to be completed.

LECTURE NOTES

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1. Hamiltonian Mechanics

Let us consider the classical harmonic oscillator

$$m\ddot{x} = -kx \qquad (x \in \mathbb{R}).$$

This is a second order differential equation in terms of Newtonian mechanics. We can convert it into 1st order ordinary differential equations by introducing the momentum

$$p = m\dot{x}; \qquad q = x.$$

Then we have

$$\dot{q} = \frac{p}{m}, \qquad \dot{p} = -kq.$$

The map $(q,p) \to \frac{p}{m}\mathbf{i} - kq\mathbf{j}$ defines a smooth vector field. The flow curve of this vector field gives the time evolution from any initial state $(q(0), p(0))$.

Let $H(q,p) = \frac{p^2}{2m} + \frac{kq^2}{2}$ be the Hamiltonian. It represents the energy function. Then

$$\frac{\partial H}{\partial q} = kq, \qquad \frac{\partial H}{\partial p} = \frac{p}{m}.$$

So we have

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}.$$

Generally speaking, for $p, q \in \mathbb{R}^n$, for $H(q,p)$ the Hamiltonian,

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}.$$

is called the Hamiltonian equation. The vector field

$$(q,p) \to (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q})$$

is called the Hamiltonian vector field, denoted by X_H. In this case, $\{(q,p)\} = \mathbb{R}^{2n}$ is called the phase space. Any (q,p) is called a state. Any functions on (q,p) is called an observable. In particular H is an observable.

Suppose that H is smooth. Then the Hamiltonian equation has local solutions. In other words, for each $(q_0, p_0) \in \mathbb{R}^{2n}$, there exists $\epsilon > 0$ and a function $(q(t), p(t))(t \in [0, \epsilon))$ such that

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t)), \dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t))$$

and $q(0) = q_0, p(0) = p_0$. Put $\phi(q_0, p_0, t) = (q(t), p(t))$. Then the function ϕ describe the time evolution

$$\phi_t : (q_0, p_0) \in \mathbb{R}^{2n} \to (q(t), p(t)).$$

ϕ_t is called the Hamiltonian flow. If ϕ exists for all t, the Hamiltonian system is said to be complete.

Homework: Give an example of a smooth Hamiltonian system that is not complete.

Properties of Hamiltonian system:

(1) The Hamiltonian remains constant under the Hamiltonian flow:

$$\frac{dH(p(t),q(t))}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = -\dot{p}\dot{q} + \dot{q}\dot{p} = 0.$$

When H is the energy function, this says that the energy is preserved.

(2) Hamiltonian vector field is divergence free

$$X_H = (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}).$$

So

$$divX_H = \frac{\partial}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial}{\partial p}\frac{\partial H}{\partial q} = 0.$$

Notice that divergence free means the flow out of a compact closed surface is equal to the flow into the compact closed surface.

Homework: Suppose that H is smooth. Show that Hamiltonian flow preserves the volume. Why?

(3) If the phase space is not the whole $\mathbb{R}^{2n}$, how do we define the Hamiltonian system? This happens when the physical system is under certain constrain, for example q satisfies a system of equations. So we have to define a Hamiltonian system on a manifold, in this course, mostly smooth algebraic varieties.

The main reference is [1].

2. Symplectic Formulation

Now we shall formulate the Hamiltonian system on manifolds. This corresponds to mechanic systems under geometric constraint.

Let V be a real vector space of dimension m. A nondegenerate skew symmetric form on V is called a symplectic form. A linear space equipped with a symplectic form is called a (linear) symplectic space.

Homework: Show that a symplectic space must be of even dimensional.

Let ω be a symplectic form on V and let V^* be the space of real linear functional on V.

(1) ω defines a canonical isomorphism $\tilde{\omega} : V \to V^*$:
$$\tilde{\omega}(x)(y) = \omega(y, x), \qquad y, x \in V.$$
(2) symplectic forms can be regarded as two forms in $\wedge^2(V^*)$.
(3) If V is of $2n$ dimensional, $\omega^n = \omega \wedge \ldots \wedge \omega$ is a top degree form on V.

