Orbits on Lagrangian Grassmanian

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The purpose of this note is to give a classification of the orbital structure of certain reductive group actions on the Lagrangian Grassmanian. The groups under consideration are $Sp \times Sp$ and $GL$. In general, the finiteness of orbits is known ([1], [5], [2], [6]). However, the homogeneous structure of each individual orbit is not known except perhaps the generic open orbit. The main results are given in Theorem 2.1, Corollary 2.5 and Theorem 3.2. This is a follow up of the author’s previous work on this topic ([3], [4]).

1 Definitions

Let $(V, \Omega)$ be a symplectic space of dimension $2m$. Let $U$ be a subspace of $V$. Define

$$U^\perp = \{ v \in V \mid \Omega(v, U) = 0 \}.$$ 

Sometimes, we also write it as $U^{\perp, \Omega}$. A subspace $U$ of $V$ is called isotropic if and only if $U^\perp \supseteq U$. Equivalently, $U$ is isotropic if $\Omega|_U$ is zero. An isotropic subspace of $V$ is called a Lagrangian subspace if $\dim U = \frac{\dim V}{2}$. Obviously, $\frac{\dim V}{2}$ is the maximal dimension of an isotropic subspace. So $U$ is Lagrangian if and only if $U^\perp = U$.

Let $Sp(V)$ be the symplectic group. Let $Iso(V, i)$ be the set of $i$-dimensional isotropic subspaces of $V$. Then $Sp(V)$ acts on $Iso(V, i)$ transitively. Fix a base point $U \in Iso(V, i)$. Let $P_U$ be the stabilizer of $U$. Then $P_U$ is a maximal parabolic subgroup and $Iso(V, i) \cong Sp(V)/P_U$.

Let $\mathcal{L}(V)$ be the set of Lagrangian subspaces of $V$. $\mathcal{L}(V)$ is called the Lagrangian Grassmanian. Let $U$ be a base point in $\mathcal{L}(V)$. Then the stabilizer $P_U$ is isomorphic to the Siegle parabolic subgroup and $\mathcal{L}(V) \cong Sp(V)/P_U$.

Unless stated otherwise, all vector spaces and groups are over $\mathbb{R}$. 

2 $Sp \times Sp$ action on Lagrangian Grassmanian

Let $m,n$ be two positive integers and $m \leq n$. Let $(V_1, \Omega_1)$ be a symplectic space of $2m$ dimension and $(V_2, \Omega_2)$ be a symplectic space of $2n$ dimension. Let $V = V_1 \oplus V_2$ and $\Omega = \Omega_1 - \Omega_2$. Then $(V, \Omega)$ is a symplectic space. Let $P_1$ be the canonical projection from $V$ to $V_1$ and $P_2$ be the canonical projection from $V$ to $V_2$. In this section, we are interested in the orbital structure of $Sp(V_1) \times Sp(V_2)$ on $L(V_1 \oplus V_2)$. It is known that there are finitely many orbits ([1]). In the case $m = n$, the orbits are classified in [4]. In this section, we will give a classification of $Sp(V_1) \times Sp(V_2)$-orbits. We will also study the structure of each orbit.

Let $U \in L(V)$. Let $U_1 = U \cap V_1$ and $U_2 = U \cap V_2$. Then

- $U_1$ is an isotropic subspace in $(V_1, \Omega_1)$.
- $U_2$ is an isotropic subspace in $(V_2, \Omega_2)$.

**Lemma 2.1** Let $U \in L(V)$. Then $P_1(U)^{\perp, \Omega_1} = U_1$ and $P_2(U)^{\perp, \Omega_2} = U_2$.

**Proof:** Suppose that $x \in V_1$ and $\Omega_1(x, P_1(U)) = 0$. Then $\Omega(x, P_1(U)) = 0$. Clearly $\Omega(x, V_2) = 0$. So $\Omega(x, V_2 + P_1(U)) = 0$. Notice that $u - P_1(u) \in V_2$ for any $u \in U$. Hence $\Omega(x, U) = 0$. This implies that $x \in U$. It follows that $x \in U \cap V_1 = U_1$. The converse is also true, that is, $\Omega_1(U_1, P_1(U)) = 0$. This follows directly from $\Omega(U_1, U) = 0$. We have thus shown that $P_1(U)^{\perp, \Omega_1} = U_1$. The second statement follows similarly. $\square$

**Corollary 2.1** The null space of $\Omega_1|_{P_1(U)} = U_1$. The null space of $\Omega_2|_{P_2(U)} = U_2$.

**Corollary 2.2** $\Omega_1|_{P_1(U)}$ reduces to a nondegenerate form $\Omega'_1$ on $P_1(U)/U_1$. $\Omega_2|_{P_2(U)}$ reduces to a nondegenerate form $\Omega'_2$ on $P_2(U)/U_2$. Furthermore, $P_1(U)/U_1 \cong U/(U_1 \oplus U_2) \cong P_2(U)/U_2$.

