Quantum Induction

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1 Introduction

Let G be a semisimple Lie group. Let P be a parabolic subgroup. Let MAN be the Langlands decomposition of P. Let τ be an irreducible unitary representation of M. Let $\mathfrak{a}$ be the Lie algebra of A. Let λ be in the real dual of $\mathfrak{a}$. Let ρ be the half sum of the positive restricted roots in $\Sigma(\mathfrak{g}, \mathfrak{a})$. Then

$$U(P, \tau, \lambda) = Ind_{MAN}^{G} \tau \otimes \exp(-\rho + i\lambda)$$

is unitary. This is the unitary parabolic induction. For "generic" λ, $U(P, \tau, \lambda)$ is irreducible.

Question: Can unitary parabolic induction produce all irreducible unitary representations?

Answer: No. There are various reasons.

1. Even for $G = SL(2, \mathbb{R})$, there are discrete series and two limit representations, and complementary series representations besides the trivial representation. There are systematic ways of constructing discrete series and complentary series.

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2. For $G = Mp_{2n}(\mathbb{R})$, the double covering of $Sp_{2n}(\mathbb{R})$, there is the Segal-Shale-Weil representation, also called the oscillator representation, the metaplectic representation. They can not be obtained from the above constructions except perhaps $n = 1$ or 2.

3. In general, for classical groups of type I, there are irreducible unitary representations of low rank in the sense of Howe. This class of representations includes most representations constructed by Kashiwara-Vergne, Howe, Li, Sahi, Tan, Binegar-Zierau, Brylinski-Kostant, Huang, Zhu (see [KV], [Ho84], [Sahi], [BZ], [HT], [BK], [ZH], [HL] and the references within them). Low rank irreducible unitary representations are classified completely by Jian-Shu Li ([Li89]).

4. There are irreducible unitary representations not on any of the lists above, as conjectured by Barbasch-Vogan and Arthur.

Our motivation is to systematically construct some of these representations in 4).

Remark: The equivalence classes of irreducible unitary representations of $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$ and $GL(n, \mathbb{H})$ are classified by D. Vogan. The equivalence classes of irreducible unitary representations of $O(n, \mathbb{C})$ and $Sp(n, \mathbb{C})$ are classified by D. Barbasch. We shall therefore focus on the rest of the classical groups.

2 Main Results

In this talk, for simplicity, I will consider $G = Mp_{2m+2n}(\mathbb{R})$. Let (p, q) be such that $p + q \leq m + n$. Consider the dual pair $(O(p, q), G)$. Let $\mathcal{E}(p, q)$ be the theta lift of trivial representation of $O(p, q)$. If $p + q$ odd, then $\mathcal{E}(p, q)$ is a genuine irreducible unitary representation of G. If $p + q$ even, then $\mathcal{E}(p, q)$ is an irreducible unitary representation of $Sp_{2n+2m}(\mathbb{R})$. These representations are studied by Howe, Li, Huang-Li and others. Fix a maximal compact subgroup K in $Mp_{2n+2m}(\mathbb{R})$.

Let $K_1 = K \cap Mp_{2m}(\mathbb{R})$ and $K_2 = K \cap Mp_{2n}(\mathbb{R})$.

Definition 1 *Let π be an irreducible unitarizable $(K_1, \mathfrak{sp}_{2\mathfrak{m}}(\mathbb{R}))$-module. Let $E(p, q)$ be the $(K, \mathfrak{sp}_{2\mathfrak{n}+2\mathfrak{m}})$-module of $\mathcal{E}(p, q)$. Formally define a Hermitian form $(,)$ on $E(p, q) \otimes \pi$ by integrating the matrix coefficients of $E(p, q)$ against the matrix coefficients of π:*

$$(\phi \otimes u, \psi \otimes v) = \int_{Mp_{2m}(\mathbb{R})} (\mathcal{E}(p, q)(g)\phi, \psi)(\pi(g)u, v)dg.$$

Suppose that $(,)$ converges. Define $\mathcal{Q}(2m; p, q; 2n)(\pi)$ to be $E(p, q) \otimes \pi$ modulo the radical of $(,)$. Then $\mathcal{Q}(2m; p, q; 2n)(\pi)$ is a $(K_2, \mathfrak{sp}_{2\mathfrak{n}})$-module.

Theorem 1 (Main Theorem: Unitarity and Irreducibility, [Heq]) *Suppose $2n - p - q \geq p + q - 2m - 2$ and $m < p \leq q$. Suppose π is a unitary representation such that every leading coefficient v satisfies*

$$\Re(v) \preceq \left(\frac{p+q}{2} - 2m - 1, \frac{p+q}{2} - 2m, \ldots, \frac{p+q}{2} - m - 2\right).$$

If $(,)$ does not vanish, then $\mathcal{Q}(2m; p, q; 2n)(\pi)$ is irreducible and unitary.

