[1] [2] [3]

ON MATRIX VALUED SQUARE INTEGRABLE POSITIVE DEFINITE FUNCTIONS

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Abstract. In this paper, we study matrix valued positive definite functions on a unimodular group. We generalize two important results of Godement on L^2 positive definite functions. We show that a matrix-valued *continuous* L^2 positive definite function can always be written as the convolution of an matrix-valued L^2 positive definite function with itself. We also prove that, given two L^2 matrix valued positive definite functions Φ and Ψ, $\int_G Tr(\Phi(g)\overline{\Psi(g)}^t)dg \geq 0$. In addition this integral equals zero if and only if $\Phi * \Psi = 0$. Our proofs are operator-theoretic and independent of the group.

1. Introduction

About 60 years ago, Godement published a paper on square integrable positive definite functions on a locally compact group ([1]). In his paper, Godement proved that every *continuous* square integrable positive definite function has an L^2-positive definite square root. He also proved, among others, that the inner product between two positive definite L^2-functions must be nonnegative. Godement's results and proofs were quite elegant. The purpose of this paper is to extend Godement's theorem to matrix-valued positive definite functions on unimodular groups. Obviously, the diagonal of matrix-valued positive definite functions must all be positive definite. Yet, there is not much to say about the off-diagonal entries and their relationship with diagonal entries. So Godement's results do not carry easily to the matrix-valued case. In this paper, we generalize Godement's theorems to matrix-valued positive definite functions ([1] and Ch 13.[2]). Our results, we believe, are new.

Let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices. For a matrix A, let $[A]_{ij}$ be the (i,j)-th entry of A. Let G be a unimodular group. A continuous function $\Phi : G \to M_n(\mathbb{C})$ is said to be *positive definite* if for any $\{\mathbf{C}_i \in \mathbb{C}^n\}_{i=1}^l$ and $\{x_i \in G\}_{i=1}^l$,

$$\sum_{i,j=1}^{l} (\mathbf{C}_i)^t \Phi(x_i^{-1}x_j)\overline{\mathbf{C}_j} \geq 0.$$

Take $x_1 = e$ and $x_2 = g$. The above inequality implies that $\Phi(g) = \overline{\Phi(g^{-1})}^t$ (See for example, Prop. 2.4.6 [7]). When $n = 1$, our definition agrees with the definition of continuous positive definite functions. We denote the set of continuous matrix-valued positive definite functions by $\mathcal{P}(G, M_n)$.

Definition 1.1. *Let $L^1_{loc}(G, M_n)$ be the set of M_n-valued locally integrable functions on G. Let $\Phi \in L^1_{loc}(G, M_n)$ act on $u \in C_c(G, \mathbb{C}^n)$ by $[\lambda(\Phi)(u)(x)]_i = \sum_{j=1}^n \int [\Phi(g)]_{ij}[u(g^{-1}x)]_j dg$. We write $\lambda(\Phi)(u)(x) =$*

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$\int \Phi(g)u(g^{-1}x)dg$. Clearly, $\lambda(\Phi)u$ is continuous. We say that Φ is positive definite if

$$\langle \lambda(\Phi)(u), u \rangle = \sum_{i=1}^{n} \int_G [\lambda(\Phi)u(g)]_i \overline{[u(g)]_i} dg \geq 0$$

for all $u \in C_c(G, \mathbb{C}^n)$.

We denote the set of matrix-valued positive definite functions by $\mathbf{P}(G, M_n)$. Clearly, $\mathbf{P}(G, M_n) \supset \mathcal{P}(G, M_n)$ (see Prop 13.4.4 [2]).

Definition 1.2. *A matrix-valued function $\Phi(x)$ is said to be square integrable, or simply L^2 if $[\Phi]_{ij}$ is in $L^2(G)$ for all (i, j). We denote the set of matrix-valued square integrable function by $L^2(G, M_n)$. Define*

$$\langle \Phi, \Psi \rangle = \int_G Tr \Phi \overline{\Psi}^t dg.$$

Put $\mathcal{P}^2(G, M_n) = L^2(G, M_n) \cap \mathcal{P}(G, M_n)$ and $\mathbf{P}^2(G, M_n) = L^2(G, M_n) \cap \mathbf{P}(G, M_n)$.

Let $\Phi, \Psi \in L^1_{loc}(G, M_n)$. Define the convolution

$$[\Phi * \Psi]_{ij} = \sum_{k=1}^{n} [\Phi]_{ik} * [\Psi]_{kj}$$

whenever the right hand side is well-defined, i.e., the convolution integral converges absolutely.

Theorem [A] *Let G be a unimodular locally compact group. Let $\Phi \in \mathcal{P}^2(G, M_n)$. Then there exists a $\Psi \in \mathbf{P}^2(G, M_n)$ such that $\Phi = \Psi * \Psi$.*

Theorem [B] *Let G be a unimodular locally compact group. Let $\Phi, \Psi \in \mathbf{P}^2(G, M_n)$. Then $\langle \Phi, \Psi \rangle \geq 0$.*

Theorem [C] *Let G be a unimodular locally compact group. Let $\Phi, \Psi \in \mathbf{P}^2(G, M_n)$. Then $\langle \Phi, \Psi \rangle = 0$ if and only of $\Phi * \Psi = 0$.*

Our motivation comes from the theory of unitary representations of Lie groups. There are representations that appear as a space of "invariant distributions "in a unitary representation $(\pi, \mathcal{H}_\pi)$. To construct a Hilbert inner product for the invariant distributions, one is often led to investigate whether

$$\int_G (\pi(g)u, u) dg \geq 0 \tag{1}$$

when $(\pi(g)u, u)$ is L^1, $u \in \mathcal{H}_\pi$ and G is unimodular ([6] [4]). For $G = \mathbb{R}$, the affirmative answer to this question is a direct consequence of Bochner's theorem, namely, the integral of a L^1 positive definite function on $\mathbb{R}$ is nonnegative. In his thesis [1], Godement raised this question for G unimodular. It is known that for G amenable, the Inequality (1) is always true (Prop 18.3.6 [2]). An amenable group is characterized by the fact that the unitary dual is weakly contained in $L^2(G)$. Therefore for G nilpotent, the Inequality (1) holds.

