

PICARD GROUPS OF SIEGEL MODULAR THREEFOLDS AND THETA LIFTING

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ABSTRACT. We show that the Humbert surfaces rationally generate the Picard groups of Siegel modular threefolds. This involves three ingredients: (1) R. Weissauer's determination of these Picard groups in terms of theta lifting from cusp forms of weight $5/2$ on $\tilde{\mathrm{SL}}_2(\mathbb{R})$ to automorphic forms on $\mathrm{Sp}_4(\mathbb{R})$. (2) The theory of special cycles due to Kudla/Millson and Tong/Wang relating cohomology defined by automorphic forms to that defined by certain geometric cycles. (3) Results of R. Howe about the structure of the oscillator representation in this situation.

1. INTRODUCTION

1.1. Let \mathfrak{H}_g be the Siegel half space of genus g and let $\Gamma \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ be a congruence subgroup. The quotient $X_\Gamma := \Gamma \backslash \mathfrak{H}_g$ is the set of complex points of a quasi-projective algebraic variety. These varieties are of considerable importance in geometry and arithmetic, but they are really only well understood for the case $g = 1$, the case of modular curves. Since the nineteenth century one has known how to compute their Betti numbers. Also in the nineteenth century it was understood that their cohomology was related to modular forms: $H^0(X_\Gamma, \Omega^1) \subset H^1(X_\Gamma, \mathbb{C})$ is canonically isomorphic to $S_2(\Gamma)$, the space of cusp forms of weight 2 for Γ . More recent is the discovery that certain special cycles, modular symbols, provide a good set of homology generators, these generators having good transformation properties with respect to the Hecke algebra. Modular symbols are of great practical value in computations with modular forms, providing the key to the algorithms of William Stein and others that are implemented in software systems. Finally, Eichler and Shimura proved that the zeta functions of modular curves are expressible in terms of L -functions of modular forms.

1.2. We know much less even for the case $g = 2$, Siegel modular threefolds. One cannot compute by any practical effective algorithm the Betti numbers of these varieties in general. Some cases where the computations have been carried out can be found in [25], [26], [9], [10], [11]. Laumon, [23], [24], has proved that the zeta functions of Siegel modular threefolds are expressible in terms of the L -functions of automorphic representations, but his theorem is limited in an important respect: neither side of this equation can be computed exactly except in a very small number of cases because the expression of those zeta functions involve multiplicities which are related to the Betti numbers of those varieties. It is true that the cohomology can be described in terms of automorphic forms, this being a general fact about

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quotients of symmetric domains by lattices, and for H^2 , one has an explicit description, due to Weissauer.

1.3. In this paper we study one piece of this H^2 , namely the Picard group. Geometrically one can view this either as the group of algebraic line bundles, or as the Chow group of codimension one algebraic cycles modulo rational equivalence. We show that these Picard groups are generated by certain special cycles which classically are known as Humbert surfaces. This is based on three key facts:

1. Weissauer has shown that $\text{Pic}(X_\Gamma) \otimes \mathbb{C} = H^{1,1}(X_\Gamma)$ and that all the cohomology classes in the complement of the canonical polarization can be represented by (\mathfrak{g}, K) -cohomology classes with values in $\theta(\sigma)$, the space of theta lifts from holomorphic cusp forms σ of weight $5/2$ for the group $\text{SL}_2(\mathbb{R})$ to the group $\text{Sp}_4(\mathbb{R}) \sim \text{SO}_0(3, 2)$.
2. The theory of special cycles, due largely to two groups: Kudla-Millson and Tong-Wang, asserts a close connection between cohomology classes defined by automorphic forms on locally symmetric varieties and classes defined by certain geometric cycles on those manifolds. In the case at hand, the connection is between theta lifts of holomorphic cusp forms of weight $5/2$ and algebraic cycles which are combinations of transforms of classical Humbert surfaces under the Hecke algebra. Here it is crucial that we are in a stable range: $1 < (3 + 2)/4$ (see theorem 9.7).
3. The main issue is then to see that the *general* theta lifts $\theta(\sigma)$ occurring in Weissauer's theorem are in the span of the *special* theta lifts $\theta_{\text{special}}(\sigma)$ occurring in the theory of KM and TW. This is a problem about the oscillator representation: the theta kernel θ_{special} is characterized by representation-theoretic properties. We apply general structure theorems about the oscillator representation and dual reductive pairs due to Howe to conclude our result.

