tilde{O}(q-p)\tilde{U}(p)$-invariant function. ■

Combining with Lemma 5.5, we obtain

Corollary 6.3. *As representations of $\widetilde{Sp}(p,\mathbb{R})\widetilde{Sp}(q,\mathbb{R})$,*

$$I_X(\epsilon,t) \cong L^2(\mathcal{M}_{\epsilon,t}, \Delta_t d[g_1]d[k_2]).$$

For each $f_1, f_2 \in I^{\infty}_{c,X_0}(\epsilon, t)$, define

$$(f_1, f_2)_{M,t} = \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(g_1k_2)\overline{f_2(g_1k_2)}\Delta_t(g_1k_2)d[g_1]d[k_2],$$

$$(f_1, f_2)_M = \int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} f_1(g_1, k_2)\overline{f_2(g_1k_2)}d[g_1]d[k_2].$$

The completion of $I^{\infty}_{c,X_0}(\epsilon, t)$ under $(\, ,\,)_{M,t}$ is $L^2(\mathcal{M}_{\epsilon,t}, \Delta_t d[g_1]d[k_2])$. We call $L^2(\mathcal{M}_{\epsilon,t}, \Delta_t d[g_1]d[k_2])$, the mixed model. We denote it by $I_M(\epsilon, t)$. On $I_M(\epsilon, t)$, the actions of $\widetilde{Sp}(p, \mathbb{R})$ and $\widetilde{Sp}(q, \mathbb{R})$ are easy to manipulate.

7. Mixed Model for Unitary Principal Series

Lemma 7.1. *If $t \in i\mathbb{R}$, then $\Delta_t(g_1, k_2)$ is a constant and $(\, ,\,)_{M,t}$ is a constant multiple of $(\, ,\,)_M$.*

Proof. Let $t \in i\mathbb{R}$. Let $f_1, f_2 \in I^{\infty}(\epsilon, t)$ and $h \in \widetilde{Sp}(p, \mathbb{R})$. Recall that X_0 is parametrized by a pair $[g_1] \in \widetilde{Sp}(p, \mathbb{R})/C$ and $[k_2] \in \tilde{U}(q)/\tilde{O}(q-p)\tilde{U}(p)$. By Lemma 6.2, we have

$$\begin{aligned} &(I(\epsilon, t)(h)f_1, I(\epsilon, t)(h)f_2)_X \\ = &\int_{X_0} f_1(h^{-1}g_1k_2)\overline{f_2(h^{-1}g_1k_2)}\Delta_t(g_1, k_2)d[g_1]d[k_2] \\ = &\int_{X_0} f_1(g_1k_2)\overline{f_2(g_1, k_2)}\Delta_t(hg_1, k_2)d[g_1]d[k_2] \end{aligned} \tag{6}$$

Since $I(\epsilon, t)$ is unitary, $(I(\epsilon, t)(h)f_1, I(\epsilon, t)(h)f_2)_X = (f_1, f_2)_X$. We have

$$\int_{X_0} f_1(g_1k_2)\overline{f_2(g_1k_2)}\Delta_t(hg_1, k_2)d[g_1]d[k_2] = \int_{X_0} f_1(g_1k_2)\overline{f_2(g_1k_2)}\Delta_t(g_1, k_2)d[g_1]d[k_2].$$

It follows that $\Delta_t(hg_1, k_2) = \Delta_t(g_1, k_2)$ for any $h \in \widetilde{Sp}(p, \mathbb{R})$. Similarly, we obtain $\Delta_t(g_1, kk_2) = \Delta(g_1, k_2)$ for any $k \in \tilde{U}(q)$. Hence, $\Delta_t(g_1, k_2)$ is a constant for purely imaginary t. ■

Combining with Corollary. 6.3, we obtain

Theorem 7.2. *Let $t \in i\mathbb{R}$. The restriction map $f \to f_{X_0}$ induces an isometry between $I(\epsilon, t)$ and $L^2(\mathcal{M}_{\epsilon,t}, d[g_1]d[k_2])$. In addition, this isometry intertwines the actions of $\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(q, \mathbb{R})$. So as $\widetilde{Sp}(p, \mathbb{R})\widetilde{Sp}(q, \mathbb{R})$ representations,*

$$I(\epsilon, t) \cong L^2(\mathcal{M}_{\epsilon,t}, d[g_1]d[k_2]);$$

and as $\widetilde{Sp}(p, \mathbb{R})\tilde{U}(q)$ representations,

$$I(\epsilon, t) \cong L^2(\mathcal{M}_{\epsilon}, d[g_1]d[k_2]).$$

Notice that $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$ does not depend on the parameter t. The following corollary is automatical.