Consider the other extreme, namely, G semisimple and noncompact. The inequality (1) is false in its full generality. Yet, applying the result of this paper, we show that Inequality (1) holds if $\mathcal{H}_\pi$ can be written as a tensor product of two L^2-representations of G and u is finite in the tensor decomposition. See Theorem 7.1. In Cor. 7.1, we give a result about a certain integral related to Howe's correspondence ([5]). It is more general than the results given in [4].

2. Convolution Algebras

Let G be a unimodular locally compact group. A matrix-valued function on G is said to be in L^p if each entry is in $L^p(G)$. Let $\Phi \in L^1(G, M_n)$. Define $\|\Phi\|_{L^1} = \sum \|[\Phi]_{ij}\|_{L^1}$. Then $L^1(G, M_n)$ becomes a Banach algebra. We have

$$L^1(G, M_n) * L^1(G, M_n) \subseteq L^1(G, M_n); \qquad C_c(G, M_n) * C_c(G, M_n) \subseteq C_c(G, M_n).$$
$$L^2(G, M_n) * L^2(G, M_n) \subseteq \mathcal{B}C(G, M_n).$$

Here $\mathcal{B}C(G, M_n)$ is the space of bounded continuous functions.

For each $u, v \in L^2(G, \mathbb{C}^n)$, define the standard inner product $\langle u, v\rangle = \int_G \sum_i [u(g)]_i \overline{[v(g)]_i} dg$. Obviously $L^1(G, M_n)$ acts on $L^2(G, \mathbb{C}^n)$ by λ. Then map

$$\lambda : L^1(G, M_n) \to \mathcal{B}(L^2(G, \mathbb{C}^n))$$

defines a bounded Banach algebra isomorphism. Notice that if $\Phi \in L^2(G, M_n)$ and $u \in L^1(G, \mathbb{C}^n)$, then $\lambda(\Phi)u \in L^2(G, \mathbb{C}^n)$.

We define the $*$ operation on $L_{loc}(G, M_n)$ by letting

$$[\Phi^*(g)]_{ij} = \overline{[\Phi(g^{-1})]_{ji}}.$$

For $u, v \in C_c(G, \mathbb{C}^n)$, we have $\langle \lambda(\Phi)u, v\rangle = \langle u, \lambda(\Phi^*)v\rangle$. If Φ is positive definite in $L_{loc}(G, M_n)$, then

$$\langle \lambda(\Phi)u, u\rangle = \overline{\langle \lambda(\Phi)u, u\rangle} = \langle u, \lambda(\Phi)u\rangle = \langle \lambda(\Phi^*)u, u\rangle.$$

Hence $\langle \lambda(\Phi - \Phi^*)u, u\rangle = 0$. Let $u = v + tw\, (t \in \mathbb{R})$. Then

$$0 = \langle \lambda(\Phi - \Phi^*)(v + tw), v + tw\rangle = \langle \lambda(\Phi - \Phi^*)v, tw\rangle + \langle \lambda(\Phi - \Phi^*)tw, v\rangle.$$

We obtain $\langle \lambda(\Phi - \Phi^*)v, w\rangle = \langle \lambda(-\Phi + \Phi^*)w, v\rangle = \langle w, \lambda(-\Phi^* + \Phi)v\rangle = \overline{\langle \lambda(\Phi - \Phi^*)v, w\rangle}$. Hence $\langle \lambda(\Phi - \Phi^*)v, w\rangle$ must be real for all $v, w \in C_c(G, \mathbb{C}^n)$. It follows that $\Phi^* = \Phi$. We have

Lemma 2.1. *Let $\Psi, \Phi \in \mathbf{P}^2(G, M_n)$. Then $\Phi = \Phi^*$, $\Psi = \Psi^*$ and $\langle \Phi, \Psi\rangle = Tr(\Phi * \Psi(e))$.*

Proof: The last statement follows from

(2)
$$Tr(\Phi * \Psi(e)) = \sum_{i,j} \int_G [\Phi]_{ij}(g)[\Psi]_{ji}(g^{-1})dg = \sum_{i,j} \int_G [\Phi]_{ij}(g)\overline{[\Psi]_{ij}(g)}dg = \int_G Tr\Phi(g)\overline{\Psi(g)^t}dg = \langle \Phi, \Psi\rangle.$$

Definition 2.1 (Ch 13. [2])**.** *Let $\Phi \in L^2(G, M_n)$. We say that Φ is moderated if $\lambda(\Phi)$ on $C_c(G, \mathbb{C}^n)$ is a bounded operator in the L^2 norm, i.e., there is a M such that*

$$\|\lambda(\Phi)u\| \leq M\|u\|$$

for any $u \in C_c(G, \mathbb{C}^n)$.

