Invariant Tensor Product

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Abstract

In this paper, we define invariant tensor product and study invariant tensor products associated with discrete series representations. Let $G(V_1) \times G(V_2)$ be a pair of classical groups diagonally embedded in $G(V_1 \oplus V_2)$. Suppose that $\dim V_1 < \dim V_2$. Let π be a discrete series representation of $G(V_1 \oplus V_2)$. We prove that the functor $\pi \otimes_{G(V_1)} *$ maps unitary representations of $G(V_1)$ to unitary representations of $G(V_2)$. Here we enlarge the definition of unitary representations by including the zero dimensional "representation".

1 Invariant Tensor Products

Various forms of invariant tensor products appeared in the literature implicitly, for example, in Schur's orthogonality for finite groups ([Se]). In many cases, they are employed to study the space $\mathrm{Hom}_G(\pi_1, \pi_2)$ where one of the representations π_1 and π_2 is irreducible. In this paper, we formulate the concept of invariant tensor product uniformly. We also study the invariant tensor functor associated with discrete series representations for classical groups. For motivations and applications, see [Li1], [He00], [GGP].

Definition 1 *Let G be a locally compact topological group and dg be a left invariant Haar measure. Let (π, H_π) and (π_1, H_{π_1}) be two unitary representations of G. Let V and V_1 be two dense subspaces of H_π and H_{π_1}. Formally, define the averaging operator*

$$\mathcal{L} : V \otimes V_1 \to (V \otimes V_1)^*_{\mathbb{R}}$$

as follows, $\forall \ u, v \in V, u_1, v_1 \in V_1$,

$$\begin{aligned} \mathcal{L}(v \otimes v_1)(u \otimes u_1) &= \int_G \big((\pi \otimes \pi_1)(g)(v \otimes v_1), (u \otimes u_1)\big) dg && (1) \\ &= \int_G (\pi(g)v, u)(\pi_1(g)v_1, u_1) dg. && (2) \end{aligned}$$

Suppose that $\mathcal{L}$ is well-defined. The image of $\mathcal{L}$ will be called the **invariant tensor product***. It will be denoted by $V \otimes_G V_1$. Whenever we use the notation $V \otimes_G V_1$,* **we assume $V \otimes_G V_1$ is**

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well-defined, *that is, the integral (1) converges for all* $u, v \in V, u_1, v_1 \in V_1$. *Denote* $\mathcal{L}(v \otimes v_1)$ *by* $v \otimes_G v_1$. *Define*

$$(v \otimes_G v_1, u \otimes_G u_1)_G = \int_G (\pi(g)v, u)(\pi_1(g)v_1, u_1) dg.$$

For any unitary representation $(\pi, \mathcal{H})$ of G, let $(\pi^c, \mathcal{H}^c)$ be the same unitary representation of G equipped with the conjugate linear multiplication. If V is a subspace of $\mathcal{H}$, let V^c be the corresponding subspace of $\mathcal{H}^c$.

Lemma 1.1 *Let G be a unimodular group. Suppose that $V \otimes_G V_1$ is well-defined. Then the form $(\, ,\,)_G$ is a well-defined Hermitian form on $V \otimes_G V_1$.*

The main result proved in this paper is as follows.

Theorem 1.1 *Let $G(m+n)$ be a classical group of type I with $m > n$. Let $(G(n), G(m))$ be diagonally embedded in G (see Def. 2). Suppose that $(\pi, \mathcal{H}_\pi)$ is a discrete series representation of $G(m+n)$ and $(\pi_1, \mathcal{H}_1)$ is a unitary representation of $G(n)$. Let $\mathcal{H}_\pi^\infty$ be the space of smooth vectors in $\mathcal{H}_\pi$. Then $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1$ is well-defined. Suppose that $H_\pi^\infty \otimes_{G(n)} \mathcal{H}_1 \neq 0$. Then $(\, ,\,)_{G(n)}$ is positive definite. Furthermore, $(H_\pi^\infty \otimes_{G(n)} \mathcal{H}_1, (\, ,\,)_{G(n)})$ completes to a unitary representation of $G(m)$.*

2 Example: π_1 Irreducible

Example I: Let G be a compact group. Let (π, H_π) and (π_1, H_{π_1}) be two unitary representations of G. Then $H_\pi \otimes_G H_{\pi_1}$ is always well-defined. Suppose that π_1 is irreducible. Then the dimension of $H_\pi \otimes_G H_{\pi_1}$ is the the multiplicty of π_1^* occuring in H_π.