Let M be a m dimensional smooth manifold. Let TM be the tangent bundle. A smooth section of the tangent bundle
$$x \in M \to T_x M \cong \mathbb{R}^m$$
is called a vector field. Let $\mathcal{X}(M)$ be the space of (smooth) vector fields on M. Let X be a smooth vector field and let $\phi_X(x,t) \in M$ be the local flow defined by X. Let $f \in C^\infty(M)$. Define a function $Xf \in C^\infty(M)$ as follows
$$Xf(x) = \frac{d}{dt}|_{t=0} f(\phi_X(x,t)).$$
Then X becomes a linear first order differential operator on M. $\mathcal{X}(M)$ has a Lie algebra structure
$$[X, Y] = XY - YX.$$
Homework: Show that $[X, Y] \in \mathcal{X}(M)$ for $M = \mathbb{R}^m$.

Definition 2.1. *Let M be a smooth manifold of $2n$ dimension. A symplectic form ω on M is a two form on the TM satisfying the following properties:*

(1) *For each $x \in M$, ω_x is a linear symplectic form on $T_x M$;*
(2) *$\omega : M \to \wedge^2(T^*M)$ is smooth and its exterior derivative vanishes. The exterior derivative is a linear operator from two forms to three forms:*
$$\begin{aligned} d\omega(X_1, X_2, X_3) &= X_1(\omega(X_2, X_3)) + X_2(\omega(X_3, X_1)) + X_3(\omega(X_1, X_2)) \\ &\quad - \omega([X_1, X_2], X_3) - \omega([X_2, X_3], X_1) - \omega([X_3, X_1], X_2), \end{aligned}$$
Where $X_1, X_2, X_3 \in \mathcal{X}(M)$.

The forms satisfies $d\omega = 0$ are called closed forms.

The closedness of ω is what is needed to attach a Poisson structure on $C^\infty(M)$. We shall see this later.

Homework: Give a general definition of d for one forms and show that $d^2 = 0$ for one forms.

Now if $f \in C^\infty(M)$, considered as a zero form, $df(X)$ is defined to be just $X(f)$ for any $X \in \mathcal{X}(M)$. If $M = \mathbb{R}^n$, then
$$df = \sum \frac{\partial f}{\partial x_i} dx_i.$$

So exterior derivative is a generalization of complete differential.

Homework: Write down $d\omega$ for $M = \mathbb{R}^{2n}$.

Notice that what we said about linear symplectic form carry over to the manifold case.

(1) $\tilde{\omega}$ defines an isomorphism between $\mathcal{X}(M)$ and one forms on M, i.e.,

$$\tilde{\omega}(X)(Y) = \omega(Y, X).$$

(2) ω^n defines a nowhere vanishing volume form on M.

2.1. **Hamiltonian Vector Field.** Let $f \in C^\infty(M)$. Then df is a smooth one-form. We consider $\tilde{\omega}^{-1}$ of such a form.

Definition 2.2. *Define X_f be the unique vector field satisfies*

$$\omega(Y, X_f) = df(Y) = Y(f) \qquad Y \in \mathcal{X}(M).$$

ω being nondegenerate guarantees that X_f is unique. X_f is called the Hamiltonian vector field associated with (the Hamiltonian) f.

For $M = \mathbb{R}^{2n}$, then $T_xM \cong M$. Retain (q, p) as coordinates. Then $\frac{\partial}{\partial q}$, $\frac{\partial}{\partial p}$ can be regarded as a basis for T_xM. A tangent vector in T_xM can still be representated by (q, p) in this basis. Define

$$\omega((q, p), (q', p')) = -qp' + pq'.$$

Suppose that $Y = \sum_{i=1}^n \phi_i \frac{\partial}{\partial q_i} + \psi_i \frac{\partial}{\partial p_i}$. Then

$$df(Y) = Y(f) = \sum_{i=1}^n \phi_i \frac{\partial f}{\partial q_i} + \psi_i \frac{\partial f}{\partial p_i}.$$

Since $\omega(Y, X_f) = Y(f)$, it is easy to check that

$$X_f = \sum \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}.$$

Again we obtain our Hamiltonian system

$$(\dot{q}, \dot{p}) = X_H.$$

So for (M, ω) a symplectic manifold, we define Halmiltonian equation to be

$$\dot{x} = X_H(x) \qquad x \in M.$$

2.2. **Poisson bracket on $C^\infty(M)$.** For $f, g \in C^\infty(M)$, define

$$\{f, g\} = \omega(X_f, X_g) = X_f(g) = -X_g(f).$$

Clearly, $\{f, g\} \in C^\infty(M)$.