Corollary 7.3. *Suppose that $p+q=n$ and $p \leq q$. For t real,*

$$I(\epsilon, it)|_{\widetilde{Sp}(p,\mathbb{R})\tilde{U}(q)} \cong I(\epsilon, 0)|_{\widetilde{Sp}(p,\mathbb{R})\tilde{U}(q)} \cong L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2]).$$

For t a nonzero real number, $\Delta_t(g,k)$ is not a constant. So $C(\epsilon,t)$ cannot be modeled naturally on $L^2(\mathcal{M}_{\epsilon,t}, d[g_1]d[k_2])$. Nevertheless, we have

Theorem 7.4 (Main Theorem). *Suppose that $p+q=n$ and $p \leq q$. Given a complementary series representation $C(\epsilon,t)$,*

$$C(\epsilon, t)|_{\widetilde{Sp}(p,\mathbb{R})\tilde{U}(q)} \cong I(\epsilon, 0)|_{\widetilde{Sp}(p,\mathbb{R})\tilde{U}(q)} \cong L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2]).$$

In other words, there is an isometry between $C(\epsilon,t)$ and $I(\epsilon,0)$ that intertwines the actions of $\tilde{U}(q)$ and of $\widetilde{Sp}(p,\mathbb{R})$.

We shall postpone the proof of this theorem to the next section. We will first derive some corollaries from Lemma 7.1 concerning Δ and $\nu(g_1)$.

Corollary 7.5. *$J([g_1],[k_2]) = c\nu(p(g_1))^{-2\rho}$ for a constant c and $\Delta_t(g_1,k_2) = c\nu(p(g_1))^{t+\bar{t}}$. So both Δ_t and $J([g_1],[k_2])$ do not depend on k_2. Furthermore,*

$$I(\epsilon, t) \cong L^2(\mathcal{M}_{\epsilon,t}, \nu(p(g_1))^{t+\bar{t}} d[g_1]d[k_2]) = I_M(\epsilon, t). \tag{7}$$

$\nu(p(g_1))$ is a function on $\widetilde{Sp}(p,\mathbb{R})/C$. So it can be regarded as a function on $Sp(p,\mathbb{R})$.

Corollary 7.6. *$\nu(p(g_1))^{-\rho} \in L^2(Sp(p,\mathbb{R}))$ and $\nu(p(g_1))^{-1}$ is a bounded positive function.*

Proof. Since X is compact,

$$\int_{Sp(p,\mathbb{R})} \nu(p(g_1))^{-2\rho} dg_1 = C\int_{\tilde{M}/\tilde{O}(q-p)\tilde{U}(p)} J([g_1],[k_2]) d[g_1]d[k_2] = C\int_{\tilde{U}(n)/\tilde{O}(n)} 1 d[k] < \infty.$$

So $\nu(p(g_1))^{-\rho} \in L^2(Sp(p,\mathbb{R}))$. Now we need to compute $\nu(g_1)$. Recall that P is defined to be the stabilizer of

$$x_0 = \mathrm{span}\{e_1+f_1, \ldots, e_p+f_p, e_1^*+f_1^*, \ldots, e_p^*+f_p^*, f_{p+1}, \ldots f_q\}.$$

So $j(g_1,1)$ is the following Lagrangian

$$\mathrm{span}\{g_1e_1+f_1, \ldots, g_1e_p+f_p, g_1e_1^*+f_1^*, \ldots, g_1e_p^*+f_p^*, f_{p+1}, \ldots f_q\}.$$