When Φ is moderated, $\lambda(\Phi)|_{C_c(G,\mathbb{C}^n)}$ can be extended to a bounded operator on $L^2(G, \mathbb{C}^n)$ which coincides with the operator $\lambda(\Phi)|_{L^2(G,\mathbb{C}^n)}$. To see this, let $u_i \to u$ under the L^2-norm with $u_i \in C_c(G, \mathbb{C}^n)$. Since $\lambda(\Phi)$ is bounded on $C_c(G, M_n)$, $\{\lambda(\Phi)u_i\}_{i=1}^{\infty}$ yields a Cauchy sequence in $L^2(G, M_n)$. Therefore $\lambda(\Phi)u_i$ converges to an L^2-function $v \in L^2(G, \mathbb{C}^n)$. In particular, $(\lambda(\Phi)u_i)(g)$ converges in L^2-norm to $v(g)$ on any compact subset K. On the other hand, $(\lambda(\Phi)u_i)(g)$ converges to $\lambda(\Phi)u(g)$ uniformly on G, in particular, on K. Hence $(\lambda(\Phi)u)(g) = v(g)$ for $g \in K$ almost everywhere. It follows that $v(g) = \lambda(\Phi)u(g)$ almost everywhere. Therefore $\lambda(\Phi)u \in L^2(G, \mathbb{C}^n)$. In short, if Φ is moderated, and $u \in L^2(G, \mathbb{C}^n)$, then $\lambda(\Phi)(u) \in L^2(G, \mathbb{C}^n)$. We retain $\lambda(\Phi)$ to denote the bounded operator on $L^2(G, \mathbb{C}^n)$. The following is obvious.

Lemma 2.2. *Φ is moderated if and only if $[\Phi]_{ij}$ are all moderated in $L^2(G)$.*

Let $M(G, M_n)$ be the space of moderated L^2 functions on G. Lemma 13.8.4 [2] asserts that $M(G) * M(G) \subseteq M(G)$ and $\lambda|_{M(G)} : M(G) \to \mathcal{B}(L^2(G))$ is an algebra homomorphism. Therefore, we obtain

Lemma 2.3. *Let $\Phi, \Psi \in L^2(G, M_n)$. If Φ and Ψ are moderated, then $\Phi * \Psi$ is also moderated. In addition $\lambda(\Phi * \Psi) = \lambda(\Phi)\lambda(\Psi)$.*

3. Matrix-Valued Positive Definite Functions

Let G be a unimodular locally compact group. Let us recall some basic result from [2]. Let $\Phi, \Psi \in \mathbf{P}(G, M_n)$. We define an ordering $\Phi \preceq \Psi$ if $\Psi - \Phi \in \mathbf{P}(G, M_n)$. An immediate consequence is that $Tr(\Phi)(e) \leq Tr(\Psi)(e)$. Clearly, $\Phi \preceq \Psi$ if and only if

$$\langle \lambda(\Phi)u, u\rangle \leq \langle \lambda(\Psi)u, u\rangle \qquad (\forall\ u \in C_c(G, \mathbb{C}^n)).$$

For two bounded operators X and Y in $\mathcal{B}(\mathcal{H})$, we say that $X \preceq Y$ if $Y - X$ is positive (Ch 2.4 [7]). $Y - X$ is positive implies that $Y - X$ is self-adjoint (Prop. 2.4.6 [7]). If Φ and Ψ are moderated, then $\Phi \preceq \Psi$ if and only if $\lambda(\Phi) \preceq \lambda(\Psi)$.

Theorem 3.1 (Prop. 16 [1])**.** *Let Φ, Ψ be two moderated elements in $\mathbf{P}^2(G, M_n)$. Suppose that $\Phi * \Psi = \Psi * \Phi$. Then $\langle \Phi, \Psi\rangle \geq 0$. Let*

$$\Phi_1 \preceq \Phi_2 \preceq \ldots \preceq \Phi_n \preceq \ldots$$

be an increasing sequence of moderated positive definite functions in $L^2(G, M_n)$. Suppose that Φ_i mutually commute. If $\sup_i \|\Phi_i\|_{L^2} < \infty$, then $\Phi = \lim \Phi_i$ exists in $\mathbf{P}^2(G, M_n)$.

The $n = 1$ case is proved as Prop. 16 in [1]. See also 13.8.5, 13.8.4 [2].

Proof of Theorem 3.1: $\Phi * \Psi = \Psi * \Phi$ implies that $\lambda(\Phi)\lambda(\Psi) = \lambda(\Psi)\lambda(\Phi)$ as bounded operators on $L^2(G, \mathbb{C}^n)$. Since $\lambda(\Phi)$ and $\lambda(\Psi)$ are both positive, they must be self-adjoint. Hence $\lambda(\Phi)\lambda(\Psi)$ must be positive and self-adjoint. In other words, $\lambda(\Phi * \Psi)$ is positive on $L^2(G, \mathbb{C}^n)$. In particular, it is positive with respect to $C_c(G, \mathbb{C}^n)$. Hence $\Phi * \Psi$, as a matrix-valued continuous function, is positive definite. $\Phi * \Psi(e)$ must be a positive semi-definite matrix. By Lemma 2.1 $\langle \Phi, \Psi\rangle = Tr(\Phi * \Psi)(e) \geq 0$.

Let $\Phi_1 \preceq \Phi_2 \preceq \ldots \preceq \Phi_n \preceq \ldots$ be an increasing sequence of moderated positive definite functions in $L^2(G, M_n)$. For $j \geq i$, notice that $\|\Phi_j\|^2 = \|\Phi_i\|^2 + \langle \Phi_i, \Phi_j - \Phi_i\rangle + \langle \Phi_j - \Phi_i, \Phi_i\rangle + \|\Phi_j - \Phi_i\|^2 \geq \|\Phi_i\|^2$. Since $\sup_i \|\Phi_i\|_{L^2} < \infty$, the sequence $\{\|\Phi_i\|\}$ is an increasing sequence bounded from above. In particular, it is a Cauchy sequence. Notice that for $j \geq i$

$$\|\Phi_j - \Phi_i\|^2 = \|\Phi_j\|^2 - \|\Phi_i\|^2 - \langle \Phi_i, \Phi_j - \Phi_i\rangle - \langle \Phi_j - \Phi_i, \Phi_i\rangle \leq \|\Phi_j\|^2 - \|\Phi_i\|^2.$$