**Proof:** The first two statements are obvious. Since $\ker P_1 = V_2$. Therefore, $P_1(U) \cong U/U \cap V_2 = U/U_2$. Our assertion follows immediately. $\square$

**Corollary 2.3** $\dim(U_2) = \dim(U_1) + n - m$.

**Proof:** Since $U_1 = P_1(U_1)^{\perp, \Omega_1}$, $\dim U_1 + \dim(P_1(U_1)) = 2m$. Notice that

$$\dim(P_1(U_1)) = \dim(U/U_2) = \dim U - \dim(V_2 \cap U) = (n + m) - (\dim U_2).$$

Hence, $\dim U_1 + n + m - \dim U_2 = 2m$. Consequently, $\dim U_1 = \dim U_2 + m - n$. $\square$

**Theorem 2.1** Let $L_i$ be the set of Lagrangian subspaces of $(V, \Omega)$ such that $\dim(U \cap V_1) = i$. 

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Then \( \dim(U \cap V_2) = i + (n - m) \) and \( \mathcal{L}_i \) is a single \( Sp_{2m}(\mathbb{R}) \times Sp_{2n}(\mathbb{R}) \) orbit.

Proof: It suffices to prove that \( \mathcal{L}_i \) is a single \( Sp_{2m}(\mathbb{R}) \times Sp_{2n}(\mathbb{R}) \)-orbit. For every \( U \in \mathcal{L}_i \), let \( \pi(U) = (U_1, U_2) \). Then \( \dim U_1 = i \) and \( \dim U_2 = n - m + i \). Clearly, \( \pi \) is a map from \( \mathcal{L}_i \) to \( Iso(2m, i) \times Iso(2n, i + n - m) \). We claim that \( \pi \) is a projection. Let us fix \( U_1, U_2 \). Consider

\[
\mathcal{L}_{U_1, U_2} = \{ U \in \mathcal{L}_i \mid U \cap V_1 = U_1, U \cap V_2 = U_2 \}.
\]

We will show that \( \mathcal{L}_{U_1, U_2} \) is nonempty and isomorphic to \( Sp_{2m-2i}(\mathbb{R}) \).

Since \( U_1, U_2 \) are fixed, \( P_1(U) = U_1^{\perp_{\Omega_1}} \) and \( P_2(U) = U_2^{\perp_{\Omega_1}} \) are known. In addition

\[
U \subseteq P_1(U) \oplus P_2(U) = U_1^{\perp_{\Omega_1}} \oplus U_2^{\perp_{\Omega_1}}.
\]

Clearly, \( U \in \mathcal{L}_{U_1, U_2} \) is uniquely determined by \( U/U_1 \oplus U_2 \), a \( 2m - 2i \) dimensional subspace in

\[
U_1^{\perp_{\Omega_1}} \oplus U_2^{\perp_{\Omega_1}}/(U_1 \oplus U_2),
\]

and vice versa. Let \( u, w \in U \). We compute

\[
0 = \Omega(u, w) = \Omega_1(P_1(u), P_1(w)) - \Omega_2(P_2(u), P_2(w)) = \Omega'_1([P_1(u)], [P_1(w)]) - \Omega'_2([P_2(u)], [P_2(w)]).
\]

Here \([P_1(u)], [P_1(w)] \in U_i^{\perp_{\Omega_i}}/U_i\) and \( \Omega'_i \) is the reduced symplectic form on \( U_i^{\perp_{\Omega_i}}/U_i \). For each \( U \in \mathcal{L}_{U_1, U_2} \), define \( P'_i : U/U_1 \oplus U_2 \rightarrow U_i^{\perp_{\Omega_i}}/U_i \) by letting \( P'_i([u]) = [P_i(u)] \). We see that

\[
\Omega'_1(P'_1([u]), P'_1([w])) = \Omega'_2(P'_2([u]), P'_2([w])).
\]

