Coadjoint Orbits of Siegel Parabolic Subgroups

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Let $P_+(n)$ be the Siegel parabolic subgroup of $O(n,n)$, and $P_-(n)$ be the Siegel parabolic subgroup of $Sp_{2n}(\mathbb{R})$. In this paper, we study the coadjoint orbits of $P_\pm(n)$. We establish a one-to-one correspondence between the real coadjoint orbits of $Sp_{2n}(\mathbb{R})$ and the principal coadjoint orbits of $P_+(2n)$, and a one-to-one correspondence between the real coadjoint orbits of $O(p, n-p)$ with $p \in [0,n]$ and the principal real coadjoint orbits of $P_-(n)$.

1 Introduction

Let G be a Lie group. Let $\mathfrak{g}$ be the real Lie algebra of G. Let Ad be the adjoint action of G on $\mathfrak{g}$. Each G-orbit in $\mathfrak{g}$ is called an adjoint orbit. Let $\mathfrak{g}^*$ be the space of real homomorphisms from $\mathfrak{g}$ to $\mathbb{R}$. $\forall\, g \in G, \phi \in \mathfrak{g}^*, x \in \mathfrak{g}$, define

$$(Ad^*(g)\phi)(x) = \phi(Ad(g^{-1})x).$$

This action is often called the coadjoint action. Each G-orbit is called a coadjoint orbit.

The motivation for classifying coadjoint orbits comes from two directions. Firstly, adjoint orbits generalize the notion of Jordan canonical forms. Notice for $G = GL_n(\mathbb{R})$, $\mathfrak{g}$ is the space of $n \times n$ matrices. The adjoint orbits of $GL(n,\mathbb{R})$ are in one-to-one correspondence with Jordan forms. Furthermore, one can identify adjoint orbits with coadjoint orbits by the trace form. In other words, for each $x \in \mathfrak{g}$, define a functional δ_x by

$$\delta_x : y \in \mathfrak{g} \to Trace(xy).$$

Then adjoint orbits become coadjoint orbits. It follows that coadjoint orbits are in one-to-one correspondence with Jordan forms. This is true for all semisimple Lie groups.

Secondly, coadjoint orbits have a profound connection with the representation theory of Lie groups. In [Kirillov], Kirillov proved that, for simply connected nilpotent groups, equivalence classes of irreducible unitary representations are in one to one correspondence with coadjoint orbits of G. Later, Auslander and Kostant extended Kirillov's result to solvable groups of type I. For semisimple groups, the structure of coadjoint orbits provides a nice channel to the construction and classification of unitary representations (see [Vogan]). In all cases, classification of coadjoint orbits is a very important problem in representation theory.

Let G be a semisimple Lie group. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form. Then adjoint orbits can be identified with coadjoint orbits. The classification of adjoint orbits for semisimple groups are known (see for example [Steinberg], [Collingwood-McGovern]). In this paper, we are interested in the classification of a selected class of coadjoint orbits of the Siegel parabolic subgroups.

All matrices in this paper are real matrices. Let

$$P_+(n) = \{X = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mid X \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix} X^t = \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix}\}$$

be the Siegel parabolic subgroup of $O(n,n)$. Let

$$P_-(n) = \{X = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mid X \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} X^t = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}\}$$

be the Siegel parabolic subgroup of $Sp_{2n}(\mathbb{R})$. An element $\eta \in \mathfrak{p}_{\pm}(n)^*$ is said to be principal if η restricted to the upper diagonal part B is nondegenerate. We say that $\mathcal{O}$ is a *principal coadjoint orbit* of $P_{\pm}(n)$ if one of its element is principal. In this paper, we prove the following theorem:

Theorem 1.1 *There exists a one-to-one correspondence j between coadjoint orbits of $Sp_{2n}(\mathbb{R})$ and principle coadjoint orbits of $P_+(2n)$. There exists a one-to-one correspondence between coadjoint orbits of $O(p, n-p)$ with $p \in [0, \frac{n}{2}]$ and principal coadjoint orbits of $P_-(n)$.*