1.4. For the convenience of the reader, sections 8 through 12 collect some background utilized in sections 2 through 7 where the proofs of the main results are given. The reader is warned that the notation sometimes changes (for instance the letter V is a local system in section 3; a real vector space of dimension $p + q$ in sections 8, 9; a rational vector space of dimension 4 in section 10; a real vector space of dimension $2n$, in section 11). Eventually these are specialized to $(p, q) = (3, 2)$, $n = 1$. This is done in part to be consistent with the notation in the references. Also, the papers of Weissauer use the adelic point of view, and the group of symplectic similitudes, whereas the papers of Kudla-Millson and Tong-Wang use the classical viewpoint of automorphic forms as functions on real Lie groups invariant under a lattice (one exception: [18], but that paper deals only with the anisotropic case, i.e., compact quotients). This necessitates a discussion of the connection between them in section 5.

2. SIEGEL MODULAR THREEFOLDS

2.1. Let $G = \text{Sp}_4(\mathbb{R})$ be the group of real symplectic matrices of size four. This acts on the Siegel space

$$\mathfrak{H}_2 = \{ \tau \in \mathbf{M}_2(\mathbb{C}) : {}^t\tau = \tau, \text{Im}(\tau) \text{ is positive definite} \}$$

via

$$g.\tau = (a\tau + b)(c\tau + d)^{-1}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The group $G/\pm 1$ is the group of holomorphic automorphisms of \mathfrak{H}_2 , and it acts transitively. The stabilizer K of the point $i\mathbf{1}_2 = \sqrt{-1}\mathbf{1}_2$ is $\{k = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\}$, which is isomorphic to the unitary group $U(2)$ via $k \mapsto a + bi$. Thus \mathfrak{H}_2 is the symmetric space G/K attached to K . Reference: [17].

2.2. Let $\Gamma \subset \mathrm{Sp}_4(\mathbb{Q})$ be a subgroup commensurable with $\mathrm{Sp}_4(\mathbb{Q})$. Then Γ is an arithmetic group. According to a theorem of Baily-Borel, $X_\Gamma = \Gamma \backslash \mathfrak{H}_2$ is the analytic space attached to the set of \mathbb{C} -points of a quasi-projective algebraic variety defined over \mathbb{C} . The principal congruence subgroup of level N , for an integer $N \geq 1$, is defined as

$$\Gamma(N) = \{\gamma \in \mathrm{Sp}_4(\mathbb{Z}) \mid \gamma \equiv \mathbf{1}_4 \pmod{N}\}.$$

Every subgroup $\Gamma \subset \mathrm{Sp}_4(\mathbb{Z})$ of finite index is a congruence subgroup in the sense that $\Gamma \supset \Gamma(N)$ for some N . The spaces X_Γ admit several compactifications: the Borel-Serre compactification (which is a manifold with corners); the Satake compactification, which is a projective variety, but usually singular; the toroidal compactifications, which are often smooth and projective. For a modern discussion of compactifications of quotients of bounded symmetric domains, see [3]. The general theory of Siegel modular varieties and their compactifications can be found in [4].

2.3. The algebraic variety X whose analytic space is $X^{an} = X(\mathbb{C}) = \Gamma \backslash \mathfrak{H}_2$, is defined a priori over \mathbb{C} , but in fact has a model defined over a number field (a finite extension of \mathbb{Q}). This can be seen from two points of view:

1. X is a moduli space for systems (A, Φ) where A is an abelian variety of dimension 2, and Φ consists of additional structures on A , typically polarizations, endomorphisms, rigidifications of points of some order N . The theory of moduli spaces then provides a structure of a scheme (or more generally, stack) over a number field.
2. X is a Shimura variety. Shimura varieties arise as quotients of hermitian symmetric spaces by arithmetic groups with certain additional properties. It is known that these have canonical models defined over algebraic number fields. For an introduction to this see [30].