This implies that $\{\Phi_i\}$ is a Cauchy sequence in $L^2(G, M_n)$. Let Φ be the L^2-limit of $\{\Phi_i\}$. For every $u = (u_p) \in C_c(G, \mathbb{C}^n)$, since $\lambda(u_p)$ is a bounded operator on $L^2(G)$, we obtain

$$([\Phi_i]_{p,q} * u_q, u_p) \to ([\Phi]_{p,q} * u_q, u_p).$$

It follows that

$$0 \leq \langle \lambda(\Phi_i)u, u\rangle \to \langle \lambda(\Phi)u, u\rangle.$$

Therefore Φ is positive definite. $\square$

Theorem 3.2 (Thm. 17 [1])**.** *Suppose that Φ is a moderated element in $\mathbf{P}^2(G, M_n)$ such that $\Phi \preceq \Theta$ with Θ a continuous positive definite function. Then there is a unique moderated element Ψ in $\mathbf{P}^2(G, M_n)$ such that $\Phi = \Psi * \Psi$ in L^2. In particular, Φ equals a continuous positive definite function almost everywhere and $\|\Psi\|^2 \leq Tr(\Theta(e))$.*

In particular, if Φ is continuous and moderated in $\mathbf{P}^2(G, M_n)$, its square root Ψ exists and is unique.

The $n = 1$ case is established by Godement. Our proof follows from the proof of Theorem 13.8.6 in [2] for the scalar-valued positive definite L^2 functions. The original idea of Godement is to construct an increasing sequence of positive definite moderated elements Ψ_k in $L^2(G)$ that approaches the square root. In Dixmier's book, $\Psi_k = \|\lambda(\Phi)\| p_k(\frac{\Phi}{\|\lambda(\Phi)\|})$. Here $\{p_k(t)\}$ is an increasing sequence of nonnegative polynomials on $[0, 1]$ such that $p_k(t) \to \sqrt{t}$ on $[0, 1]$ and $p_k(0) = 0$. We shall supply a proof of this fact before we carry out the proof of Theorem 3.2.

Lemma 3.1. *There exists a sequence of polynomials*

$$0 \leq p_1(t) \leq p_2(t) \leq \ldots \leq p_k(t) \leq \ldots \leq \sqrt{t} \qquad (t \in [0, 1]),$$

such that $p_k(t) \to \sqrt{t}$ uniformly on $[0, 1]$.

Proof: Consider the function $t^{-\frac{1}{2}}$, $(t \in [0, 1])$. Let $q_k(t)$ be the k-th Taylor polynomial at $t = 1$. Clearly

$$q_{k+1}(t) = q_k(t) + \frac{(\frac{1}{2})(\frac{3}{2}) \ldots (\frac{2k+1}{2})}{(k+1)!}(1-t)^{k+1}.$$

Let $p_k(t) = tq_k(t)$. Clearly $p_k(t)$ is an increasing sequence of non-negative continuous functions with limit $\sqrt{t}$ over the interval $[0, 1]$. By Taylor's theorem, $p_k(t) \to \sqrt{t}$ uniformly on $[\epsilon, 1]$. On $[0, \epsilon]$, $\sqrt{t} - p_k(t) < \sqrt{t} \leq \sqrt{\epsilon}$. Hence $p_k(t) \to \sqrt{t}$ uniformly on $[0, 1]$. $\square$

Proof of Theorem 3.2: Let Φ be a moderated element in $\mathbf{P}^2(G, M_n)$ such that $\Phi \preceq \Theta$ with Θ a continuous positive definite function. Without loss of generality, suppose the operator norm $\|\lambda(\Phi)\| = 1$. For any polynomial $p(t) = \sum_{i=0}^r a_i t^i$, define

$$p(\Phi) = \sum_{i=0}^{r} a_i \overbrace{\Phi * \Phi * \ldots * \Phi}^{i}.$$

Let $\Psi_k = p_k(\Phi)$ with $p_k(t)$ defined in the last lemma. Essentially by functional calculus, we will have

$$\lambda(\Psi_k) \preceq \lambda(\Psi_{k+1}), \qquad \lambda(\Psi_k)\lambda(\Psi_k) \preceq \lambda(\Phi).$$

It follows that $\Psi_k \preceq \Psi_{k+1}$ and $\Psi_k * \Psi_k \preceq \Phi \preceq \Theta$. By taking the value at e, we have $Tr(\Psi_k * \Psi_k(e)) \leq Tr(\Theta(e))$. By Lemma 2.1, $\|\Psi_k\|$ is bounded by $\sqrt{Tr\Theta(e)}$. Since $\{\Psi_k\}$ mutually commutes and is an increasing sequence, by Theorem 3.1, the L^2-limit of Ψ_k exists. Put $\Psi = \lim_{k\to\infty} \Psi_k$. By Theorem 3.1, $\Psi \in \mathbf{P}^2(G, M_n)$. We have

$$\Psi * \Psi(g) = \lim_{k\to\infty} \Psi_k * \Psi_k(g),$$

pointwise. Since $\lim_{k\to\infty} \lambda(\Psi_k * \Psi_k - \Phi) = 0$ in the operator norm, for $u \in C_c(G, \mathbb{C}^n)$, we have

$$\lim_{k\to\infty} \lambda(\Psi_k * \Psi_k)u = \lambda(\Phi)u$$

in $L^2(G,\mathbb{C}^n)$. However, the pointwise limit of the left hand side is obviously $\lambda(\Psi*\Psi)u$. Hence $\lambda(\Psi*\Psi-\Phi)u=0$ for every $u\in C_c(G,\mathbb{C}^n)$. Hence $\Psi*\Psi(g)=\Phi(g)$ almost everywhere. In particular, $\Phi(g)$ is equal to a continuous positive definite function almost everywhere.