The action of $\tilde{U}(n)$ will not change the volume of the n-dimensional cube spanned by the basis above. So $\nu(p(g_1))$, as the determinant character, is equal to the volume of the n-dimensional cube, up to a constant. Hence

$$\nu(p(g_1)) = [2^{-n}\det(g_1g_1^t + I)]^{\frac{1}{2}}.$$

Clearly, $\nu(p(g_1))^{-1}$ is bounded and positive. ■

This corollary is easy to understand in terms of compactification. Notice that the map j, without the covering,

$$Sp(p,\mathbb{R})U(q)/U(p)O(q-p) \to U(n)/O(n)$$

is an analytic compactification. Hence the Jacobian $J(g_1,[k_2])$ should be positive and bounded above. Since $J(g_1,[k_2]) = c\nu(p(g_1))^{-2\rho}$, $\nu(p(g_1))^{-1}$ must also be positive and bounded above. The situation here is similar to [He02] (see Appendix) and [He06] (Theorem 2.3). It is not clear that $j(g_1,1)$ gets mapped onto $U(2p)/O(2p)$ though.

If $f \in I_M(\epsilon,t_1)$ and $h>0$, by Cor. 7.6 and Equation (7), we have $\|f\|_{M,t_1-h} \leq C\|f\|_{M,t_1}$. So $I_M(\epsilon,t_1) \subset I_M(\epsilon,t_1-h)$.

Corollary 7.7. *Suppose that $h>0$. Then $I_M(\epsilon,t_1) \subset I_M(\epsilon,t_1-h)$.*

8. "Square Root "of the Intertwining Operator

Suppose from now on $t \in \mathbb{R}$. For $f \in I^\infty(\epsilon,t)|_{\tilde{M}}$, define a function on $\tilde{M}$,

$$(A_M(\epsilon,t)f)(g_1k_2) = A(\epsilon,t)f(g_1k_2) \qquad (g_1 \in \widetilde{Sp}(p,\mathbb{R}), k_2 \in \tilde{U}(q)).$$

So $A_M(\epsilon,t)$ is the "restriction "of $A(\epsilon,t)$ onto $\tilde{M}$. $A_M(\epsilon,t)$ is not yet an unbounded operator on $I_M(\epsilon,t)$. In fact, for $t>0$, $A_M(\epsilon,t)$ does not behave well and it is not clear whether $A_M(\epsilon,t)$ can be realized as an unbounded operator on $I_M(\epsilon,t)$. The function $A_M(\epsilon,t)f$ differs from $A_X(\epsilon,t)f$.

Lemma 8.1. *For $t \in \mathbb{R}$ and $f \in I^\infty(\epsilon,t)$,*

$$(A_M(\epsilon,t)f|_{\tilde{M}})(g_1k_2) = (A_X(\epsilon,t)f)(g_1k_2)\nu(p(g_1))^{2t} = (A_X(\epsilon,t)f)(g_1k_2)\Delta_t(g_1,k_2).$$

This Lemma is due to the fact that $A_X(\epsilon,t)f \in I(\epsilon,t)$ but $A(\epsilon,t)f \in I(\epsilon,-t)$.

Let $f \in I^\infty(\epsilon,t)$. In terms of the mixed model, the invariant Hermitian form $(\,,\,)_{\epsilon,t}$ can be written as follows:

$$(f,f)_{\epsilon,t} = (A_X(\epsilon,t)f,f)_X = \int_{\tilde{M}/\tilde{O}(q-p)U(p)} A_M(\epsilon,t)\, f|_{\tilde{M}}\, \overline{f}|_{\tilde{M}} d[g_1]d[k_2].$$

This follows from Lemma 8.1 and Lemma 6.2. We obtain

Lemma 8.2. *For $f_1,f_2 \in I^\infty(\epsilon,t)$, $(f_1,f_2)_{\epsilon,t} = (A_M(\epsilon,t)f_1|_{\tilde{M}}, f_2|_{\tilde{M}})_M$.*