3. COHOMOLOGY AND AUTOMORPHIC FORMS

3.1. Let $X_\Gamma = \Gamma \backslash \underline{G}(\mathbb{R})/K$, where \underline{G} is a semisimple (more generally: reductive) algebraic group defined over \mathbb{Q} , $K \subset \underline{G}(\mathbb{R})$ is a maximal compact subgroup, and $\Gamma \subset \underline{G}(\mathbb{Q})$ is a congruence subgroup. If \underline{V} is a local system of complex vector spaces coming from a rational finite dimensional representation $V = V_\mu$ of \underline{G} , then it is known that the cohomology $H^n(X_\Gamma, \underline{V})$ is computable in terms of automorphic forms. There is a canonical isomorphism

$$H^n(X_\Gamma, \underline{V}) = H^n(\Gamma, V_\mu) = H^n(\mathfrak{g}, K; C^\infty(\Gamma \backslash G) \otimes V_\mu)$$

where the right-hand side is relative Lie algebra cohomology (see [1]). The major result, due Jens Franke, [5], built on earlier works by Borel, Casselman, Garland, Wallach, is that these spaces are all isomorphic to $H^n(\mathfrak{g}, K; \mathcal{A}(\Gamma \backslash G) \otimes V_\mu)$, where $\mathcal{A}(\Gamma \backslash G) \subset C^\infty(\Gamma \backslash G)$ is the subspace of automorphic forms (see [2] for the definition of this space; see [29] and [36] for a description and survey of this result, and [6] and [31] for refinements and generalizations).

3.2. As before, $G = \underline{G}(\mathbb{R})$ for a semisimple algebraic group \underline{G} , and let $\Gamma \subset G$ be a lattice (a discrete, finite covolume subgroup). Let $L_{\text{disc}}^2(\Gamma \backslash G)$ be the discrete part of the G -module $L^2(\Gamma \backslash G)$, so that we have

$$L_{\text{disc}}^2(\Gamma \backslash G) = \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma) H_{\pi}$$

where the sum is over the irreducible unitary representations, and the multiplicities $m(\pi, \Gamma)$ are finite. If Γ is cocompact or G has a compact Cartan subgroup, there is an isomorphism

$$\text{IH}^n(X_{\Gamma}, \underline{V}) = \text{H}_{(2)}^n(X_{\Gamma}, \underline{V}) = \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma) \text{H}^n(\mathfrak{g}, K; H_{\pi} \otimes V_{\mu})$$

where the left-hand term is intersection cohomology, the middle term is L^2 -cohomology; the first equality records the solution to the Zucker conjecture. The determination of which irreducible unitary π have nonzero (\mathfrak{g}, K) -cohomology is due to Vogan and Zuckerman, [35]. These are the representations denoted by $A_{\mathfrak{q}}(\lambda)$ for θ -stable parabolic subalgebras $\mathfrak{q} \subset \mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$, $\mathfrak{g}_0 = \text{Lie}(G)$, and certain linear forms λ on a Levi factor $\mathfrak{l} \subset \mathfrak{q}$.

3.3. For $G = \text{GSp}_4(\mathbb{R})$, which has a compact Cartan subgroup, the representations with nonzero (\mathfrak{g}, K) -cohomology have been determined. The list, without proofs, can be found in [33]. Because of the isogeny $\text{Sp}_4(\mathbb{R}) \sim \text{SO}_0(3, 2)$ (see section 10) these representations can be described in orthogonal language and have been listed in [8], [29]. The important case for us are those that contribute to the to the Hodge (1, 1) part. These are: the trivial representation, and two others π^{\pm} . These last two differ by twist: $\pi^{-} = \pi^{+} \otimes (\text{sgn} \circ \nu)$ where $\nu : \text{GSp}_4 \rightarrow \mathbf{G}_m$ is the canonical character with kernel Sp_4 . For more details, see section 12.