Notice that \( P'_i \) is injective. An easy computation shows that

\[
\dim(U/U_1 \oplus U_2) = 2m - 2i = \dim(U_1^{\perp_{\Omega_1}}/U_1) = \dim(U_2^{\perp_{\Omega_2}}).
\]

So \( P'_i \) is surjective. It follows that \( U \in \mathcal{L}_{U_1, U_2} \) is in one-to-one correspondence with \( P'_i(P'_i)^{-1} \), a symplectic isomorphism between \( (U_1^{\perp_{\Omega_1}}/U_1, \Omega'_1) \) and \( (U_2^{\perp_{\Omega_2}}/U_2, \Omega'_2) \). The correspondence is explicitly given by

\[
\phi \in Sp(U_1^{\perp_{\Omega_1}}/U_1, U_2^{\perp_{\Omega_2}}/U_2) \rightarrow \{(u_1), \phi(u_1)\} \in U_1^{\perp_{\Omega_1}}/U_1 \oplus U_2^{\perp_{\Omega_2}}/U_2
\]

\[
\rightarrow \{(u_1, u_2) \mid u_1 \in U_1^{\perp_{\Omega_1}}, u_2 \in U_2^{\perp_{\Omega_2}}, u_1 + u_2 = \phi(u_1 + U_1)\}.
\]

We have thus obtained a principal fibration

\[
Sp_{2m-2i}(\mathbb{R}) \rightarrow \mathcal{L}_i \rightarrow Iso(2m, i) \times Iso(2n, n - m + i).
\]

Observe that the stabilizer of \( U_1 \) in \( Sp(V_1) \), \( P_{U_1} \), acts on \( U_1 \) and \( U_1^{\perp_{\Omega_1}} \). \( P_{U_1} \) is of the form \( Sp_{2m-2i}(\mathbb{R})GL_iN_{i, 2m-2i} \). It acts on \( U_1^{\perp_{\Omega_1}}/U_1 \) by the \( Sp_{2m-2i}(\mathbb{R}) \) factor. By the correspondence 1, \( P_{U_1} \) acts on \( \mathcal{L}_{U_1, U_2} \) transitively. It follows that \( \mathcal{L}_i \) is a single \( Sp_{2m}(\mathbb{R}) \times Sp_{2n}(\mathbb{R}) \) orbit. \( \square \)

The following corollaries are more or less obvious.
Let $0 \leq i \leq m$. $U \in L_i$ if and only if there exist $U_1 \in Iso(2m, i)$, $U_2 \in Iso(2n, n - m + i)$ and a symplectic isomorphism between $(U_1^+ \cup \Omega_1)$ and $(U_2^+ \cup \Omega_2)$ such that $U_1 = U \cap V_1$ and $U_2 = U \cap V_2$ and
\[ U = \{(u_1, u_2) \in U_1^+ \cup \Omega_1 \oplus U_2^+ \cup \Omega_2 \mid \phi(u_1 + U_1) = u_2 + U_2\}. \]
Here $\Omega'_i$ is the reduced symplectic form on $U_i^+ \cup \Omega_i / U_i$.

**Corollary 2.5** Let $i \leq m$. Let $P_{i, 2m - 2i}$ be a maximal parabolic subgroup of $Sp_{2m}(\mathbb{R})$ preserving an $i$-dimensional isotropic subspace. Let $P_{n - m + i, 2m - 2i}$ be a maximal parabolic subgroup of $Sp_{2n}(\mathbb{R})$ preserving an $n - m + i$-dimensional isotropic subspace. Let $GL_iSp_{2m - 2i}N_{i, 2m - 2i}$ be the Langlands decomposition of $P_{i, 2m - 2i}$. Let $GL_{n - m + i}Sp_{2m - 2i}(\mathbb{R})N_{n - m + i, 2m - 2i}$ be the Langlands decomposition of $P_{n - m + i, 2m - 2i}$. Let
\[ H_i = \{(m_1u_1, m_2u_2) \mid m_1 \in GL_i, m_2 \in GL_{n - m + i}, u \in Sp_{2m - 2i}(\mathbb{R}), n_1 \in N_{i, 2m - 2i}, n_2 \in N_{n - m + i, 2m - 2i}\}. \]
Then $L_i \cong Sp_{2m}(\mathbb{R}) \times Sp_{2n}(\mathbb{R}) / H_i$. In particular, if $m \leq n$, then
\[ L_0 \cong Sp_{2n}(\mathbb{R}) / GL_{n - m}N_{n - m, 2m}. \]