Lemma 2.1.

$$[X_f, X_g] = X_{\{f,g\}}.$$

Proof: It suffice to show that for any $Y \in \mathcal{X}(M)$,

$$\omega([X_f, X_g], Y) = \omega(X_{\{f,g\}}, Y).$$

Since ω is a closed tow form, we have $0 = d\omega(X_f, X_g, Y)$. So

$$\begin{aligned}
&\omega([X_f, X_g], Y)\\
=&X_f(\omega(X_g, Y)) + X_g(\omega(Y, X_f)) + Y(\omega(X_f, X_g)) - \omega([X_g, Y], X_f) - \omega([Y, X_f], X_g)\\
=&-X_f(Y(g)) + X_g(Y(f)) + Y(X_f(g)) - [X_g, Y](f) - [Y, X_f](g)\\
=&-X_f(Y(g)) + X_g(Y(f)) + Y(X_f(g)) - X_g(Y(f)) + Y(X_g(f)) - Y(X_f(g)) + X_f(Y(g)) \quad (1)\\
=&Y(X_g(f))\\
=&\omega(Y, \{g, f\})\\
=&\omega(\{f, g\}, Y)
\end{aligned}$$

Lemma 2.2. *$\{,\}$ defines a Lie bracket on $C^\infty(M)$ and the canonical map $f \to X_f$ is a Lie algebra homomorphism. The kernel is the space of constant functions.*

2.3. **Poisson Algebra** $(C^\infty(M), \{,\})$.

Definition 2.3. *A Poisson algebra A is an associative algebra equipped with a Poisson bracket $\{,\}$ such that*

(1) *$(A, \{,\})$ is a Lie algebra;*
(2) *The Poisson bracket with any element is a derivation, i.e.,*

$$\{x, yz\} = \{x, y\}z + y\{x, z\}.$$

Lemma 2.3. *Let M be a symplectic manifold. Let $C^\infty(M)$ be equipped with the ordinary multiplication and Poisson bracket $\{,\}$. $C^\infty(M)$ is a Poisson algebra.*

Proof: $\{f, gh\} = X_f(gh) = X_f(g)h + gX_f(h) = \{f, g\}h + g\{f, h\}$. So (2) is proved.

The associative algebra we are interested will be commutative, coincides with "regular functions" in a proper category. Generally speaking, the associative algebra does not have to be commutative.

Lastly, for $M = \mathbb{R}^{2n}$,

$$\{f, g\} = \sum \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}.$$

3. Harmonic Oscillator and Quantum Mechanics

Recall that the Hamiltonian equation for the classical harmonic oscillator is given by

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}.$$

with $H(q,p) = \frac{p^2}{2m} + \frac{kq^2}{2}$. In the classical system, a state is a point in the phase space and the value of an observable is completely determined for each state. In a quantum mechanic system, a state is a projective line in a Hilbert space $\mathcal{H}$. Observables are certain self adjoint operator commuting with the Hamiltonian.

Set the Planck constant to be 1. For the quantum harmonic oscillator, the Hamiltonian is given by

$$\mathbf{H} = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + \frac{k}{2}x^2.$$

It is self adjoint operator. The states are in $\mathbb{P}\mathcal{H}$ with $\mathcal{H} = L^2(\mathbb{R})$. The operator p, q are given by

$$p \to \frac{d}{dx}, \qquad q \to m(x).$$

Here $m(x)$ is the multiplication operator by x. Sometimes, we will write just x.

A state in $\mathbb{P}\mathcal{H}$ can be represented by a unit vector $u \in \mathcal{H}$. One start with an initial ψ_0, a unit vector in $L^2(\mathbb{R})$. The evolution of the Harmonic oscillator is given by the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = \mathbf{H}\psi = -\frac{1}{2m}\frac{\partial^2 \psi}{\partial x^2} + \frac{k}{2}x^2\psi,$$

with $\psi(x,0) = \psi_0(x)$. $\psi(x,t)$ is called the wave function. It is **complex** valued.