4. WEISSAUER'S THEOREMS

For Siegel modular threefolds, and $\underline{V}_{\mu} = \mathbb{C}$, Weissauer has completely analyzed $\text{H}^2(X_{\Gamma}, \mathbb{C})$ in terms of automorphic forms. See his papers [39], [41]. First, the cohomology is all square-integrable, in fact:

Theorem 4.1.

$$\text{H}^2(X_{\Gamma}, \mathbb{C}) = \text{H}_{(2)}^2(X_{\Gamma}, \mathbb{C}) = \text{IH}^2(X_{\Gamma}, \mathbb{C}).$$

This theorem shows in particular that $\text{H}^2(X_{\Gamma}, \mathbb{C})$ has a pure Hodge structure of weight 2. Consider the Hodge decomposition

$$\text{H}^2(X_{\Gamma}, \mathbb{C}) = \text{H}^{2,0}(X_{\Gamma}) \oplus \text{H}^{1,1}(X_{\Gamma}) \oplus \text{H}^{0,2}(X_{\Gamma}), \quad \text{H}^{0,2} = \overline{\text{H}^{2,0}}.$$

Via the isomorphism $\text{H}^2(X_{\Gamma}, \mathbb{C}) = \text{H}^2(\mathfrak{g}, K; \mathcal{A}(\Gamma \backslash G))$ recalled in section 3.1, we can describe the cohomology in degree 2 as certain kinds of closed differential forms on \mathfrak{H}_2 with automorphic form coefficients. The automorphic forms that contribute are square-integrable. More specifically one has:

Theorem 4.2. *The automorphic forms contributing to $\text{H}^{2,0}(X_{\Gamma})$ are given by theta lifting from dual reductive pairs $(\text{GO}(b), \text{Sp}_4(\mathbb{R}))$ where b are two dimensional positive-definite quadratic forms defined over \mathbb{Q} . This allows for an explicit computation of $\dim \text{H}^{2,0}(X_{\Gamma})$. See [40].*

This has been generalized in part by Jian-Shu Li to $\text{H}^{g,0}(X_{\Gamma})$ for quotients of \mathfrak{H}_g , see [28].

Theorem 4.3. 1. *Siegel modular threefolds have maximal Picard number:*

$$\text{H}^{1,1}(X_{\Gamma}) = \text{Pic}(X_{\Gamma}) \otimes \mathbb{C}.$$

2. *There is a canonical decomposition $\text{Pic}(X_\Gamma) \otimes \mathbb{C} = \mathbb{C} \cdot [\mathcal{L}] \oplus \text{Pic}(X_\Gamma)_0$ where $[\mathcal{L}]$ is the Lefschetz class. Then:*

2.1 *$[\mathcal{L}]$ corresponds to the trivial automorphic representation of Sp_4 .*

2.2 *The automorphic forms in $\text{Pic}(X_\Gamma)_0$ are given by theta lifting from the dual reductive pair $(\text{SL}(2, \mathbb{R}), \text{SO}_0(3, 2) \sim \text{Sp}_4(\mathbb{R}))$. More precisely, they are all given by lifting of weight $5/2$ holomorphic cusp forms on $\text{SL}(2, \mathbb{R})$ to the unique automorphic representation of $\text{SO}_0(3, 2)$ that contributes to $\text{Pic}(X_\Gamma)_0$ by Vogan-Zuckerman theory.*

5. ADELIC FORMULATION

Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ be the ring of adèles of the rational field \mathbb{Q} ; $\mathbb{A}_f = \mathbb{Q} \otimes \prod_p \mathbb{Z}_p$ is the ring of finite adèles.

5.1. Let $G = \text{GSp}_4$ be the algebraic group over \mathbb{Q} of symplectic similitudes, i.e., of 4×4 matrices g such that

$${}^t g \Psi g = \nu(g) \Psi, \quad \Psi = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}.$$

There is an exact sequence

$$0 \longrightarrow \text{Sp}_4 \longrightarrow \text{GSp}_4 \xrightarrow{\nu} \mathbb{G}_m \longrightarrow 0.$$