Now $\lambda(\Phi)=\lambda(\Psi)^2$ on $C_c(G,\mathbb{C}^n)$. For any $u\in C_c(G,\mathbb{C}^n)$, we have

$$\|\lambda(\Psi)u\|^2=\langle\lambda(\Psi)u,\lambda(\Psi)u\rangle=\langle\lambda(\Psi)^2u,u\rangle=\langle\lambda(\Phi)u,u\rangle\leq\|\lambda(\Phi)\|\|u\|^2.$$

Hence $\lambda(\Psi)$ is bounded on $C_c(G,\mathbb{C}^n)$ and the function Ψ is moderated. By Lemma 2.3, $\lambda(\Phi)=\lambda(\Psi)^2$, as bounded self-adjoint operators on $L^2(G,\mathbb{C}^n)$. Since $\lambda(\Psi)$ is positive, $\lambda(\Psi)$ is unique as a bounded operator on $L^2(G,\mathbb{C}^n)$. In particular, $\lambda(\Psi)$ is uniquely defined on $C_c(G,\mathbb{C}^n)$. Then Ψ must be unique. □

4. Square Roots: Proof of Theorem A

Let G be a unimodular locally compact group. Let $\Phi\in\mathcal{P}^2(G,M_n)$. Now we would like to give a proof of Theorem A. Our proof is somewhat different from the proof of Theorem 13.8.6 given in [2]. The basic idea is the same, namely, to construct a sequence of moderated continuous positive definite functions $\Phi_k\to\Phi$. Let Ψ_k be the square root of Φ_k. Then the square root of Φ can be obtained as the L^2-limit of Ψ_k. The construction is canonical. In our proof, the continuity of Φ_k is given by Theorem 3.2. We do not use Cor. 13.7.11 in [2] which requires several more pages of argument. We also wish to point out a major difference. In the scalar case $\lambda(\Phi_k)$ acts on $L^2(G)$ and in our case $\lambda(\Phi_k)$ acts on $L^2(G,\mathbb{C}^n)$ not on $L^2(G,M_n)$.

Proof of Theorem [A]: Let $x\in G$. Let $\rho(x)$ act on $L^2(G,\mathbb{C}^n)$ by $(\rho(x)u)(g)=u(gx)$. The action ρ is simply the right regular action. Hence $\rho(x)$ is a unitary operator on $L^2(G,\mathbb{C}^n)$. If $\Phi\in L_{loc}(G,M_n)$, then obviously

$$\rho(x)\lambda(\Phi)\rho(x^{-1})=\lambda(\Phi) \tag{3}$$

on $C_c(G,\mathbb{C}^n)$.

Let $\Phi\in\mathcal{P}^2(G,M_n)$. Then $\lambda(\Phi)|_{C_c(G,\mathbb{C}^n)}$ is a positive symmetric operator densely define on $L^2(G,\mathbb{C}^n)$, by the definition of positive definiteness of Φ. Let $\Lambda(\Phi)$ be the Friedrichs extension of $\lambda(\Phi)|_{C_c(G,\mathbb{C}^n)}$. Then $\Lambda(\Phi)$ is an (unbounded) positive and self-adjoint operator (Ch. 5.6. [7]). By Equation (3), we must have $\rho(x)\Lambda(\Phi)\rho(x^{-1})=\Lambda(\Phi)$.

Let $\Lambda(\Phi)=\int_0^\infty t\,dP$ be the spectral decomposition. Here P is a projection-valued measure on the Borel subsets of $\mathbb{R}$. In other words, for every B a Borel subset of $\mathbb{R}$, there is a projection $P(B)$ on $L^2(G,\mathbb{C}^n)$. Then $\rho(x)\Lambda(\Phi)\rho(x^{-1})=\int_0^\infty t\,d[\rho(x)P\rho(x^{-1})]$. Notice here that $\rho(x)$ is unitary. Hence $\rho(x)P(B)\rho(x^{-1})$ remains a projection. The uniqueness of the spectral decomposition of self-adjoint operators implies that $\rho(x)P(B)\rho(x^{-1})=P(B)$. Since $P(B)$ is bounded, we have $\rho(x)P(B)=P(B)\rho(x)$ for any Borel subset B and for any $x\in G$.

Let $[\Phi]_{*j}$ be the j-th column vector of Φ. Fix a Borel subset B. Define Φ_B by letting the j-th column vector to be $[\Phi_B]_{*j}=P(B)[\Phi]_{*j}$. Clearly $\Phi_B\in L^2(G,M_n)$.

Claim 1: $\lambda(\Phi_B)=P(B)\lambda(\Phi)$ on $C_c(G,\mathbb{C}^n)$.

Proof: Let $u \in C_c(G, \mathbb{C}^n)$. Then

$$[\lambda(\Phi)u](g) = \sum_j \int_{x\in G} [\Phi]_{*j}(gx^{-1})[u]_j(x)dx = \sum_j \int_{x\in G} (\rho(x^{-1})[\Phi]_{*j})(g)[u]_j(x)dx.$$

For any $x \in G$, we have

$$(P(B)\rho(x^{-1})[\Phi]_{*j})(g) = (\rho(x^{-1})P(B)[\Phi]_{*j})(g) = (P(B)[\Phi]_{*j})(gx^{-1}).$$

Since $P(B)$ is a bounded operator on $L^2(G, \mathbb{C}^n)$, $[\Phi]_{*j} \in L^2(G, \mathbb{C}^n)$ and $[u]_j(x) \in L^1(G)$, we have

$$\begin{aligned}
[P(B)(\lambda(\Phi)u)](g) =& P(B) \int \sum_j (\rho(x^{-1})[\Phi]_{*j})(g)[u]_j(x)dx \\
=& \int \sum_j (P(B)\rho(x^{-1})[\Phi]_{*j})(g)[u]_j(x)dx \\
=& \int \sum_j (\rho(x^{-1})P(B)[\Phi]_{*j})(g)[u]_j(x)dx \\
=& \int \sum_j (P(B)[\Phi]_{*j})(gx^{-1})[u]_j(x)dx \\
=& \int \sum_j ([\Phi_B]_{*j})(gx^{-1})[u]_j(x)dx \\
=& \int \Phi_B(gx^{-1})u(x)dx \\
=& (\lambda(\Phi_B)u)(g)
\end{aligned} \tag{4}$$

Our claim is proved.