The orbital correspondence in this paper is an analogy of the orbital induction (see [Richardson], [Kem]). It is parallel to Mackey's induction on the representation level as explored by Howe (see [Howe]). Throughout this paper, the group action will mostly be the matrix multiplication. If a group G acts on a set X and $x \in X$, we use G_x to denote the subgroup preserving x. We use $M(n)$ to denote the space of $n \times n$ real matrices, $\mathcal{A}(n)$ to denote the space of $n \times n$ antisymmetric matrices and $\mathcal{S}(n)$ to denote the space of $n \times n$ symmetric matrices. Unless stated otherwise, all our vectors will be column vectors.

2 Orbital Correspondence for $P_+(2n)$ and $Sp_{2n}(\mathbb{R})$

Let $S_{2n,2n} = \begin{pmatrix} 0_{2n} & I_{2n} \\ I_{2n} & 0_{2n} \end{pmatrix}$. We define a real symmetric form

$$(x, y) = x^t S_{2n,2n} y \qquad (x, y \in \mathbb{R}^{4n}).$$

Let $O_{2n,2n}$ be the isometric group fixing the symmetric form $(,)$. The Siegel parabolic subgroup $P_+(2n)$ of $O(2n,2n)$ will be denoted by P_+ for simplicity. Then the Lie algebra

$$\mathfrak{p}_+ = \{\begin{pmatrix} X & Y \\ 0 & -X^t \end{pmatrix} \mid X, Y \in M(2n); Y^t = -Y\}.$$

Let

$$N = \{\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \mid B^t = -B, B \in M(2n)\}.$$

Then P admits a Levi decomposition $P = GL(2n)N$. Here the Levi factor $GL(2n)$ is simply

$$\{\begin{pmatrix} X & 0 \\ 0 & (X^{-1})^t \end{pmatrix} \mid X \in GL(2n)\}.$$

We have the following exact sequence

$$1 \rightarrow N \rightarrow P_+ \rightarrow GL(2n) \rightarrow 1.$$

Lemma 2.1 *The matrix* $\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix}$ *is in* P_+ *if and only if* AB^t *is skew-symmetric, i.e.,* $BA^t = -AB^t$. *We have*

$$\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & B^t \\ 0 & A^t \end{pmatrix}$$

Proof: The lemma follows from the following computation.

$$\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix} S_{n,n} \begin{pmatrix} A^t & 0 \\ B^t & A^{-1} \end{pmatrix} = \begin{pmatrix} BA^t + AB^t & I \\ I & 0 \end{pmatrix} \tag{1}$$

Q.E.D.

Therefore,

$$P_+ = \{\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix} \mid A \in GL(2n), B \in M(2n), BA^t = -AB^t\}.$$

We parameterize P_+ by a pair (A, B) such that $BA^t = -AB^t$ and parameterize $\mathfrak{p}_+$ by a pair (U, V) with $U \in M(2n)$ and $V \in \mathcal{A}(2n)$. $\mathfrak{p}_+ = \mathfrak{gl}(2n) \oplus \mathcal{A}(2n)$.

Consider the Siegel parabolic subalgebra $\mathfrak{p}_+$. Every element in the dual of $\mathfrak{p}_+$ can be represented by a matrix through the trace form:

$$\begin{pmatrix} X & [*] \\ Y & -X^t \end{pmatrix} : \begin{pmatrix} U & V \\ 0 & -U^t \end{pmatrix} \rightarrow Tr(XU + YV + X^tU^t) \qquad (X, Y \in M(2n), Y^t = -Y).$$

Notice that changes of $[*]$ do not effect the linear functional it represents. We compute the action of P_+ on $\mathfrak{p}_+^*$ as follows.