Let

$$h : \mathbb{S} := \text{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_m) \rightarrow \text{GSp}_4$$

be the morphism defined over \mathbb{R} with the property that $x + iy \in \mathbb{C}^\times = \mathbb{S}(\mathbb{R})$ maps to

$$\begin{pmatrix} x1_2 & y1_2 \\ -y1_2 & x1_2 \end{pmatrix}.$$

Let $K_\infty \subset \text{GSp}_4(\mathbb{R})$ be the stabilizer of h . Then $K_\infty = Z_{\mathbb{R}} \cdot K'_\infty$, where $Z_{\mathbb{R}} \subset \text{GSp}_4(\mathbb{R})$ is the center and $K'_\infty \subset \text{Sp}_4(\mathbb{R})$ is a maximal compact subgroup. For any open subgroup of finite index $L \subset \text{GSp}_4(\mathbb{A}_f)$ we define

$$M_L(\mathbb{C}) = M_L(\text{GSp}_4(\mathbb{Q}), h)_{\text{an}} = \text{GSp}_4(\mathbb{Q}) \backslash \text{GSp}_4(\mathbb{A}) / K_\infty L.$$

This is the set of complex points of a quasiprojective algebraic variety M_L defined over a number field. This is a disjoint union of spaces of the type X_Γ discussed above, for various arithmetic subgroups $\Gamma \subset \text{Sp}_4(\mathbb{Q})$. For instance, if we take, for an integer $N \geq 1$,

$$L_N = \left\{ k \in \prod_p G(\mathbb{Z}_p) \mid k \cong 1_4 \pmod{N} \right\}$$

then $M_{L_N}(\mathbb{C}) := M_N(\mathbb{C})$ is a disjoint union of $\phi(N)$ copies of $\Gamma(N) \backslash \mathfrak{H}_2$. The variety M_N is defined over \mathbb{Q} , and each connected component is defined over $\mathbb{Q}(\zeta_N)$, $\zeta_N = \exp(2\pi i/N)$.

5.2. Recall that $G = \text{GSp}_4$. We define

$$\text{H}^i(\text{Sh}(G), \mathbb{C}) := \varinjlim_L \text{H}^i(M_L(\mathbb{C}), \mathbb{C})$$

which is in a canonical way an admissible $\pi_0(G(\mathbb{R})) \times G(\mathbb{A}_f)$ -module. For any compact open subgroup L we have

$$\text{H}^i(\text{Sh}(G), \mathbb{C})^L = \text{H}^i(M_L(\mathbb{C}), \mathbb{C}).$$

This is a module for the Hecke algebra $\mathcal{H}_L = C_c(G(\mathbb{A}_f)/L)$ of \mathbb{C} -valued compactly supported L -biinvariant functions on $G(\mathbb{A}_f)$, which is an algebra for the convolution product, once a Haar measure is fixed on $G(\mathbb{A}_f)$. The major result is that there is a canonical isomorphism

$$H^i(\mathrm{Sh}(G), \mathbb{C}) = H^i(\mathfrak{g}, K_\infty; \mathcal{A}(G)),$$

where $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R})) = \mathfrak{gsp}_4(\mathbb{R})$, K_∞ is defined in section 5.1, $\mathcal{A}(G)$ is the space of automorphic forms on $G(\mathbb{A})$, and the right-hand side is relative Lie algebra cohomology. This is an isomorphism of $G(\mathbb{A}_f)$ -modules, for the canonical structures on both sides. In this case, the above isomorphism can be refined to an isomorphism of Hodge (p, q) -components.

5.3. Weissauer's theorems are the following:

5.3.1. $H^2(\mathrm{Sh}(G), \mathbb{C}) = H^2_{(2)}(G, \mathbb{C})$. Therefore we have, for each Hodge index (p, q) with $p + q = 2$,

$$H^{p,q}(M_L(\mathbb{C})) = \bigoplus_{\pi_\infty \in \mathrm{Coh}^{p,q}} m(\pi) H^{p,q}(\mathfrak{g}, K_\infty; \pi_\infty) \otimes \pi_f^L$$

where the sum ranges over all the irreducible automorphic representations $\pi = \pi_\infty \otimes \pi_f$ which occur in the discrete spectrum

$$L^2_d(G(\mathbb{Q})Z(\mathbb{R})^\circ \backslash G(\mathbb{A}), dg)$$

where $Z(\mathbb{R})^\circ \subset G(\mathbb{R})$ is the connected component of the center. The set $\mathrm{Coh}^{p,q}$ is the finite set of unitary representations of $G(\mathbb{R})$ with trivial central character and with nonzero (\mathfrak{g}, K_∞) -cohomology in dimension (p, q) .