Observe that $P(B)\lambda(\Phi) = P(B)\Lambda(\Phi)$ on $C_c(G, \mathbb{C}^n)$ and $P(B)\Lambda(\Phi)$ is positive and bounded. Therefore $\lambda(\Phi_B) = P(B)\lambda(\Phi)$ is bounded on $C_c(G, \mathbb{C}^n)$ and positive with respect to $C_c(G, \mathbb{C}^n)$. Hence Φ_B is moderated and positive definite. We must have $\lambda(\Phi_B) = P(B)\Lambda(\Phi)$ on $L^2(G, \mathbb{C}^n)$. In addition if $B_1 \supset B_2$

$$\lambda(\Phi_{B_1} - \Phi_{B_2}) = (P(B_1) - P(B_2))\Lambda(\Phi)$$

on $C_c(G, \mathbb{C}^n)$ and the right hand side is positive and self adjoint. Hence $\Phi_{B_1} \succeq \Phi_{B_2}$. Similarly $\Phi_{B_1} \preceq \Phi$.

For each positive integer k, define $\Phi_k = \Phi_{[0,k]}$. We then obtain an increasing sequence of moderated positive definite functions

$$\Phi_1 \preceq \Phi_2 \preceq \ldots \preceq \Phi_k \preceq \ldots (\preceq \Phi).$$

Due to the way $[\Phi_k]_{*j}$ are defined, $\Phi_k \to \Phi$ in L^2-norm. We have

Lemma 4.1. *Let G be a unimodular group. Every $\Phi \in \mathbf{P}^2(G, M_n)$ is a L^2-limit of an increasing sequence of mutually commutative moderated elements in $\mathbf{P}^2(G, M_n)$.*

The $n = 1$ case was proved by Godement as Prop. 14 in [1].

Since Φ_k is moderated in $\mathbf{P}^2(G, M_n)$ with $\Phi_k \preceq \Phi$, by Theorem 3.2, there is a moderated element $\Psi_k \in \mathbf{P}^2(G, M_n)$ such that $\Phi_k = \Psi_k * \Psi_k$ almost everywhere. Without loss of generality, suppose that

$\Phi_k = \Psi_k * \Psi_k$ pointwise. Notice that both $\lambda(\Phi_k)$ and $\lambda(\Psi_k)$ can be regarded as positive bounded self-adjoint operators on the Hilbert space $L^2(G, \mathbb{C}^n)$. By Lemma 2.3, as bounded self-adjoint operators on $L^2(G, \mathbb{C}^n)$, $\lambda(\Phi_k) = \lambda(\Psi_k)^2$. We have

$$\lambda(\Psi_k) = \int_0^k \sqrt{t} dP(t). \tag{5}$$

In particular, $\lambda(\Psi_k)$ is uniquely defined on $C_c(G, \mathbb{C}^n)$. Therefore Ψ_k is unique and satisfies Equation 5. By functional calculus, $\{\lambda(\Psi_k)\}$ mutually commute and yield an increasing sequence of positive bounded self-adjoint operators on $L^2(G, \mathbb{C}^n)$. Restricted to $C_c(G, \mathbb{C}^n)$, it is easy to see that $\{\Psi_k\}$ must mutually commute and

$$\Psi_1 \preceq \Psi_2 \preceq \ldots \preceq \Psi_k \preceq \ldots.$$

Observe that $\|\Psi_k\|^2 = Tr(\Psi_k * \Psi_k(e)) \leq Tr(\Phi(e))$. By Theorem 3.1, $\{\Psi_k\}$ converges in $L^2(G, M_n)$. Let $\Psi_k \to \Psi$ in $L^2(G, M_n)$. By Theorem 3.1, $\Psi \in \mathbf{P}^2(G, M_n)$. Notice that $\Psi_k \in L^2(G, M_n)$. Then $\Phi_k = \Psi_k * \Psi_k$ converges uniformly to $\Psi * \Psi$. Since $\Phi_k|_K \to \Phi|_K$ in $L^2(K, M_n)$ for any compact set K, $\Phi|_K = \Psi * \Psi|_K$ almost everywhere. Therefore $\Phi = \Psi * \Psi$ almost everywhere. Since Φ is continuous, $\Phi = \Psi * \Psi$. Theorem A is proved. $\square$

5. Nonnegative Integral: Proof of Theorem B

Let G be a unimodular group. Let $\Phi, \Gamma \in \mathbf{P}^2(G, M_n)$. We want to prove that

$$\langle \Phi, \Gamma \rangle \geq 0.$$

The main idea of the proof here is essentially due to Godement (Prop.18 [1]). We start with the following lemma.

Lemma 5.1. *Let G be a unimodular group. Every Φ in $\mathbf{P}^2(G, M_n)$ is a limit of an increasing sequence of moderated elements in $\mathcal{P}^2(G, M_n)$ under the L^2 norm.*

Proof: By Lemma 4.1, it suffices to show that every moderated element Φ in $\mathbf{P}^2(G, M_n)$ is the L^2 limit of an increasing sequence of moderated elements in $\mathcal{P}^2(G, M_n)$. Without loss of generality, suppose that $\|\lambda(\Phi)\| = 1$. Let $r_k(t)$ be the $k-$th Taylor polynomial of $\frac{1}{t}$ at $t = 1$. We define $q_k(t) = t^2 r_k(t)$. Then $q_k(t)$ is an increasing sequence of nonnegative polynomial functions on $[0, 1]$ such that $q_k(t) \to t$ uniformly on $[0, 1]$ (c.f. Lemma 3.1).