Example II: Let G be a real reductive group. Let π and π_1 be two discrete series representations. Then $H_\pi \otimes_G H_{\pi_1}$ is always well-defined. It is one dimensional if and only if $\pi_1 \cong \pi^*$. Otherwise, it is zero dimensional.

Theorem 2.1 *Let π_1 be an irreducible unitary representation of G. Suppose that V_1 and V are both closed under the action of G. Suppose that $V_1 \otimes_G V$ is well-defined. Then $\mathcal{L}$ induces an injection from $V_1 \otimes_G V$ to $\mathrm{Hom}_G(V^c, V_1^h)$, the space of G-equivariant homomorphisms from V^c to the Hermitian dual V_1^h.*

Proof: For each $v_1 \in V_1$, $v, u \in V$, define $\mathcal{L}(v_1 \otimes v)(u) \in V_1^h$ as follows:

$$\forall\, u_1 \in V_1, \qquad \mathcal{L}(v_1 \otimes v)(u)(u_1) = (v_1 \otimes v, u_1 \otimes u)_G.$$

We have for every $\lambda \in \mathbb{C}$,

$$\mathcal{L}(\lambda v_1 \otimes v)(u)(u_1) = \lambda \mathcal{L}(v_1 \otimes v)(u)(u_1);$$

$$\mathcal{L}(v_1 \otimes \lambda v)(u)(u_1) = \lambda \mathcal{L}(v_1 \otimes v)(u)(u_1);$$
$$\mathcal{L}(v_1 \otimes v)(\lambda u)(u_1) = \overline{\lambda} \mathcal{L}(v_1 \otimes v)(u)(u_1);$$
$$\mathcal{L}(v_1 \otimes v)(u)(\lambda u_1) = \overline{\lambda} \mathcal{L}(v_1 \otimes v)(u)(u_1).$$

We see that $\mathcal{L}(v_1 \otimes v)(u)$ is in the Hermitian dual of V_1. In addition, $\mathcal{L}(v_1 \otimes v)$ is G-equivariant:

$$\begin{aligned}
\mathcal{L}(v_1 \otimes v)(\pi(g)u)(u_1) &= \int_{h \in G} (\pi_1(h)v_1, u_1)(\pi(h)v, \pi(g)u)dh && (3) \\
&= \int_{h \in G} (\pi_1(h)v_1, u_1)(\pi(g^{-1}h)v, u)dh && (4) \\
&= \int_{h \in G} (\pi_1(gh)v_1, u_1)(\pi(h)v, u)dh && (5) \\
&= \int_{h \in G} (\pi_1(h)v_1, \pi_1(g^{-1})u_1)(\pi(h)v, u)dh && (6) \\
&= \mathcal{L}(v_1 \otimes v)(u)(\pi_1(g^{-1})u_1) && (7) \\
&= [\pi_1^h(g)\mathcal{L}(v_1 \otimes v)(u)](u_1). && (8)
\end{aligned}$$

Here $d\,h$ is a left invariant measure if G is not unimodular. Now it is easy to see that $\mathcal{L}(v_1 \otimes v)(u) = 0$ for every u if and only if $v_1 \otimes_G v = 0$. So

$$\mathcal{L} : V_1 \otimes_G V \to \mathrm{Hom_G}(\mathrm{V^c}, \mathrm{V_1^h})$$

is an injection. $\square$

Corollary 2.1 *Under the same assumption as in Theorem 2.1, let G be a real reductive group and K a maximal compact subgroup of G. Suppose that V and V_1 are both smooth and K-finite. Then $\mathcal{L}$ induces an injection from $V_1 \otimes_G V$ into* $\mathrm{Hom}_{\mathfrak{g},\mathrm{K}}(\mathrm{V^c}, \mathrm{V_1})$.

Proof: When V is K-finite, $\mathcal{L}(v_1 \otimes v)(u)$ will land in the K-finite subspaces of V_1^h which is isomorphic to V_1. $\square$

3 A Geometric Realization

Let G be a Lie group and dg a left invariant Haar measure. Let X be a manifold with a continuous free (right) G action. Suppose that X/G is a smooth manifold. Let $(\pi, \mathcal{H})$ be a unitary representation of G. For any $f \in C_c(X)$, $v \in \mathcal{H}$, define

$$\mathcal{L}^0(f \otimes v)(x) = \int_G f(xg)\pi(g)vdg.$$

Then $\mathcal{L}^0(f \otimes v)$ is a $\mathcal{H}$-valued function on X. We shall see that it realizes $f \otimes_G v$ in the following sense.