Since $i\mathbf{H}$ is skew-self adjoint operator, $\psi(x,t)$ can be written as $e^{-it\mathbf{H}}\psi_0$ where $e^{-it\mathbf{H}}$ is a one parameter group of unitary operators generated by $it\mathbf{H}$. The existence of the one parameter group $e^{-it\mathbf{H}}$ for self-adjoint $\mathbf{H}$ is known as the Stone's theorem. So $\psi(x,t)$ are states for all t. In particular, let $\rho(x,t) = \|\psi(x,t)\|^2$. Then

$$\int_x \rho(x,t)dx = 1.$$

The function $\rho(x,t)$ can be interpreted as the probability density for observing the oscillator at the position x at time t.

If $\mathbf{H}\psi = E\psi$ for $\psi \in \mathbb{R}$, i.e., ψ is an eigenfunction of $\mathbf{H}$, ψ is called a stationary state. ψ is often denoted by ψ_E. In higher dimensional case, where the eigenspace of $\mathbf{H}$ is more than 1 dimensional, a commuting set of self-adjoint operators will be needed to parametrize the stationary states. So E can be a set of parameters.

Now consider the Schrödinger equation

$$i\frac{d}{dt}\psi(x,t) = \mathbf{H}\psi(x,t).$$

We try the separation of variables by letting $\psi(x,t) = \psi_E(x)A_E(t)$. Then we have

$$i\frac{dA(t)}{de}\psi_E(x) = A(t)\mathbf{H}\psi_E(x) = A(t)E\psi_E(t).$$

So $\frac{dA(t)}{dt} = -iEA(t)$. We see that

$$A(t) = e^{-iEt}.$$

So $\psi(x,t) = \psi_E(x)e^{-iEt}$ is a wave function. $\psi(x,t_0)$ differs from $\psi(x,0)$ by a phase factor, which means that in $\mathbb{P}\mathcal{H}$, they are identical. So $\psi_E(x)$ is called stationary.

This method can be used to solve the Schrödinger equation by expanding $\phi_0 = \sum_E a_E \psi_E(x)$. Then

$$\phi(x,t) = \sum_E a_E e^{-iEt} \psi_E(x).$$

3.1. **Observables and Spectral Theory.** The broader definition of an observable is any self-adjoint operator commuting with the Hamiltonian. Two observables A, B are said to be incompatible if $AB \neq BA$. Physically, making the observation A after the measure B will have different results from making the observation B after A.

Given an observable A and a state u, the value of A at u is

$$(Au, u) = (u, Au) = \langle u \mid A \mid u \rangle.$$

Here u is a unit vector. The value of A is independent of the phase factor. Also, if $A = \mathbf{H}$, then the value of $\mathbf{H}$ at ψ_E is exactly the energy.

There is yet a narrower definition of observables by enforcing all observables commuting with each other. See [8].

Discuss projection valued measure here.

4. Formulation of Quantization

Now we go back to the basic question of deriving quantum mechanic system from classical Hamiltonian system, in particular, the Schrödinger equations from Hamiltonian equations. The key question is how one can quantize the classical Hamiltonian

$$H \to \mathbf{H}$$

and how one can quantize the phase space

$$M \to L^2(X).$$

In the harmonic oscillator case, $M = \mathbb{R}^{2n}$, and $X = \mathbb{R}^n$.

It seems that no matter how you formulate the quantization process, you will always encounter inconsistency. See [4] for his formulation of quantization in terms of constraint. In this section, I will give a formulation that applies to affine algebraic varieties with a symplectic structure, which seems to be one focus of [4].