Main idea of the Proof: Under our hypothesis $\mathcal{Q}(2m; p, q; 2n) = \theta(p, q; 2n)\theta(2m; p, q)$ (see [Heq] and [Heu]).

Similar to parabolic induced representation $Ind_P^G \tau \otimes \exp(-\rho + i\lambda)$ whose vectors are in

$$Hom_P(C_c^\infty(G), \tau \otimes \exp(-\rho + i\lambda)),$$

quantum induced $\mathcal{Q}(2m; p, q; 2n)(\pi)$ lies in

$$Hom_{K_1, \mathfrak{sp}_{2m}}(E(p,q), \pi).$$

However, $Ind_P^G \tau \otimes \exp(-\rho + i\lambda)$ has a nice geometric description. It consists of sections of some homogeneous vector bundle over G/P. In contrast, quantum induction does not possess this kind of classical interpretation except for the limit case $p + q = n + m + 1$.

Theorem 2 (Quantum Induction and Parabolic Induction, [1])

$$Ind_{MSp_{2m}(\mathbb{R})GL_{n-m}N}^{Mp_{2n}(\mathbb{R})} \pi \otimes \chi^{\alpha} = \oplus_{p+q=m+n+1,\ p-q \equiv \alpha \pmod 4} \mathcal{Q}(2m; p, q; 2n)(\pi).$$

This theorem is proved by using a theorem of Kudla-Rallis which is more explicitly given in [LZ].

Theorem 3 (Infinitesimal Character) *Under the same hypothesis as in the Main Theorem, suppose $Q(*)(\pi) \neq 0$.*

If $p+q$ is even, then

$$\begin{aligned}\mathcal{I}(Q(2m; p, q; 2n)(\pi)) =& \mathcal{I}(\pi) \oplus (\frac{p+q}{2} - m - 1, \frac{p+q}{2} - m - 2, \ldots, 0) \\ & \oplus (n - \frac{p+q}{2}, n - \frac{p+q}{2} - 1, \ldots, 1)\end{aligned} \tag{1}$$

If $p+q$ is odd, then

$$\begin{aligned}\mathcal{I}(Q(2m; p, q; 2n)(\pi)) =& \mathcal{I}(\pi) \oplus (\frac{p+q}{2} - m - 1, \frac{p+q}{2} - m - 2, \ldots, \frac{1}{2}) \\ & \oplus (n - \frac{p+q}{2}, n - \frac{p+q}{2} - 1, \ldots, \frac{1}{2})\end{aligned} \tag{2}$$

This is proved in [Heq] using a theorem of Przebinda regarding the daulity correspondence of infinitesimal characters [PR96], see also [Li99]. This theorem is consistent with the behavior of infinitesimal characters under parabolic induction. Recall that for parabolic induction the infinitesimal character

$$\mathcal{I}(U(P, \tau, \lambda)) = \mathcal{I}(\tau) \oplus i\lambda.$$

(see Theorem 8.22 of [KN], for example).

3 Some Problems

Conjecture 1 *The Main Theorem and the theorem concerning infinitesimal character hold without the assumption on the leading exponents of* π.

Our definition of quantum induction is analytic. Motivated by the work of Howe [Ho89], one may attempt to give a purely algebraic definition.

Conjecture 2 *Let* $\mathcal{R}(Mp_{2m}(\mathbb{R}), Mp_{2n}(\mathbb{R}))$ *be the irreducible* $(\mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2n}, K_1 K_2)$*-modules occuring as a quotient in* $E(p,q)$*. Let* $\mathcal{R}(Mp_{2m}(\mathbb{R}))$ *be the irreducible* $(\mathfrak{sp}_{2m}, K_1)$*-modules occuring as a quotient in* $E(p,q)$*. Let* $\mathcal{R}(Mp_{2n}(\mathbb{R}))$ *be the irreducible* $(\mathfrak{sp}_{2n}, K_2)$*-modules occuring as a quotient in* $E(p,q)$*. Then* $\mathcal{R}(Mp_{2m}(\mathbb{R}), Mp_{2n}(\mathbb{R}))$ *gives a one-to-one correspondence between* $\mathcal{R}(Mp_{2m}(\mathbb{R})$ *and* $\mathcal{R}(Mp_{2n}(\mathbb{R}))$.

This one-to-one correspondence, if proved, can be regarded as the algebraic version of $\mathcal{Q}(2m; p, q; 2n)$.

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