Theorem 3.1 *Let G be a Lie group and dg a left invariant Haar measure. Let X be a manifold with a continuous free (right) G action such that the topological quotient X/G is a smooth manifold. Suppose there exist measures (X, μ) and $(X/G, d[x])$ such that*

$$\int_X f(x)d\mu(x) = \int_{[x] \in X/G} \int_G f(xg)dgd[x].$$

Let $C_c(X)$ be the set of continuous functions with compact support. Let $(\pi, \mathcal{H})$ be a representation of G. Then $\mathcal{L}^0(f \otimes v) \in C_c(X \times_G \mathcal{H}, X/G)$ where $C_c(X \times_G \mathcal{H}, X/G)$ is the set of continuous compactly supported sections of the vector bundle

$$X \times_G \mathcal{H} \to X/G.$$

Furthermore,

$$C_c(X) \otimes_G \mathcal{H} \cong \mathcal{L}^o(C_c(X) \otimes \mathcal{H}),$$

and for every $f \in C_c(X)$ *and* $v \in \mathcal{H}$*,*

$$(f \otimes_G v, f \otimes_G v)_G = (\mathcal{L}^0(f \otimes v), \mathcal{L}^0(f \otimes v))_{X/G}.$$

Proof: Let $f \in C_c(X)$ and $v \in \mathcal{H}$. It is easy to see that $\mathcal{L}^0(f \otimes v)$ is compactly supported in X/G. In addition

$$\mathcal{L}^0(f \otimes v)(xg_1) = \int_G f(xg_1 g)\pi(g) v dg = \int_G f(xg)\pi(g_1^{-1} g) v dg = \pi(g_1)^{-1}\mathcal{L}^0(f \otimes v)(x).$$

So $\mathcal{L}^0(f \otimes v) \in C_c(X \times_G \mathcal{H}, X/G)$. Observe that

$$
\begin{aligned}
& (f \otimes_G v, f \otimes_G v)_G && (9)\\
= \;& \textstyle\int_G (R(g)f, f)(\pi(g)v, v) dg && (10)\\
= \;& \textstyle\int_G \int_X f(xg)\overline{f(x)} dx (\pi(g)v, v) dg && (11)\\
= \;& \textstyle\int_G \int_{X/G} \int_G f(xg_1 g)\overline{f(xg_1)}(\pi(g)v, v) dg_1 d[x] dg && (12)\\
= \;& \textstyle\int_{X/G} \int_G \int_G f(xg)\overline{f(xg_1)}(\pi(g_1^{-1} g)v, v) dg_1 dg d[x] && (13)\\
= \;& \textstyle\int_{X/G} \int_G \int_G f(xg)\overline{f(xg_1)}(\pi(g)v, \pi(g_1)v) dg_1 dg d[x] && (14)\\
= \;& \textstyle\int_{X/G} (\int_G f(xg)\pi(g) v dg, \int_G f(xg_1)\pi(g_1) v dg_1) d[x] && (15)\\
& && (16)
\end{aligned}
$$

Absolute convergence are guaranteed since $f(g)$ is compactly supported. Notice that

$$\mathcal{L}^0(f \otimes v)(x) = \int_G f(xg)\pi(g) v dg.$$

We have

$$(f \otimes_G v, f \otimes_G v)_G = (\mathcal{L}^0(f \otimes v), \mathcal{L}^0(f \otimes v))_{X/G}.$$

Clearly, $C_c(X) \otimes_G \mathcal{H} \cong \mathcal{L}^o(C_c(X) \otimes \mathcal{H})$. $\square$

4 Invariant Tensor Product and Representation Theory

Definition 2 *Let* G *be a classical group that preserves a nondegenerate sesquilinear form* Ω*. Write* $G = G(V, \Omega)$ *or simply* $G(V)$*, where* V *is a vector field over* $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ *equipped with the nondegenerate sesquilinear form* Ω*. Let* $V = V_1 \oplus V_2$ *such that* $\Omega(V_1, V_2) = 0$*. Let* $G_1 = G(V_1, \Omega|_{V_1})$ *and* $G_2 = G(V_2, \Omega|_{V_2})$*. For each* $g_1 \in G_1, g_2 \in G_2$*, let* (g_1, g_2) *acts on* $V_1 \oplus V_2 = V$ *diagonally. We say that* $G_1 \times G_2$ *is diagonally embedded in* G*.*