Recall that the classical Hamiltonian $H(q,p) = \frac{p^2}{2m} + kq^2/2$ and the Schrödinger operator $\mathbf{H}(\psi) = -\frac{\mathbf{1}}{\mathbf{2m}}\frac{\partial^2 \psi}{\partial \mathbf{x^2}} + \mathbf{kx^2/2}\psi$. So it seems that the quantization process can be achieved by

$$p \to i\frac{\partial}{\partial x}; \qquad q \to x.$$

This is far from the case. In fact, if $H(q,p) = \frac{p^2}{2m} + kq^2p^2$, one is facing serious problem in choosing $\mathbf{H}$. For example, is

$$\mathbf{H} = -\frac{1}{2m}\frac{\partial^2 \psi}{\partial x^2} - k\frac{\partial^2}{\partial x^2}x^2/2\psi - kx^2/2\frac{\partial^2}{\partial x^2}\psi,$$

or

$$\mathbf{H} = -\frac{1}{2m}\frac{\partial^2 \psi}{\partial x^2} - k\frac{\partial}{\partial x}x^2\frac{\partial}{\partial x}\psi$$

a better choice? Are there canonical choices?

4.1. Weyl Correspondence.

Yes, there are canonical choices:

$$H \in \mathcal{P}(\mathbb{R}^{2n}) \to \mathbf{H} \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)).$$

Here $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decaying functions, namely the topological vector space defined by the seminorm $\|.\|_{\alpha,\beta}$

$$\|f\|_{\alpha,\beta} = \|x^\alpha \partial^\beta f\|_{sup}.$$

Let $\mathcal{S}'$ be the dual of $\mathcal{S}$, namely the space of continuous functionals on $\mathcal{S}$ equipped with the weak star topology. Elements in $\mathcal{S}'$ are called tempered distributions. They behave like generalized functions. Notice that even though differential operators with slowly increasing smooth coefficients are not bounded operators on $L^2(\mathbb{R}^n)$, they are continuous operators in $L(\mathcal{S}, \mathcal{S})$. So the Schwartz space is very convenient in our setting.

One classical result is that the Fourier transform operator $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ and $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ are bijective.

For any slowly increasing continuous function H, Define a kernel

$$K_H(x,y) = \int H(\xi, \frac{1}{2}(x+y)) \exp 2\pi i(x-y)\xi d\xi$$

This is a well-defined tempered distribution. For any $f \in \mathcal{S}$, define

$$W_H f(x) = \int K_H(x,y) f(y) dy = \int\int H(\xi, \frac{1}{2}(x+y)) f(y) \exp 2\pi i (x-y)\xi dy d\xi.$$

It is not hard to check that W_H is a well-defined continuous map from $\mathcal{S}$ to $\mathcal{S}$ for any $H \in \mathcal{S}'(\mathbb{R}^{2n})$. The well-known Schwartz kernel theorem then states that every continuous operator on $\mathcal{S}$ can be written as W_H for some $H \in \mathcal{S}'(\mathbb{R}^{2n})$. The map

$$H \in \mathcal{S}(\mathbb{R}^{2n}) \to W_H \in L(\mathcal{S}, \mathcal{S})$$

is called the Weyl correspondence and the function H is called the Weyl symbol of the operator W_H.

We shall now show that the function $H(\xi, x) = x$ is the symbol for the multiplication operator $m(x)$ and $H(\xi, x) = 2\pi i \xi$ is the symbol for the operator $\frac{d}{dx}$. We will prove it for the one variable case. For every $F \in \mathcal{S}$, we formally compute

$$\begin{aligned} W_H f(x) =& \int\int 2\pi i \xi f(y) \exp 2\pi i (x-y)\xi dy d\xi \\ =& \int\int -f(y) \frac{d \exp 2\pi i (x-y)\xi}{dy} dy d\xi \\ =& \int\int \exp 2\pi i (x-y)\xi \frac{df}{dy} dy d\xi \\ =& \int \delta_x(y) \frac{df}{dy} dy \\ =& \frac{df}{dx} \end{aligned} \tag{2}$$