**Corollary 2.6** Let $i \leq m \leq n$. Then $\dim(L_i) = \frac{(n + m + 1)(n + m)}{2} - i^2 - ni + mi$. There are total of $m + 1$ $Sp_{2n}(\mathbb{R}) \times Sp_{2m}(\mathbb{R})$ orbits in $L(V)$ and
\[ L_{i+1} \subseteq cl(L_i). \]

### 3 $GL_n$-orbits

Let $\Omega$ be a symplectic form on a $2n$ dimensional real vector space $V$. Let $L$ be the Lagrangian Grassmannian. Fix two Lagrangian subspaces $W_1$ and $W_2$ such that
\[ W_1 \oplus W_2 = V. \]

Clearly, $\Omega$ identifies $W_2$ with $W_1^*$, the dual of $W_1$. Let $GL_n$ be the subgroup of $Sp(V, \Omega)$ preserving $W_1$ and $W_2$. Let $g \in GL_n$. Then $g$ acts on $U \in L$ by
\[ gU = \{(gx_1, g^*x_2) \mid (x_1, x_2) \in U\}. \]

Now we want to study the $GL_n$ orbits on $L$.

Let $e_1, e_2, \ldots, e_n$ be a basic for $W_1$. Let $f_1, f_2, \ldots, f_n$ be a basic for $W_2$ such that $\Omega(e_i, f_j) = \delta^i_j$ where $\delta^i_j$ is the Kronecker symbol. Clearly, $f_i$ can be identified with $e^*_i$. Define $J \in \text{End}(V)$ by setting
\[ Je_i = -f_i, \quad Jf_i = e_i. \]
Then
\[ \Omega(x, y) = (x, Jy) = -(Jx, y), \]
where \((, )\) is the standard inner product. Furthermore \(Jgx_1 = g^*Jx_1\).

Put \(U_1 = U \cap W_1\) and \(U_2 = U \cap W_2\). Let
\[ L_{i,j} = \{ U \in L \mid \dim(U \cap W_1) = i, \dim(U \cap W_2) = j \}. \]

We have

**Lemma 3.1** \(L_{0,0}\) consists of \(n + 1\) \(GL_n\) orbits:

\[ L_{0,0}^0, \ldots, L_{0,0}^i, \ldots, L_{0,0}^n, \]

where \(L_{0,0}^i \cong GL_n/O(i, n - i)\).

**Proof:** For every \(U \in L_{0,0}\), define \(\phi_U(x_1) = x_2\) if and only if \((x_1, x_2) \in U\). Since \(\dim U = \dim W_1 = \dim W_2 = n\) and \(\dim U \cap W_1 = \dim U \cap W_2 = 0\), \(\phi_U\) is a well-defined isomorphism from \(W_1\) to \(W_2\). Intuitively, \(U\) is the graph of \(\phi_U\). Let \((x_1, \phi_U(x_1)), (y_1, \phi_U(y_2)) \in U\). Then
\[ 0 = \Omega((x_1, \phi_U(x_1)), (y_1, \phi_U(y_1))) = \Omega(x_1, \phi_U(y_1)) + \Omega(\phi_U(x_1), y_1) \]
\[ = \Omega(x_1, \phi_U(y_1)) - \Omega(y_1, \phi_U(x_1)) = (x_1, J\phi_U(y_1)) - (y_1, J\phi_U(x_1)). \]

Therefore \((x_1, J\phi_U(y_1)) = (y_1, J\phi_U(x_1))\) for all \(x_1, y_1 \in W_1\). It follows that \((x_1, J\phi_U(y_1))\) is a Hermitian form on \(W_1\).

Let \(g \in GL_n\). Then
\[ gU = \{ (gx_1, g^*\phi_U(x_1) \mid x_1 \in W_1) = \{ (x_1, g^*\phi_U g^{-1}x_1) \mid x_1 \in W_1 \}. \]

So \(\phi_{gU} = g^*\phi_U g^{-1}\). The Hermitian form
\[ (x_1, J\phi_{gU}(y_1)) = (x_1, Jg^*\phi_U g^{-1}y_1) = (x_1, g^*J\phi_U g^{-1}y_1). \]

Clearly, \(GL_n\) acts on the non-degenerate Hermitian forms with \(n + 1\) orbits. Each orbit consists of Hermitian forms with a fixed signature. Let \(L_{0,0}^i\) be the set of \(U\) such that \((x_1, J\phi_U(y_1))\) has signature \((i, n - i)\). Then \(L_{0,0}^i\) is a single \(GL_n\) orbit and \(L_{0,0}^i \cong GL_n/O(i, n - i)\). \(\square\)

The case \((\dim U_1, \dim U_2) \neq (0, 0)\) is considerably more difficult. One needs to carry out two symplectic reductions. However, the ending result is easy to state.