Definition 3 *Let* (G_1, G_2) *be diagonally embedded in* G*. Let* $(\pi, \mathcal{H}_\pi)$ *be a unitary representation of* G *and* $(\pi_1, \mathcal{H}_{\pi_1})$ *be a unitary representation of* G_1*. Let* V *be a subspace of* $\mathcal{H}_\pi^\infty$ *that is invariant under* G_2*. Let* V_1 *be a subspace of* $\mathcal{H}_{\pi_1}^\infty$ *such that* $V \otimes_{G_1} V_1$ *is well-defined. Define a linear* G_2*-representation* $(\pi \otimes_G \pi_1, V \otimes_{G_1} V_1)$ *as follows:*

$$(\pi \otimes_{G_1} \pi_1)(g_2)(u \otimes_{G_1} u_1) = \pi(g_2) u \otimes_{G_1} u_1 \qquad (g_2 \in G_2, u \in V, u_1 \in V_1).$$

Since the Lie group action of G_2 *commutes with the integration over* G_1*, the action of* G_2 *on* $V \otimes_{G_1} V_1$ *is well-defined.*

The linear representation $(\pi\otimes_{G_1}\pi_1, V\otimes_{G_1} V_1)$ is not neccessarily continuous because no topology has been defined on $V\otimes_{G_1} V_1$.

Lemma 4.1 *The form $(\ ,\ )_{G_1}$ on $V\otimes_{G_1} V_1$ is G_2-invariant.*

Proof: Let $u, v \in V; u_1, v_1 \in V_1$ and $g_2 \in G_2$. Write $\sigma = \pi\otimes_{G_1}\pi_1$. Then

$$\begin{aligned}
& (\sigma(g_2)(u\otimes_{G_1} u_1), v\otimes_{G_1} v_1)_{G_1} && (17)\\
= & \int_{G_1}(\pi(g_1)\pi(g_2)u, v)(\pi_1(g_1)u_1, v_1)dg_1 && (18)\\
= & \int_{G_1}(\pi(g_2)\pi(g_1)u, v)(\pi_1(g_1)u_1, v_1)dg_1 && (19)\\
= & \int_{G_1}(\pi(g_1)u, \pi(g_2^{-1})v)(\pi_1(g_1)u_1, v_1)dg_1 && (20)\\
= & (u\otimes_{G_1} u_1, \pi(g_2^{-1})v\otimes_{G_1} v_1)_{G_1} && (21)\\
= & (u\otimes_{G_1} u_1, \sigma(g_2^{-1})(v\otimes_{G_1} v_1))_{G_1} && (22)
\end{aligned}$$

Hence $(\ ,\ )_{G_1}$ is G_2-invariant. $\square$

5 ITP associated with Discrete Series Representations

Let $G(m+n)$ be a classical group preserving a nondegenerate sesquilinear form. Let $(G(n), G(m))$ be diagonally embedded in G. For any irreducible unitary representation π of $G(m+n)$, let $\mathcal{H}_\pi^\infty$ be the Frechet space of smooth vectors.

Theorem 5.1 *Suppose that $(\pi, \mathcal{H}_\pi)$ is a discrete series representation of $G(m+n)$. Suppose that $m > n$ and $(\pi_1, \mathcal{H}_1)$ is a unitary representation of $G(n)$. Then $\mathcal{H}_\pi^\infty\otimes_{G(n)}\mathcal{H}_1$ is well-defined. Suppose that $\mathcal{H}_\pi^\infty\otimes_{G(n)}\mathcal{H}_1 \neq 0$. Then $(\ ,\ )_{G(n)}$ is positive definite. Furthermore, $(\mathcal{H}_\pi^\infty\otimes_{G(n)}\mathcal{H}_1, (\ ,\ )_{G(n)})$ completes to a unitary representation of $G(m)$.*