Notice here we have $\int \exp 2\pi i \xi x d\xi = \delta_x$. Similarly, for $H(\xi, x) = x$, we have

$$\begin{aligned} W_H f(x) =& \int\int H(\xi, \frac{1}{2}(x+y)) f(y) \exp 2\pi i (x-y)\xi \; dy \; d\xi \\ =& \int\int \frac{1}{2}(x+y) f(y) \exp 2\pi i (x-y)\xi \; dy \; d\xi \\ =& \int\int \frac{1}{2}(2x-z) \exp(2\pi i z\xi) f(x-z) \; dz \; d\xi \\ =& \int \frac{1}{2}(2x-z) \int \exp(2\pi i z\xi) \; d\xi f(x-z) \; dz \\ =& \int \frac{1}{2}(2x-z) f(x-z) \delta_z(0) \\ =& x f(x) \end{aligned} \tag{3}$$

All the computation makes sense as distributions when $f \in \mathcal{S}(\mathbb{R}^n)$. Thus, Weyl correspondence establishes a one-to-one correspodence from

$$\mathcal{S}'(\mathbb{R}^{2n}) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^n)),$$

that meets certain requirements from quantization. In addition, there is a Weyl convolution

$$H_1 \sharp H_2(u) = \int\int H_1(v) H_2(w) \exp 4\pi i u(u-w, u-v) dv dw,$$

such that $W_{H_1 \sharp H_2} = W_{H_1} W_{H_2}$.

The problem now is how we can generalize the Weyl correspondence to algebraic symplectic manifold. Let (M^{2n}, ω) be an algebraic symplectic manifold of dimension $2n$. We are seeking a manifold X of dimension n, such that there is a correspondence between a subspace of smooth functions $\mathcal{Q}^{\infty}(M^{2n})$ and a space of formally skew self-adjoint (unbounded) operators on $L^2(X)$ with respect to a certain measure. A good choice of $\mathcal{Q}^{\infty}(M^{2n})$ will be real valued algebraic functions on M^{2n}. One nice feature of these functions is the slowing increasing property. The question is then what condition we shall impose on this correspondence to make it meets the requirement of quantization.

4.2. **Symbol Calculus.** We discuss the symbol calculus on $\mathbb{R}^n$. It applies to any smooth manifold M. Let D be a smooth differential operator on $\mathbb{R}^n$. For each $D = \sum_{\alpha} a_{\alpha} D^{\alpha}$ of order less or equal to m, we define a (principal) symbol

$$\sigma_m(D) = \sum_{\|\alpha\|=m} a_{\alpha}(x)\xi^{\alpha}.$$

If no ambiguity occurs, we will write the symbol map as σ.

Homework Show that σ_m here is linear. But σ is not.

When M is a manifold, one can choose a coordinate atlas $\{U_{\alpha}\}$. Then on each coordinate patch U_{α}, $\sigma(D)$ will be a function on the cotangent bundle $T^*(U_{\alpha})$ with local coordinate (x, ξ). So $\sigma(D)$ will be a function on $T * M$. Over each cotangtant space, $\sigma(D)$ is a polynomial of degree equal to the order of the differential operator.

Homework Show that the principal symbol σ is independent of the coordinate system.

Let $\mathcal{D}_{\leq m}$ be the linear space of smooth differential operators of order less or equal to m. If $D_1 \in \mathcal{D}_{\leq m_1}$ and $D_2 \in \mathcal{D}_{\leq m_2}$, then $D_1D_2, D_2D_1 \in \mathcal{D}_{\leq m_1+m_2}$. In this case,

$$\sigma_{m_1+m_2}(D_1D_2) = \sigma_{m_1+m_2}(D_2D_1) = \sigma_{m_1}(D_1)\sigma_{m_2}(D_2).$$

This is the naive way of quantization. But it does not address the noncommutativity we encounter in the earlier example. To address the noncommutativity, we consider the Lie bracket $[D_1, D_2]$.

Homework: Show that $[D_1, D_2] \in \mathcal{D}_{\leq m_1+m_2-1}$.