**Theorem 3.1** Each \(L_{i,j}\) consists of \(n - i - j + 1\) \(GL_n\)-orbits.
We shall now give a proof of this theorem and the structure of each orbit will be stated at the end of this discussion.

Let \( U \in \mathcal{L}_{i,0} \). Let \( U_1 = U \cap W_1 \). Then \( U_1 \) is an isotropic subspace. Fix an \( U_1 \subseteq W_1 \) and let
\[
\mathcal{L}_{U_1,0} = \{ U \in \mathcal{L} \mid U \cap W_1 = U_1, U \cap W_2 = 0 \}.
\]
We have the fiberation
\[
\mathcal{L}_{U_1,0} \rightarrow \mathcal{L}_{i,0} \rightarrow G(n, i),
\]
where \( G(n, i) \) is the \( i \)-th Grassmanian of \( W_1 \). Notice that \( GL_n \) acts on \( G(n, i) \). Let \( P_{U_1} \) be the stabilizer of \( U_1 \in G(n, i) \). The \( P_{U_1} \) is isomorphic to the block-wise upper triangular matrices of size \((n - i, i)\). It suffices to study the action of \( P_{U_1} \) on \( \mathcal{L}_{U_1,0} \).

Consider now \( U_{1}^{\perp} \), with respect to \( \Omega \). Let \( \Omega' \) be the reduced symplectic form on \( U_{1}^{\perp}/U_1 \). Since \( U_1 \subseteq U \), \( U \subseteq U_{1}^{\perp} \). We see that
\[
[U]^{\perp, \Omega'} = U_{1}^{\perp}/U_1 = U/U_1.
\]
So \( U/U_1 \) is a Lagrangian subspace of \( U_{1}^{\perp}/U_1 \). Then \( \mathcal{L}_{U_{1}, 0} \) can be identified with a subset of \( \mathcal{L}(U_{1}^{+}/U_1) \). Furthermore, \( P_{U_1} \) acts on \( U_1 \) and \( U_{1}^{\perp} \). The \( GL(i) \) factor acts on \( U_1 \) naturally. The nilradical \( N \) acts on \( U_{1}^{\perp} \) by shifts along directions in \( U_1 \). We thus obtain a \( GL_{n-i} \) action on \( \mathcal{L}(U_{1}^{+}/U_1) \). By choosing \( U_1 \) to be the linear space spanned by \( e_1, \ldots, e_i \), the \( GL_{n-i} \) action on \( \mathcal{L}(U_{1}^{+}/U_1) \) can be identified with the standard \( GL_{n-i} \) action on \( \mathcal{L}(\mathbb{R}^{2n-2i}) \).

Let \( W_2' = W_2 \cap U_{1}^{\perp} \). Then \( (W_2')^{\perp} = W_2 \oplus U_1 \) and \( \dim(W_2') = n - i \). Furthermore
\[
(W_2' \oplus U_1)^{\perp} = (W_2 \oplus U_1) \cap U_{1}^{\perp} = W_2' \oplus U_1.
\]
We see that \( W_2' \oplus U_1 \) is a Lagrangian subspace in \( U_{1}^{\perp}/U_1 \). Since \( W_2 \cap U = 0 \), \( W_2' + U_1 \cap U = U_1 \). Since \( W_1 \subseteq U_{1}^{\perp} \) and \( W_1 \) is Lagrangian, \( W_1/U_1 \) is a Lagrangian subspace of \( U_{1}^{\perp}/U_1 \). Furthermore,
\[
U_{1}^{\perp}/U_1 = (W_2' + U_1)/U_1 \oplus W_1/U_1.
\]
Finally, notice that \( U/U_1 \cap (W_2' + U_1)/U_1 = [0] \) and \( U/U_1 \cap W_1/U_1 = [0] \). We see that
\[
\mathcal{L}_{U_{1}, 0} \cong \mathcal{L}_{0, 0}(\Omega', (W_2' + U_1)/U_1 \oplus W_1/U_1).
\]
By Lemma 3.1, we obtain