$$\begin{aligned} & \begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix} \begin{pmatrix} X & [*] \\ Y & -X^t \end{pmatrix} \begin{pmatrix} A^{-1} & B^t \\ 0 & A^t \end{pmatrix} \\ = & \begin{pmatrix} AX + BY & A[*] - BX^t \\ (A^t)^{-1}Y & -(A^{-1})^t X^t \end{pmatrix} \begin{pmatrix} A^{-1} & B^t \\ 0 & A^t \end{pmatrix} \\ = & \begin{pmatrix} AXA^{-1} + BYA^{-1} & [*] \\ ((A^{-1})^t Y A^{-1}) & (A^{-1})^t Y B^t - (A^{-1})^t X^t A^t \end{pmatrix} \end{aligned} \tag{2}$$

Therefore, when we represent $\begin{pmatrix} X & [*] \\ Y & -X^t \end{pmatrix}$ by a pair of $2n \times 2n$ matrices (X, Y) such that $Y \in \mathcal{A}(2n)$, and represent $\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix} \in P$ by a pair of $2n \times 2n$ matrices (A, B) such that BA^t is antisymmetric, then the coadjoint action is given by

$$Ad(A, B)(X, Y) = (AXA^{-1} + BYA^{-1}, (A^{-1})^t Y A^{-1}).$$

Clearly $(A^{-1})^t Y A^{-1}$ is antisymmetric as well.

Since $rank(Y)$ is fixed by the action of A, we can define the **rank** of a coadjoint orbit $\mathcal{O} = Ad(P_+)(X, Y)$ to be $rank(Y)$. We say that $\mathcal{O}$ is a **principal orbit** if $rank(Y) = 2n$. Now let

$$W = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

Recall that

$$\mathfrak{sp}_{2n} = \{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A, B, C \in M(n), B^t = B, C^t = C \}.$$

Equip $\mathfrak{gl}(2n)$ with trace form $(X, Y) = TrXY^t$. Then $\mathfrak{gl}(2n)$ can be decomposed as a direct sum

$$\mathfrak{sp}_{2n}(\mathbb{R}) \oplus \{ \begin{pmatrix} A & B \\ C & A^t \end{pmatrix} \mid (A \in M(n), B, C \in \mathcal{A}(n)) \}.$$

Since $\mathfrak{p}_+ = \mathfrak{gl}(2n) \oplus \mathcal{A}(2n)$, the embedding of $\mathfrak{s}p_{2n}$ to $\mathfrak{gl}(2n)$ induced an embedding of $\mathfrak{s}p_{2n}(\mathbb{R})$ to $\mathfrak{p}_+$. On the group level, $g \in Sp_{2n}(\mathbb{R})$ is embedded into P_+ as $(g, 0)$.

Theorem 2.1 *The map*

$$j : Ad(Sp_{2n}(\mathbb{R}))X \to Ad(P_+)(X, W) \qquad (X \in \mathfrak{sp}_{2n}(\mathbb{R}))$$

defines a one-to-one correspondence between the real coadjoint orbits of $Sp_{2n}(\mathbb{R})$ and the real principal orbits of $P_+(2n)$. Furthermore, $j(Ad(Sp_{2n}(\mathbb{R}))X) \cong P_+(2n)/(Sp_{2n}(\mathbb{R}))_X$.

Proof: Fix an arbitrary orbit $\mathcal{O} = Ad(Sp_{2n}(\mathbb{R}))X$. First, the map j does not depend on the choices of X. Notice that $\forall \;\; g \in Sp_{2n}(\mathbb{R})$,

$$Ad(g,0)(X,W) = (gXg^{-1}, (g^{-1})^t W g^{-1}) = (Ad(g)X, W).$$

By Lemma 2.1, $(g,0) \in P_+$. Therefore

$$Ad(P_+)(X,W) = Ad(P_+)(Ad(g,0)(X,W)) = Ad(P_+)(Ad(g)X,W).$$

So choosing $Ad(g)X$ instead of X will not change the image of j. Therefore j is a well-defined map from coadjoint orbits of $Sp_{2n}(\mathbb{R})$ to coadjoint orbits of P_+.