Theorem 8.3. *If $t<0$ and $C(\epsilon,t)$ is a complementary series representation, then $A_M(\epsilon,t)$ is a positive and densely defined symmetric operator. Its self-adjoint-extension $(A_M(\epsilon,t)+I)_0-I$ has a unique square root which extends to an isometry from $C(\epsilon,t)$ onto*

$$L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2]).$$

Proof. Let $t<0$. Put

$$\mathcal{H}=L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2]).$$

Let $f\in I^\infty(\epsilon,t)$. Then $A_M(\epsilon,t)(f|_{\tilde M})(g_1k_2)=\nu(p(g_1))^{2t}A_X(\epsilon,t)f(g_1k_2)$. By Lemma 8.1, Cor. 7.6 and Lemma 6.2, we have

$$\begin{aligned}
&\int_{\tilde M/\widetilde{O}(q-p)\tilde U(p)} A_M(\epsilon,t)(f|_{\tilde M})\overline{A_M(\epsilon,t)(f|_{\tilde M})}d[g_1]d[k_2]\\
=&\int_{\tilde M/\widetilde{O}(q-p)\tilde U(p)} \nu(p(g_1))^{2t}|(A_X(\epsilon,t)f)(g_1k_2)|^2\nu(p(g_1))^{2t}d[g_1]d[k_2]\\
=&\int_{\tilde M/\widetilde{O}(q-p)\tilde U(p)} \nu(p(g_1))^{2t}|(A_X(\epsilon,t)f)(g_1k_2)|^2\Delta_t(g_1,k_2)d[g_1]d[k_2]\\
\leq& C\int_{\tilde M/\widetilde{O}(q-p)\tilde U(p)} |A_X(\epsilon,t)f(g_1k_2)|^2\Delta_t(g_1,k_2)d[g_1]d[k_2]\\
=&C(A_X(\epsilon,t)f, A_X(\epsilon,t)f)_X<\infty.
\end{aligned}\tag{8}$$

Therefore, $A_M(\epsilon,t)(f|_{\tilde M})\in\mathcal{H}$. Let $\mathcal{D}=I^\infty(\epsilon,t)|_{\tilde M}$. Clearly, $\mathcal{D}$ is dense in $\mathcal{H}$. So $A_M(\epsilon,t)$ is a densely defined unbounded operator. It is positive and symmetric by Lemma 8.2.

Definition 8.4. Define $\mathcal{U}(\epsilon,t)=((A_M(\epsilon,t)+I)_0-I)^{\frac12}$.

Now $(f,g)_{\epsilon,t}=(A_M(\epsilon,t)f|_{\tilde M}, g|_{\tilde M})_M$ for any $f,g\in I^\infty(\epsilon,t)$. So $C(\epsilon,t)=\mathcal{H}_{A_M(\epsilon,t)}$. By Lemma 2.1, $\mathcal{U}(\epsilon,t)$, mapping from $C(\epsilon,t)$ into $\mathcal{H}$, is an isometry.

Suppose that $\mathcal{U}(\epsilon,t)$ is not onto. Let $f\in\mathcal{H}$ such that for any $u\in\mathcal{D}(\mathcal{U}(\epsilon,t))$,

$$(f,\mathcal{U}(\epsilon,t)u)_M=0.$$

Notice that

$$I^\infty(\epsilon,t)|_{\tilde M}\subset\mathcal{D}((A_M(\epsilon,t)+I)_0-I)\subset\mathcal{D}(\mathcal{U}(\epsilon,t)),$$

and

$$\mathcal{U}(\epsilon,t)\mathcal{U}(\epsilon,t)=(A_M(\epsilon,t)+I)_0-I.$$

In particular,

$$\mathcal{U}(\epsilon,t)I^\infty(\epsilon,t)|_{\tilde M}\subset\mathcal{D}(\mathcal{U}(\epsilon,t)).$$

It follows that

$$\begin{aligned}
&(f, A_M(\epsilon,t)I^\infty(\epsilon,t)|_{\tilde M})_M\\
=&(f,((A_M(\epsilon,t)+I)_0-I)I^\infty(\epsilon,t)|_{\tilde M})_M\\
=&(f,\mathcal{U}(\epsilon,t)\mathcal{U}(\epsilon,t)I^\infty(\epsilon,t)|_{\tilde M})_M\\
=&0.
\end{aligned}\tag{9}$$

Let $f_{\epsilon,t}$ be a function such that $f_{\epsilon,t}|_{\tilde M}=f$ and

$$f_{\epsilon,t}(gln)=(\mu^\epsilon\otimes\nu^{t+\rho})(l^{-1})f_{\epsilon,t}(g)\qquad(l\in\tilde L, n\in N).$$