Let $\Phi_k = q_k(\Phi)$. Then $\lambda(\Phi_k) = q_k(\lambda(\Phi))$ is an increasing sequence of positive self-adjoint operators that approaches $\lambda(\Phi)$. Obviously, $\Phi_k(g)$ is positive definite. Since $\lambda(\Phi)$ extends to a bounded operator on $L^2(G, \mathbb{C}^n)$, $\lambda(\Phi_k)$ also extends to a bounded operator on $L^2(G, \mathbb{C}^n)$. Hence Φ_k is moderated. Since $\Phi * \Phi$ is continuous, $\Phi_k = q_k(\Phi)$ is always continuous. Therefore, $\{\Phi_k\}$ is an increasing sequence of continuous moderated positive definite functions.

Notice that $\lambda(\Phi_k)$, $\lambda(\Phi)$ all mutually commute. Since $\lambda(\Phi_k) \preceq \lambda(\Phi)$, $(\lambda(\Phi_k))^k \preceq (\lambda(\Phi))^k$. Hence $\Phi_k * \Phi_k \preceq \Phi * \Phi$. This implies $Tr(\Phi_k * \Phi_k(e)) \leq Tr(\Phi * \Phi(e))$. By a similar argument in the proof of Theorem 3.1, $\|\Phi_k\| \leq \|\Phi\|$. By Theorem 3.1, let Ψ be the L^2-limit of Φ_k. For any $u \in C_c(G, M_n)$, $\lambda(\Psi)u = \lim_{k \to \infty} \lambda(\Phi_k)u$ pointwise, and $\lim_{k \to \infty} \lambda(\Phi_k)u = \lambda(\Phi)u$ in L^2-norm. It follows that $\Psi = \Phi$ almost everywhere. Therefore $\Phi_k \to \Phi$ in L^2-norm.

We have obtained an increasing sequence of moderated elements in $\mathcal{P}^2(G, M_n)$ such that $\Phi_k \to \Phi$ in L^2-norm. $\square$

Lemma 5.2. *Let Φ_1 be a moderated element in $\mathbf{P}^2(G, M_n)$ and $\Phi_2 \in \mathcal{P}^2(G, M_n)$. We have*

$$\langle \Phi_1, \Phi_2 \rangle \geq 0.$$

Proof: Suppose that $\Phi_2 = \Psi * \Psi$ with $\Psi \in \mathbf{P}^2(G, M_n)$. Then

$$\langle \Phi_1, \Phi_2 \rangle = Tr(\Phi_1 * \Phi_2(e)) = Tr(\Phi_1 * \Psi * \Psi(e)) = \langle \lambda(\Phi_1)\Psi, \Psi \rangle = \sum_{i=1}^{n} \langle \lambda(\Phi_1)[\Psi]_{*i}, [\Psi]_{*i} \rangle.$$

Notice that $\lambda(\Phi_1)$ is a bounded positive self adjoint operator. We have $\langle \Phi_1, \Phi_2 \rangle \geq 0$. $\square$

Proof of Theorem B: For $\Phi, \Gamma \in \mathbf{P}^2(G, M_n)$, let Φ_α be a sequence of moderated element in $\mathbf{P}^2(G, M_n)$ with L^2-limit Φ and Γ_β be a sequence of elements in $\mathcal{P}^2(G, M_n)$ with L^2-limit Γ. Then we have

$$\langle \Phi, \Gamma \rangle = \lim_{\alpha, \beta \to \infty} \langle \Phi_\alpha, \Gamma_\beta \rangle \geq 0.$$

Theorem B is proved. $\square$

6. Zero Integral

Let $\Phi, \Psi \in \mathbf{P}^2(G, M_n)$. If $\Phi * \Psi = 0$, we have $\langle \Phi, \Psi \rangle = Tr(\Phi * \Psi(e)) = 0$. Now we would like to show that the converse is also true.

Theorem 6.1. *Let G be a unimodular locally compact group. Let $\Phi, \Psi \in \mathbf{P}^2(G, M_n)$. If $\langle \Phi, \Psi \rangle = 0$, then $\Phi * \Psi = 0$.*

Proof: By Lemma 4.1, let Φ_m be an increasing sequence of moderated elements in $\mathbf{P}^2(G, M_n)$ such that $\|\Phi_m - \Phi\| \leq \frac{1}{m}$. By Lemma 5.1, let Ψ_p be an increasing sequence in $\mathcal{P}^2(G, M_n)$ such that $\|\Psi_p - \Psi\| \leq \frac{1}{p}$. Then

$$0 = \langle \Phi, \Psi \rangle = \langle \Phi - \Phi_m, \Psi \rangle + \langle \Phi_m, \Psi \rangle \geq \langle \Phi_m, \Psi \rangle \geq \langle \Phi_m, \Psi_p \rangle \geq 0.$$

Hence all the inequalities here must be equalities. Suppose that $\Psi_p = \Theta_p * \Theta_p$ with $\Theta_p \in \mathbf{P}^2(G, M_n)$. Then

$$0 = \langle \Phi_m, \Psi_p \rangle = Tr(\Phi_m * \Theta_p * \Theta_p(e)) = \sum_i \langle \lambda(\Phi_m)[\Theta_p]_{*i}, [\Theta_p]_{*i} \rangle.$$