Definition 4.1. *Let $\mathcal{P}_{\leq m}(x, \xi)$ be smooth functions on (x, ξ) that are polynomials of degree less or equal to m for ξ. Let $\mathcal{P}_m(x, \xi)$ be those of degree m for ξ. Let*

$$\mathcal{P}(x, \xi) = \cup \mathcal{P}_{\leq m}(x, \xi) = \oplus \mathcal{P}_m(x, \xi).$$

Define a Lie bracket on $\mathcal{P}(x, \xi)$ as follows. Let $D_1, D_2 \in \mathcal{D}_{\leq m_1}, \mathcal{D}_{\leq m_2}$. Define

$$\{\sigma_{m_1}(D_1), \sigma_{m_2}(D_2)\} = \sigma_{m_1+m_2-1}([D_1, D_2]).$$

Theorem 4.1. *$\{,\}$ is well-defined. $(\mathcal{P}(x, \xi), \{,\})$ is a Poisson algebra under the ordinary multiplication.*

Proof: Let $D_1 \in \mathcal{D}_{m_1}$, $D_2 \in \mathcal{D}_{m_2}$ and $D_3 \in \mathcal{D}_{m_3}$. It suffices to show that the anti-commutativity, Jacobian identity and derivation requirement hold for $\sigma_{m_1}(D_1)$, $\sigma_{m_2}(D_2)$ and $\sigma_{m_3}(D_3)$.

(1) $\{\sigma_{m_1}(D_1), \sigma_{m_2}(D_2)\} = \sigma_{m_1+m_2-1}([D_1, D_2]) = -\sigma_{m_1+m_2-1}([D_2, D_1]) = -\{\sigma_{m_2}(D_2), \sigma_{m_1}(D_1), \}$.

(2) (Jacobian Identity)

(4)
$$
\begin{aligned}
&\{\sigma_{m_1}(D_1),\{\sigma_{m_2}(D_2),\sigma_{m_3}(D_3)\}\}+\{\sigma_{m_2}(D_2),\{\sigma_{m_3}(D_3),\sigma_{m_1}(D_1)\}\}+\{\sigma_{m_3}(D_3),\{\sigma_{m_1}(D_1),\sigma_{m_2}(D_2)\}\}\\
=&\{\sigma_{m_1}(D_1),\sigma_{m_2+m_3-1}([D_2,D_3])\}+\{\sigma_{m_2}(D_2),\sigma_{m_1+m_3-1}([D_3,D_1])\}+\{\sigma_{m_3}(D_3),\sigma_{m_2+m_1-1}([D_1,D_2])\}\\
=&\sigma_{m_1+m_2+m_3-2}([D_1,[D_2,D_3]])+\sigma_{m_1+m_2+m_3-2}([D_2,[D_3,D_1]])+\sigma_{m_1+m_2+m_3-2}([D_3,[D_1,D_2]])\\
=&\sigma_{m_1+m_2+m_3-2}([D_1,[D_2,D_3]]+[D_2,[D_3,D_1]]+[D_3,[D_1,D_2]])\\
=&0
\end{aligned}
$$

(3) (Derivation)

$$
\begin{aligned}
&\{\sigma_{m_1}(D_1),\sigma_{m_2}(D_2)\sigma_{m_3}(D_3)\}\\
=&\{\sigma_{m_1}(D_1),\sigma_{m_2+m_3}(D_2D_3)\}\\
=&\sigma_{m_1+m_2+m_3-1}([D_1,D_2D_3])\\
=&\sigma_{m_1+m_2+m_3-1}(D_1D_2D_3-D_2D_3D_1)\\
=&\sigma_{m_1+m_2+m_3-1}([D_1,D_2]D_3+D_2[D_1,D_3])\\
=&\sigma_{m_1+m-2-1}([D_1,D_2])\sigma_{m_3}(D_3)+\sigma_{m_2}(D_2)\sigma_{m_1+m_3-1}([D_1,D_3])\\
=&\{\sigma_{m_1}(D_1),\sigma_{m_2}(D_2)\}\sigma_{m_3}(D_3)+\sigma_{m_2}(D_2)\{\sigma_{m_1}(D_1),\sigma_{m_3}(D_3)\}.
\end{aligned}
\tag{5}
$$

Notice that $T^*(\mathbb{R}^n)$ has a cannonical symplectic form. Identifty the tangent space $T_a(\mathbb{R}^n)$ with $x \in \mathbb{R}^n$. Then $\xi \in T_a^*(\mathbb{R}^n)$ can be regarded as a function on x. Define

$$\omega((x,\xi),(x',\xi')) = \xi'(x) - \xi(x').$$

Then $\mathcal{P}(x,\xi)$ has a natural Poisson structure induced from $(T^*(\mathbb{R}^n),\omega)$. The Poisson bracket is simply

$$\{f,g\} = \sum \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i}.$$

Theorem 4.2. *The two different Poisson structures on $\mathcal{P}(x,\xi)$ are identical.*