The key of the proof is to realize $\mathcal{H}_\pi^\infty\otimes_{G(n)}\mathcal{H}_1$ as a subspace of the L^2-sections of the Hilbert bundle

$$\mathcal{H}_1\times_{G(n)} G(m+n)\rightarrow G(n)\backslash G(m+n).$$

Proof: Write $G = G(m+n)$. Fix a maximal compact subgroup K of G such that

$$K(m) = K\cap G(m), \qquad K(n) = K\cap G(n)$$

are maximal compact subgroups of $G(m)$ and $G(n)$ respectively. Let $\mathfrak{a}$ be a maximal Abelian subalgebra in the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form $(\ ,\ )_\kappa$, such that

$$\mathfrak{a} = (\mathfrak{a}\cap\mathfrak{g}(m))\oplus(\mathfrak{a}\cap\mathfrak{g}(n)).$$

Let A be the analytic group generated by $\mathfrak{a}$. The function $\log : A\rightarrow\mathfrak{a}$ is well-defined. Let $\|H\|^2 = (H, H)_\kappa$ for each $H\in\mathfrak{a}$.

Since $(\pi, \mathcal{H})$ is a discrete series representation, without loss of generality, realize $\mathcal{H}$ on a right K-finite subspace of $L^2(G)$. So $\mathcal{H}\subseteq L^2(G)_K$.

Let $\Xi_G(g)$ be Harish-Chandra's basic spherical function. Let $\mathcal{H}CS(G)$ be the space of Harish-Chandra's Schwartz space. It is well-known that every $f\in\mathcal{H}_\pi^\infty\subseteq\mathcal{H}CS(G)$ satisfies $f(g)\leq C_f\Xi_G(g)$ for some C_f (see for example Ch. 12.4 [?]). For every $h\in G$, $|f(hg)|\leq\Xi_G(hg)\leq$

$C_h C_f \Xi_G(g)$ for a constant C_h. Observe that $\mathcal{H}_\pi^\infty$ is $G(m)$-invariant.

Fix a positive root system in $\Sigma(\mathfrak{g}, \mathfrak{a})$. Let A^+ be the corresponding closed Weyl Chamber. Let ρ be the half sum of positive roots. Let $u, v \in \mathcal{H}_\pi^\infty \subseteq L^2(G)$. Then $|(L(g)u, v)| \leq C_{u,v} \Xi_G(g)$ for a positive constant $C_{u,v}$ ([CHH] [HE]). Notice that for $a \in A^+$, $k_1, k_2 \in K$,

$$\Xi_G(k_1 a k_2) \leq C(1 + \| \log a \|)^q \exp(-\rho(\log a))$$

for some $q \geq 0$ and $C > 0$. Let $\rho(n)$ be the half sum of positive roots of the restricted root system $\Sigma(\mathfrak{g}(n), \mathfrak{a} \cap \mathfrak{g}(n))$. Let $(\mathfrak{a} \cap \mathfrak{g}(n))^+$ be the positive Weyl chamber of $\mathfrak{a} \cap \mathfrak{g}(n)$ with respect to the root system $\Sigma(\mathfrak{g}(n), \mathfrak{a} \cap \mathfrak{g}(n))$. Since

$$\rho|_{\mathfrak{a} \cap \mathfrak{g}(n)}(H) > 2\rho(n)(H) \qquad (H \in (\mathfrak{a} \cap \mathfrak{g}(n))^+),$$

$\Xi_G(g)|_{G(n)} \in L^1(G(n))$. It follows that $(L(g)u, v)|_{G(n)} \in L^1(G(n))$ for every $u, v \in \mathcal{H}_\pi^\infty$. Notice that $g_1 \in G(n) \rightarrow (\pi_1(g_1)u_1, v_1)$ is always bounded for $u_1, v_1 \in \mathcal{H}_1$. We see that

$$\int_{G(n)} (\pi(g_1)u, v)(\pi_1(g_1)u_1, v_1) dg_1$$

always converges. So $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1$ is well-defined. Now suppose that $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1 \neq 0$.