Since $\lambda(\Phi_m)$ is a positive operator on $L^2(G, \mathbb{C}^n)$, $\langle \lambda(\Phi_m)[\Theta_p]_{*i}, [\Theta_p]_{*i} \rangle = 0$. Thus $\lambda(\Phi_m)[\Theta_p]_{*i} = 0$ in $L^2(G, \mathbb{C}^n)$. It follows that $\Phi_m * \Theta_p = 0$ in $L^2(G, M_n)$. Since $\Phi_m * \Theta_p(g)$ is a continuous function, $\Phi_m * \Theta_p(g) = 0$ for all $g \in G$. Hence $\Phi_m * \Psi_p(g) = \Phi_m * \Theta_p * \Theta_p(g) = 0$. Since $\Phi_m \to \Phi$ and $\Psi_p \to \Psi$ in $L^2(G, M_n)$, we have $\Phi_m * \Psi_p(g) \to \Phi * \Psi(g)$. Therefore $\Phi * \Psi(g) = 0$ for all g. $\square$

Corollary 6.1. *Let G be a locally compact unimodular group. Let $\Phi, \Psi \in \mathbf{P}^2(G, M_n)$. Then $\langle \Phi, \Psi \rangle \geq 0$ and $\langle \Phi, \Psi \rangle = 0$ if and only if $\Phi * \Psi = 0$.*

7. Applications in Representation Theory

Let G be a unimodular group. We call a unitary representation $(\pi, \mathcal{H})$ of G L^p if there is a cyclic vector u in $\mathcal{H}$ such that $(\pi(g)u, u)$ is L^p. A L^p unitary representation has a G-invariant dense subspace with L^p-matrix coefficients.

Theorem 7.1. *Let G be a unimodular locally compact group and $(\pi, \mathcal{H})$ be a unitary representation of G. Suppose that $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are two L^2-unitary representations of G such that*

$$(\pi, \mathcal{H}) \cong (\pi_1 \otimes \pi_2, \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2).$$

Let $u = \sum_{i=1}^n u_1^{(i)} \otimes u_2^{(i)}$ such that matrix coefficients with respect to $\{u_1^{(i)}\}$ and $\{u_2^{(i)}\}$ are all L^2. Then

$$\int_G (\pi(g)u, u) dg = \sum_{i,j=1}^n \int_G (\pi_1(g)u_1^{(i)}, u_1^{(j)})(\pi_2(g)u_2^{(i)}, u_2^{(j)}) dg \geq 0.$$

Proof: Observe that Φ_1 defined by $[\Phi_1]_{ij} = (\pi_1(g)u_1^{(i)}, u_1^{(j)})$ is square integrable and positive definite. Similarly, $\Phi_2 \in L^2(G, M_n)$ defined by $[\Phi_2]_{ij} = (u_2^{(i)}, \pi_2(g)u_2^{(j)})$ is square integrable and positive definite. This theorem follows easily from Theorem B. $\square$

Now we shall apply our result to Howe's correspondence ([5]). Let $(G(m), G'(n))$ be a dual reductive pair in Sp. Let $(G'(n_1), G'(n_2))$ be two G'-groups diagonally embedded in $G'(n)$ with $n_1 + n_2 = n$. Then $(G(m), G'(n_i))$ is a dual reductive pair in some $Sp^{(i)}$ such that $(Sp^{(1)}, Sp^{(2)})$ are diagonally embedded in Sp. Let ω_i be the oscillator representation of $\widetilde{Sp^{(i)}}$. Let ω be the oscillator representation of Sp. Then ω can be identified with $\omega_1 \otimes \omega_2$. This identification preserves that actions of $G(m)$ and $G'(n_i)$.

Now suppose that the matrix coefficients of $\omega_1|_{\tilde{G}(m)}$ with respect to the Schwartz space are L^2. Let π be an irreducible unitary representation of $\tilde{G}(m)$. Suppose that the matrix coefficients for $\omega_2^\infty|_{\tilde{G}(m)} \otimes \pi^\infty$ are all square integrable. Then for any $v \in \pi^\infty$, $u_1^{(j)} \in \omega_1^\infty, u_2^{(j)} \in \omega_2^\infty$ with $j \in [1, N]$, we have

$$\begin{aligned} &\int_{\tilde{G}(m)} (\omega(g)(\sum u_1^{(j)} \otimes u_2^{(j)}), (\sum u_1^{(k)} \otimes u_2^{(k)}))(\pi(g)v, v) dg \\ =&\sum_{j,k} \int_{\tilde{G}(m)} (\omega_1(g)u_1^{(j)}, u_1^{(k)})(\omega_2(g)u_2^{(j)}, u_2^{(k)})(\pi(g)v, v) dg. \end{aligned} \tag{6}$$

By Theorem 7.1, this integral must be nonnegative.

Corollary 7.1. *Consider a dual reductive pair $(G(m), G'(n))$ in Sp. Let $n = n_1 + n_2$. Let $(G(m), G'(n_i))$ be a dual reductive pair in $Sp^{(i)}$. Let ω_i be the oscillator representation of $Sp^{(i)}$. Let π be an irreducible unitary representation of $\tilde{G}(m)$. Suppose that the matrix coefficients with respect to $\omega_1^\infty|_{\tilde{G}(m)}$ and $\omega_2^\infty|_{\tilde{G}(m)} \otimes \pi^\infty$ are square integrable. Let $\xi \in \omega_1^\infty \otimes \omega_2^\infty$ and $u \in \pi^\infty$, then*

$$\int_{\tilde{G}(m)} (\omega(g)\xi, \xi)(\pi(g)u, u) dg \geq 0.$$

This Corollary holds for both p-adic groups and real groups. See [6] [3] [4] for the importance of this integral in Howe's correspondence ([5]). In particular, under the hypothesis of the Corollary, Howe's correspondence preserves unitarity.

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