Secondly, j is one-to-one. Suppose that

$$j(Ad(Sp_{2n}(\mathbb{R}))(X)) = j(Ad(Sp_{2n}(\mathbb{R})(X'))).$$

Then there exists $(A,B) \in P_+$ such that $Ad(A,B)(X,W) = (X',W) \in \mathfrak{p}^*$. In other words,

$$(AXA^{-1} + BWA^{-1}, (A^{-1})^t W A^{-1}) = (X', W).$$

It follows that $(A^{-1})^t W A^{-1} = W$. So $A \in Sp_{2n}(\mathbb{R})$. We obtain

$$X' = AXA^{-1} + BWA^{-1} = AXA^{-1} + BA^t W.$$

By Lemma 2.1, BA^t is skew-symmetric. So BA^tW is of the form

$$\begin{pmatrix} M & N \\ L & M^t \end{pmatrix} \qquad (M, N, L \in M(n), N^t = -N, L^t = -L)$$

which is perpendicular to $\mathfrak{sp}_{2n}(\mathbb{R})$ with respect to the trace form. Since $X' \in \mathfrak{s}p_{2n}(\mathbb{R})$, we must have

$$BA^t W = 0, \qquad AXA^{-1} = X'.$$

This shows that X and X' is on the same Sp_{2n}-orbit and $B = 0$. Furthermore, if we take $X' = X$, we see that the isotropy subgroup of P_+ fixing (X,W) is equal to $(Sp_{2n}(\mathbb{R}))_X$. This implies that

$$j(\mathcal{O}_X) \cong P/(Sp_{2n}(\mathbb{R}))_X.$$

Here $(Sp_{2n}(\mathbb{R}))_X$ is embedded in P_+ as $((Sp_{2n}(\mathbb{R}))_X, 0)$.

Lastly, j is also onto. For any principal orbit $Ad(P_+)(X,Y)$, Y must be skew-symmetric and of rank $2n$. We can choose a $A \in GL(2n)$ such that $(A^{-1})^t Y A^{-1} = W$. By Equation 2, we may assume $Y = W$. Now by the same equation, if we take $A = I_{2n} \in Sp_{2n}(\mathbb{R})$ and B skew symmetric, we have $(I_{2n}, B) \in P_+$ and

$$Ad(I_{2n}, B)(X,W) = (X + BW, W).$$

Notice that BW is of the form

$$\begin{pmatrix} M & N \\ L & M^t \end{pmatrix} \qquad (L, M, N \in M(n), L^t = -L, N^t = -N).$$

We can choose a B such that

$$X + BW \in \mathfrak{sp}_{2n}.$$

This shows that in any principal orbit $Ad(P_+)(X,Y)$, there is an element (X', W) such that $X' \in \mathfrak{sp}_{2n}$. So j is onto.

Therefore j defines a one-to-one correspondence between coadjoint orbit of $Sp_{2n}(\mathbb{R})$ and principal coadjoint orbit of P_+. Q.E.D.

We can now compute the dimension of $j(\mathcal{O}_X)$. Notice that

$$\dim(\mathcal{O}_X) = \dim(Sp_{2n}(\mathbb{R})) - \dim((Sp_{2n}(\mathbb{R}))_X) = 2n^2 + n - \dim((Sp_{2n}(\mathbb{R}))_X)$$

$$\dim P_+(2n) = \dim(GL(2n)) + \dim(\mathcal{A}_{2n}) = 4n^2 + 2n^2 - n = 6n^2 - n.$$

Therefore, we have

$$\dim(j(\mathcal{O}_X)) = \dim(P_+(2n)) - \dim((Sp_{2n}(\mathbb{R}))_X) = 4n^2 - 2n + \dim(\mathcal{O}_X).$$