Notice that $u \in \mathcal{H}_\pi^\infty \subseteq L^2(G)_K$ is bounded by a multiple of $\Xi_G(g)$. So $u|_{G(n)} \in L^1(G(n))$. For each $u \in \mathcal{H}_\pi^\infty$ and $u_1 \in \mathcal{H}_1$, define $\mathcal{L}^0(u \otimes u_1)$ to be the $\mathcal{H}_1$-valued function on G:

$$g \in G \rightarrow \int_{g_1 \in G(n)} [L(g_1)u](g) \pi_1(g_1) u_1 \, dg_1 = \int_{g_1 \in G(n)} u(g_1^{-1} g) \pi_1(g_1) u_1 dg_1$$

in the strong sense. Notice that for $g \in G, h_1 \in G(n)$,

$$\begin{aligned}
\mathcal{L}^0(u \otimes u_1)(h_1 g) &= \int_{g_1 \in G(n)} [L(g_1)u](h_1 g) \pi_1(g_1) u_1 \, dg_1 && (23) \\
&= \int_{g_1 \in G(n)} u(g_1^{-1} h_1 g) \pi_1(g_1) u_1 \, dg_1 && (24) \\
&= \int_{g_1 \in G(n)} u(g_1^{-1} g) \pi_1(h_1 g_1) u_1 \, dg_1 && (25) \\
&= \pi(h_1) [\int_{g_1 \in G(n)} L(g_1) u(g) \pi_1(g_1) u_1 \, dg_1] && (26) \\
&= \pi(h_1) L^0(u \otimes u_1)(g) && (27)
\end{aligned}$$

So $\mathcal{L}^0(u \otimes u_1)$ can be regarded as a section of the Hilbert bundle

$$\mathcal{H}_1 \ {}_{G(n)}\!\times G \rightarrow G(n) \backslash G.$$

In addition, we have

$$
\begin{aligned}
& (u \otimes_{G(n)} u_1, v \otimes_{G(n)} v_1)_{G(n)} && (28)\\
= & \int_{G(n)} (L(g_1)u, v)(\pi_1(g_1)u_1, v_1) dg_1 && (29)\\
= & \int_{G(n)} \int_G u(g_1^{-1}g)\overline{v(g)} dg (\pi_1(g_1)u_1, v_1) dg_1 && (30)\\
= & \int_{G(n)} \int_{G(n)\backslash G} \int_{G(n)} u(g_1^{-1}h_1 g)\overline{v(h_1 g)} dh_1 d[g] (\pi_1(g_1)u_1, v_1) dg_1 && (31)\\
= & \int_{G(n)\backslash G} \int_{G(n)\times G(n)} u(g_1^{-1}h_1 g)\overline{v(h_1 g)} (\pi_1(g_1)u_1, v_1) dg_1 dh_1 d[g] \qquad (g_1 = h_1 g_1) && (32)\\
= & \int_{G(n)\backslash G} \int_{G(n)\times G(n)} u(g_1^{-1}g)\overline{v(h_1 g)} (\pi_1(h_1 g_1)u_1, v_1) dg_1 dh_1 d[g] && (33)\\
= & \int_{G(n)\backslash G} \int_{G(n)\times G(n)} u(g_1^{-1}g)\overline{v(h_1 g)} (\pi_1(g_1)u_1, \pi(h_1^{-1})v_1) dg_1 dh_1 d[g] && (34)\\
= & \int_{G(n)\backslash G} \int_{G(n)\times G(n)} u(g_1^{-1}g)\overline{v(h_1^{-1} g)} (\pi_1(g_1)u_1, \pi_1(h_1)v_1) dg_1 dh_1 d[g] && (35)\\
= & (\mathcal{L}^0(u \otimes u_1), \mathcal{L}^0(v \otimes v_1))_{G(n)\backslash G} && (36)
\end{aligned}
$$

where $G(n)\backslash G$ is equipped with a right G invariant measure. Eqn. (32) is valid because the integral Eqn. (30) converges absolutely. In fact, we have

$$\int_{G(n)\times G} |u(h_1 g)v(g)| dg dh_1 < \infty.$$

To see this, recall that $u(g), v(g) \in \mathcal{HCS}(G)$. In particular, for any $N > 0$ and $a \in A^+$, $k_1, k_2 \in K$, there exists $C_{u,N} > 0$ such that

$$|u(k_1 a k_2)| \leq C_{u,N} \| \log a \|^{-N} \Xi_G(k_1 a k_2).$$

Write $W_N(g) = \| \log a \|^{-N} \Xi_G(k_1 a k_2)$ for $g = k_1 a k_2$. Then there also exists $C_{v,N} > 0$ such that

$$|v(g)| \leq C_{v,N} W_N(g).$$

Fix an N such that $W_N(G) \in L^2(G)$. In particular, $W_N(G) \in {}_K L^2(G)_K$. Observe that the function

$$h \in G \to (L(h)|u|(g), |v|(g))$$

is bounded by a multiple of $(L(h)W_N(g), W_N(g))$, which, by a Theorem of Cowling-Haagerup-Howe ([CHH]), is bounded by a multiple of $\Xi_G(g)$. Hence

$$\int_{G(n)\times G} |u(h_1 g)v(g)| dg dh_1 < \int_{G(n)} (\int_G |u(h_1 g)||v(g)| dg) dh_1 < \int_{G(n)} C \Xi_G(h_1) dh_1 < \infty.$$

Eqn. (30) converges absolutely. Therefore Eqn. (32) holds.