By Lemma 8.2, $\forall\, u \in V(\epsilon, t)$,

$$0 = (f, A_M(\epsilon, t)(u|_{\tilde{M}}))_M = (f_{\epsilon,t}, A_X(\epsilon, t)u)_X = (f_{\epsilon,t}, u)_{\epsilon,t}.$$

This equality is to be interpreted as an equality of integrals according to the definitions of $(\ ,\ )_M$ and $(\ ,\ )_X$. Since $A_X(\epsilon, t)$ acts on $\tilde{U}(n)$-types in $V(\epsilon, t)$ as scalars, $A_X(\epsilon, t)V(\epsilon, t) = V(\epsilon, t)$. We now have

$$(f_{\epsilon,t}, V(\epsilon, t))_X = 0.$$

In particular, $f_{\epsilon,t}|_{\tilde{U}(n)} \in L^1(X)$. By Peter-Weyl Theorem, $f_{\epsilon,t} = 0$. We see that $\mathcal{U}(\epsilon, t)$ is an isometry from $C(\epsilon, t)$ onto $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$. ■

The Hilbert space $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$ is the mixed model for $I(\epsilon, 0)$ restricted to $\tilde{M}$. We now obtain an isometry from $C(\epsilon, t)$ onto $I(\epsilon, 0)$. Within the mixed model, the action of $I(\epsilon, t)(g_1k_2)$ is simply the left regular action and it is independent of t. We obtain

Lemma 8.5. *Suppose $t < 0$. Let $g \in \tilde{U}(q)$. Let $L(g)$ be the left regular action on $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$. As an operator on $I^\infty(\epsilon, t)|_{\tilde{M}}$, $L(g)$ commutes with $A_M(\epsilon, t)$. Furthermore, $L(g)$ commutes with $(A_M(\epsilon, t)+I)_0 - I$. Similar statement holds for $g \in \widetilde{Sp}(p, \mathbb{R})$.*

Proof. Let $g \in \tilde{M}$. Both $A_M(\epsilon, t)$ and $L(g)$ are well-defined operator on $I^\infty(\epsilon, t)|_{\tilde{M}}$. Regarding $A(\epsilon, t)I(\epsilon, t)(g) = I(\epsilon, -t)(g)A(\epsilon, t)$ as operators on the mixed model $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$, we have

$$A_M(\epsilon, t)L(g) = L(g)A_M(\epsilon, t).$$

It follows that

$$L(g)^{-1}(A_M(\epsilon, t) + I)L(g) = (A_M(\epsilon, t) + I).$$

Since $L(g)$ is unitary, $L(g)^{-1}(A_M(\epsilon, t) + I)_0 L(g) = (A_M(\epsilon, t) + I)_0$. In fact, $(A_M(\epsilon, t) + I)_0$ can be defined as the inverse of $(A_M(\epsilon, t) + I)^{-1}$, which exists and is bounded. So $L(g)$ commutes with both $(A_M(\epsilon, t)+I)^{-1}$ and $(A_M(\epsilon, t)+I)_0$. ■

Lemma 8.6. *We have, for $g \in \tilde{M}$, $\mathcal{U}(\epsilon, t)I(\epsilon, t)(g) = I(\epsilon, 0)(g)\mathcal{U}(\epsilon, t)$.*

Proof. Recall from Theorem 7.2 that the action of $\tilde{M}$ on the mixed model is independent of t. It suffices to show that on the mixed model, $\mathcal{U}(\epsilon, t)$ commutes with $L(g)$ for any $g \in \tilde{M}$. By Lemma 8.5,

$$L(g)^{-1}[(A_M(\epsilon, t) + I)_0 - I]L(g) = (A_M(\epsilon, t) + I)_0 - I.$$

Since $L(g)$ is unitary on $L^2(\mathcal{M}_\epsilon, d[g_1]d[k_2])$, both sides are positive self-adjoint operators. Taking square roots, we obtain $L(g)^{-1}\mathcal{U}(\epsilon, t)L(g) = \mathcal{U}(\epsilon, t)$. ■

Theorem 7.2 is proved.

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