**Lemma 3.2** \( \mathcal{L}_{i, 0} \) consists of \( n - i + 1 \) \( GL_n \) orbits:
\[
\mathcal{L}_{i, 0}^{0}, \ldots, \mathcal{L}_{i, 0}^{n-i}.
\]
Let \( P_i \) be the maximal parabolic subgroup with Langlands decomposition \( GL_iGL_{n-i}N_{i,n-i} \). Then \( \mathcal{L}_{i, 0} \cong GL_n/GL_iO(j, n - i - j)N_{i,n-i} \).
If \(ij \neq 0\), we are yet to apply another symplectic reduction. Fix a \(U_2\), a \(j\)-dimensional subspace of \(W_2\). \(U_2\) is isotropic automatically. Let \(L_{i,U_2}\) be the set of Lagrangian subspaces satisfying
\[
U \cap W_2 = U_2, \quad \dim(U \cap W_1) = i.
\]
Then we have a fibration
\[
L_{i,U_2} \rightarrow L_{i,j} \rightarrow G(n, j).
\]
Let \(P_{U_2}\) be the stabilizer of \(U_2\) in \(GL_n\). Then \(P_{U_2}\) is isomorphic to \(GL_jGL_{n-j}N_{j,n-j}\).

Again, we construct the reduced symplectic form \(\Omega_2\) on \(U_2^\perp/U_2\). Let \(W'_1 = W_1 \cap U_2^\perp\). Then \((W'_1)^\perp = W_1 \oplus U_2\) and \(\dim(W'_1) = n - j\). Notice that
\[
(W'_1 \oplus U_2)^\perp = (W_1 \oplus U_2) \cap U_2^\perp = W'_1 \oplus U_2.
\]
We see that \((W'_1 + U_2)/U_2\) is a Lagrangian subspace of \(U_2^\perp/U_2\). \(W_2/U_2\) is also a Lagrangian subspace of \(U_2^\perp/U_2\). In addition
\[
U_2^\perp/U_2 = (W'_1 + U_2)/U_2 \oplus W_2/U_2.
\]
The group \(P_{U_2}\) acts on \(U_2^\perp/U_2\) via the factor \(GL_{n-j}\).

Since \(U\) is Lagrangian, \(U/U_2\) is a Lagrangian subspace in \(U_2^\perp/U_2\). Furthermore \(\dim(U/U_2 \cap W_2/U_2) = 0\). Observe that
\[
U/U_2 \cap (W'_1 + U_2/U_2) = U \cap (W'_1 + U_2)/U_2 = (U \cap W'_1) + U_2/U_2
\]
and
\[
(U \cap W'_1)^\perp = U^\perp + (W'_1)^\perp = U + W_1 + U = U + W_1 = (U \cap W_1)^\perp.
\]
So \(U \cap W'_1 = U \cap W_1 = U_1\). It follows that
\[
U/U_2 \cap (W'_1 + U_2/U_2) = (U \cap W'_1) + U_2/U_2 = U_1 + U_2/U_2.
\]
Consequently, \(\dim(U/U_2 \cap W'_1 + U_2/U_2) = i\). Now we can identify \(L_{i,U_2}\) with
\[
L_{i,0}(U_2^\perp/U_2, W_2/U_2 \oplus W'_1/U_2)
\]
by
\[
U \leftrightarrow U/U_2.
\]
By Lemma 3.2, \(GL_{n-j}\) acts on \(L_{i,0}(U_2^\perp/U_2)\) with \(n - j - i + 1\)-orbits:
\[
L^0_{i,0}(U_2^\perp/U_2), \ldots, L^i_{i,0}(U_2^\perp/U_2)
\]
and
\[
L^k_{i,0}(U_2^\perp/U_2) \cong GL_{n-j}/GL_iO(k, n - j - k)N_{n-i-j,i}.
\]
To summarize, we have the following theorem.
Theorem 3.2  Each $L_{i,j}$ consists of $n - i - j + 1$ $GL_n$-orbits

$$L_{i,j}^0, \ldots, L_{i,j}^{n-i-j}. $$

Let $P_{i,j,n-i-j}$ be the block-wise upper triangular matrices with size $(i, j, n - i - j)$. Let $N$ be the nilradical. Then

$$L_{i,j}^k \cong GL_n/GL_jGL_iO(k, n - i - j - k)N.$$

References