Now we have

$$(u \otimes_{G(n)} u_1, v \otimes_{G(n)} v_1)_{G(n)} = (\mathcal{L}^0(u \otimes u_1), \mathcal{L}^0(v \otimes v_1))_{G(n)\backslash G}.$$

It follows that $\mathcal{L}^0(\mathcal{H}_\pi^\infty \otimes \mathcal{H}_1) \cong \mathcal{L}(\mathcal{H}_\pi^\infty \otimes \mathcal{H}_1)$. Realize $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1$ as $\mathcal{L}^0(\mathcal{H}_\pi^\infty \otimes \mathcal{H}_1)$, which is a subspace of L^2-sections of the Hilbert bundle

$$\mathcal{H}_1 \ {}_{G(n)}\!\times G \to G(n)\backslash G.$$

Clearly $(\,,\,)_{G(n)}$ is positive definite. Let $\overline{\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1}$ be the completion of $(\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1, (\,,\,)_{G(n)})$.

Since $G(m)$ acts on $\mathcal{H}_\pi^\infty$ and it commutes with $G(n)$, $G(m)$ acts on $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1$ and it preserves $(\,,\,)_{G(n)}$. So the action of each $g_2 \in G(m)$ can be extended into a unitary operator on $\overline{\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1}$. The group structure is kept in this completion essentially due to the fact that each extension is unique. Therefore $(\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1, (\,,\,)_{G(n)})$ completes to a unitary representation of $G(m)$. $\square$

Definition 4 *Let $\Pi_u(G)$ be the unitary dual of G. Suppose that $m > n$. Let π be a discrete series representation of $G(m+n)$. We denote the functor from π_1 to the completion of $(\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1, (\,,\,)_{G(n)})$ by IT_π. If $IT_\pi(\pi_1) \neq 0$, $IT_\pi(\pi_1)$ is a unitary representation of $G(m)$. Regarding the zero dimensional representation as a unitary representation, IT_π defines a functor from unitary representations of $G(n)$ to unitary representations of $G(m)$.*

One natural question arises. That is, if π_1 is irreducible, is $IT_\pi(\pi_1)$ irreducible? This is beyond the scope of this pape. In fact, this problem is quite difficult. In general, $IT_\pi(\pi_1)$ is not irreducible. However, for a certain holomorphic discrete series representation π, $IT_\pi(\pi_1)$ will indeed be irreducible. For the time being, it is not clear which discrete series representation π has such a property. This question may be intrinsically related to the cohomology induction (see [kv]).

References

[CHH] M. Cowling, U. Haagerup, R. Howe, "Almost L^2 matrix coefficients" *J. Reine Angew. Math.* Vol 387, 1988, (97-110).

[HE] H. He, "Bounds on Smooth Matrix Coefficients of L^2 spaces, "*Selecta Mathematica* Vol 15 No. 3, 2009 (419-433).

[He00] H. He, "Theta Correspondence I-Semistable Range: Construction and Irreducibility ", *Communications in Contemporary Mathematics* (Vol. 2), (2000), 255-283.

[GGP] H. He, "On the Gan-Gross-Prasad Conjecture for $U(p,q)$," to appear.

[Li1] J.-S. Li, "Theta Lifting for Unitary Representations with Nonzero Cohomology ", *Duke Mathematical Journal*, No. 3 V. 61, (1990), 913-937.

[MI] D. Miličić, "Asymptotic Behaviour of Matrix Coefficients of The Discrete Series ", *Duke Mathematical Journal*, Vol 44, 1977, (59-88).

[kv] A. Knapp, D. Vogan *Cohomological induction and unitary representations.*, Princeton University Press, Princeton, NJ, 1995.

[Se] J.-P. Serre *Linear Representations of Finite Groups*, Springer-Verlag 1977.