Theorem 2.2 *We have*

$$\dim(j(\mathcal{O}_X)) = 4n^2 - 2n + \dim(\mathcal{O}_X).$$

3 Orbital Correspondence for $O(p,q)$ and $P_-(p+q)$

Let $A_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. In this section, we define $O(p,q)$ to be the group preserving the quadratic form defined by $A_{p,q}$. Put $n = p + q$. Let

$$P_- = \{X = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mid X \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} X^t = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}\}$$

be the Siegel parabolic subgroup of $Sp_{2n}(\mathbb{R})$. The Siegel parabolic subalgebra

$$\mathfrak{p}_- = \{\begin{pmatrix} X & B \\ 0 & -X^t \end{pmatrix} \mid X \in \mathfrak{gl}(n), B \in \mathcal{S}(n)\}.$$

Lemma 3.1 *The matrix* $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in P_-$ *if and only if* $AC^t = I$ *and* $BA^t = AB^t$*. We have*

$$\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -B^t \\ 0 & A^t \end{pmatrix}$$

Proof: The lemma follows from the following computation:

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} W \begin{pmatrix} A^t & 0 \\ B^t & C^t \end{pmatrix} = \begin{pmatrix} -BA^t + AB^t & AC^t \\ -CA^t & 0 \end{pmatrix} = W. \tag{3}$$

Q.E.D.

Therefore,

$$P_- = \{ \begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix} \mid A \in GL(n), BA^t = AB^t \}.$$

Parameterize the matrix $\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix} \in P_-$ by a pair of $n \times n$ matrices (A, B) with AB^t symmetric.

The trace form on $\mathfrak{sp}_{2n}(\mathbb{R})$

$$\kappa(X, Y) = Tr(XY)$$

is nondegenerate. It identifies the dual of $\mathfrak{sp}_{2n}(\mathbb{R})$ with $\mathfrak{sp}_{2n}(\mathbb{R})$. Thus, an element in $\mathfrak{p}_-^*$ can be identified with

$$\begin{pmatrix} X & * \\ Y & -X^t \end{pmatrix} \qquad (X \in M(n), Y \in \mathcal{S}(n)).$$

Using (X, Y) to parameterize $\mathfrak{p}_-^*$, we may compute the action of $(A, B) \in P_-$ on $(X, Y) \in \mathfrak{p}_-^*$ as in Equation 2. We obtain

$$Ad(A, B)(X, Y) = (AXA^{-1} + BYA^{-1}, (A^{-1})^t Y A^{-1}). \tag{4}$$

Since $rank(Y)$ is fixed by the action of (A, B), we define the **rank** of a coadjoint orbit $\mathcal{O} = Ad(P_-)(X, Y)$ to be $rank(Y)$. We say that the orbit $Ad(P_-)(X, Y))$ is a **principal orbit** if $rank(Y)$ is n. Notice that $\mathfrak{gl}(n)$ can be decomposed as the direct sum

$$\mathfrak{o}(p, n-p) \oplus \{ \left(\begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \right) \mid A \in \mathcal{S}(p), C \in \mathcal{S}(n-p) \}.$$

The embedding of $\mathfrak{o}(p, n-p)$ in $\mathfrak{g}l(n)$ induces an embedding of $\mathfrak{o}(p, q)$ into $\mathfrak{p}_-$. The group $O(p, n-p)$ becomes a subgroup of P_-, precisely in the form $(O(p, n-p), 0)$. We have the following theorem.

Theorem 3.1 *The real adjoint orbits of $O(p, n-p)$ for all $p \in [0, n]$ are in one-to-one correspondence with the real principal coadjoint orbits of $P_-(n)$. The correspondence is given by*

$$j_p : Ad(O(p, n-p))X \to Ad(P_-)(X, A_{p,n-p}) \qquad (X \in \mathfrak{o}(p, n-p)).$$

Furthermore, $j_p(\mathcal{O}_X) \cong P_-/(O(p, n-p))_X$.