Proof: We shall prove this for $n = 1$. By linearlity, it suffice to prove that the two Poission struction for $a(x)\xi^i$ and $b(x)\xi^j$. If $i = j = 0$, then $\{a(x), b(x)\} = 0$ for both Poission structures. Without loss generality suppose $i + j \geq 1$. For the Poisson structure induced from the symplectic form, we have

$$\{a(x)\xi^i, b(x)\xi^j\} = (ia(x)b'(x) - ja'(x)b(x)\xi^{i+j-1}.$$

Now let us compute the Poisson structure induced from symbol calculus. Let $D_1 = a(x)\frac{d^i}{dx^i}$ and $D(2) = b(x)\frac{d^j}{dx^j}$. Then

$$
\begin{aligned}
&[D_1,D_2]f\\
=&a(x)\frac{d^i}{dx^i}(b(x)\frac{d^jf}{dx^j})-b(x)\frac{d^j}{dx^j}(a(x)\frac{d^if}{dx^i})\\
=&a(x)\sum_{k=0}^{i}\binom{i}{k}\frac{d^kb(x)}{dx^k}\frac{d^{i-k+j}f}{dx^{i-k+j}}-b(x)\sum_{k=0}^{j}\binom{j}{k}\frac{d^ka(x)}{dx^k}\frac{d^{j-k+i}f}{dx^{j-k+1}}
\end{aligned}
\tag{6}
$$

So we have

$$\{a(x)\xi^i, b(x)\xi^j\} = \sigma_{i+j-1}([D_1,D_2]) = (ia(x)b'(x) - ja'(x)b(x)(\xi^{i+j-1}).$$

□

Roughly speaking, this allows us to quantize the Hamiltonians in a way that preserves the principal part of the commutant. Similarly, this process can be viewed as quantization of Hamiltonian vector field, which is closely related to representation theory.

I shall mention that there are quantization processes that are based on forms or bundles. Our discussion shall be modified to fit these situations.

4.3. **The Two Poisson algebras: Manifold Case.** I shall state the result in the manifold case for the two Poisson algebras: one induced from the canonical symplectic form; the other induced from the principal symbols.

4.4. **Quantization of Affine Symplectic Varieties.** The quantization process for $\mathbb{R}^{2n}$ has been studied for many years. Our formulation offers nothing new in this case. It becomes more interesting when M is an affine symplectic variety. Much less is known. This is the quantization under geometric constrain.

Let M be an affine variety. Let $\mathcal{P}(T^*(M))$ be the algebraic functions on T^*M that are polynomials restricted to every cotangent space. Symbol maps induces a Poisson structure on $\mathcal{P}(T^*(M))$.

Let us now state the problem. Let X be a (real) smooth affine variety, equipped with a symplectic form ω. Smoothness in not necessary because one can always choose a smooth open manifold. We require that the symplectic form evaluated on algebraic vector fields gives algebraic functions. This is often the case. Now the regular functions on X become a Poisson algebra under $\{,\}$. **The problem** is to find a real affine algebraic variety X and to construct a map

$$\mathcal{O}(X) \to \mathcal{P}(T^*M)$$

such that the Poisson structure is preserved. In addition, to make the choice unique, one often requires that the map preserves certain group action.

5. Example I: Harmonic Oscillator Revisited

I shall discuss how one arrives at the quantization of harmonic oscillator in light of our discussion in the last lecture.

5.1. **Schrödinger Model.**

5.2. **Segal-Bargmann Model.**

5.3. **Tensor Products.**

6. Example II: Kobayashi-Orsted Quantization

This is another example that

7. Open Problems

7.1. **Kostant-Sekiguchi-Vergne correspondence.** There are certain homogeneous symplectic manifolds that are also complex homogeneous.

7.2. **Real Analytic Method.**

7.3. **Complex Analytic Method.**

7.4. **Bergman Reproducing Kernel for algebraic varieties.**

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