Proof: Let $0 \leq p \leq n$. First of all, $O(p, n-p)$ is a subgroup of P_-. Let $X \in \mathfrak{o}(p, n-p)$. For every $g \in O(p, n-p)$, we have

$$Ad(g, 0)(X, A_{p,n-p}) = (Ad(g)X, A_{p,n-p}).$$

So $Ad(P_-)(X, A_{p,n-p}) = Ad(P_-)(Ad(g)X, A_{p,n-p})$. Therefore, $j_p(Ad(O(p, n-p))X)$ is unique.

Let us show that j_p is injective. By Equation 4, the action of P_- on Y does not change the signature of the symmetric matrix Y. So the images of j_p for different $O(p, n-p)$'s are distinct. Suppose that $Ad(A, B)(X, A_{p,n-p}) = (X', A_{p,n-p}) \in \mathfrak{p}_-^*$ and $X, X' \in \mathfrak{o}(p, q)$. Then $(A^{-1})^t A_{p,n-p} A^{-1} = A_{p,n-p}$ implies that $A \in O(p, n-p)$. Hence we have

$$X' = AXA^{-1} + BA_{p,n-p}A^{-1} = AXA^{-1} + BA^t A_{p,n-p}.$$

Since, BA^t is symmetric, $BA^t A_{p,n-p}$ are perpendicular to $\mathfrak{o}(p, q)$ under the Trace form. Therefore we have

$$X' = AXA^{-1}, \qquad BA^t A_{p,n-p} = 0.$$

This shows that X and X' is on the same $O(p, q)$-orbit and j_p is injective. Furthermore, if we take $X = X'$, we see that the isotropy group fixing $(X, A_{p,n-p})$ is equal to

$$\{(A, 0) \mid A \in O(p, n-p)_X\}$$

Thus $j_p(\mathcal{O}_X) \cong P_-(n)/O(p, n-p)_X$.

On the other hand, the disjoint union of j_p is also surjective. Suppose that $Ad(P)(X, Y)$ is a principal orbit. By definition, $rank(Y) = n$. There exists a $A \in GL(n)$ such that $(A^{-1})^t Y A^{-1} = A_{p,n-p}$. Therefore we may assume $Y = A_{p,n-p}$. Then for every $B \in \mathcal{S}(n)$

$$Ad(I_n, B)(X, A_{p,n-p}) = (X + BA_{p,n-p}, A_{p,n-p}).$$

Since $\mathfrak{gl}(n) = \mathfrak{o}(p, n-p) \oplus \mathcal{S}(n)A_{p,n-p}$, we may choose proper B such that $X + BA_{p,n-p} \in \mathfrak{o}(p, n-p)$. This shows that in any principal orbit $Ad(P_-)(X, Y)$, there exists an element $(X', A_{p,n-p})$ such that $X' \in \mathfrak{o}(p, n-p)$. Then

$$Ad(P_-)(X, Y) = j_p(Ad(O(p, n-p)X')).$$

Q.E.D.

Now we can compute the dimension of $j(O_X)$. Notice that

$$\dim(\mathcal{O}_X) = \dim(O(p, n-p)) - \dim(O(p, q)_X) = \frac{n^2 - n}{2} - \dim(O(p, q)_X)$$

$$\dim(P_-(n)) = \dim(GL(n)) + \dim(\mathcal{S}(n)) = n^2 + \frac{n^2 + n}{2}$$

Therefore we have

$$\dim(j(\mathcal{O}_X)) = \dim(P_-(n)) - \dim(O(p, q)_X) = n^2 + n + \dim(\mathcal{O}_X)$$

Theorem 3.2 *We have*

$$\dim(j(\mathcal{O}_X)) = n^2 + n + \dim(\mathcal{O}_X).$$

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