5.3.2. There is only one element of $\mathrm{Coh}^{2,0} = \mathrm{Coh}^{0,2}$, call it $\pi^{2,0} = \pi^{0,2}$. Then $H^{p,q}(\mathfrak{g}, K_\infty; \pi^{2,0})$ is one-dimensional for $(p, q) = (2, 0)$ or $(0, 2)$. Every automorphic representation contributing to $H^{2,0}(M_L(\mathbb{C}), \mathbb{C})$ is in the image of the theta lifting from the orthogonal similitude group $\mathrm{GO}(b)$ as b ranges over the positive-definite binary quadratic forms over \mathbb{Q} .

5.3.3. $\mathrm{Coh}^{1,1} = \{1, \pi^\pm\}$, where 1 is the trivial one-dimensional representation, and $\pi^- = \pi^+ \otimes \mathrm{sgn}$, where $\mathrm{sgn} : \mathrm{GSp}_4(\mathbb{R}) \rightarrow \{\pm 1\}$ is the sign character. One has

$$\mathrm{Pic}(M_L(\mathbb{C})) \otimes \mathbb{C} = H^{1,1}(M_L(\mathbb{C}))$$

and we can canonically decompose this as

$$\mathrm{Pic}(M_L(\mathbb{C})) \otimes \mathbb{C} = \mathbb{C}[\mathcal{L}] \oplus \mathrm{Pic}(M_L(\mathbb{C}))_0 \otimes \mathbb{C}$$

where \mathcal{L} is the canonical polarization ("Lefschetz class"). This term corresponds to the automorphic representation 1. Weissauer showed that the classes in $\mathrm{Pic}(M_L(\mathbb{C}))_0 \otimes \mathbb{C} = H^{1,1}(M_L(\mathbb{C}))_0$ in the complement of the Lefschetz class are generated by the images of $H^{1,1}(\mathfrak{g}, K; \theta(\sigma, \psi) \otimes (\chi \circ \nu))$. Here, $\psi : \mathbb{A}/\mathbb{Q} \rightarrow \mathbb{C}$ is a nontrivial additive character, σ is an irreducible (anti)holomorphic cuspidal automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ of weight $5/2$, $\theta(\sigma, \psi)$ is the theta lifting with respect to the Weil representation ω_ψ to an automorphic representation to $\mathrm{PGSp}_4(\mathbb{A})$ viewed as a representation of $\mathrm{GSp}_4(\mathbb{A})$, $\chi : \mathbb{A}^*/\mathbb{Q}^* \mathbb{R}_{>0}^* \rightarrow \mathbb{C}^*$ is an idele class (Dirichlet) character, and $\nu : \mathrm{GSp}_4 \rightarrow \mathbf{G}_m$ is the canonical character with kernel Sp_4 .

5.4. We get the Picard group by varying all the data in the above. First note that we can fix one choice of nontrivial additive character ψ . The reason is that, every other nontrivial additive character is of the form ψ_t for a $t \in \mathbb{Q}^*$, where $\psi_t(x) = \psi(tx)$. It is known that $\theta(\sigma, \psi_t) = \theta(\sigma_t, \psi)$ ([13], 1.8), where σ_t is the automorphic representation

$$g \mapsto \sigma \left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right).$$

So from now on, we drop explicit reference to ψ .

5.5. For each integer $N \geq 1$ let $M_N = M_{K_N}$, which is a scheme over \mathbb{Q} . $M_N \otimes \overline{\mathbb{Q}}$ has $\phi(N)$ (Euler phi) connected components, defined and all isomorphic over $\mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N^{th} root of unity. These are permuted simply transitively by $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. We denote any one of these components by M_N^0 . We have $M_N^0(\mathbb{C}) = \Gamma(N) \backslash \mathfrak{H}_2$. The group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\text{Pic}(M_N) \otimes \mathbb{C}$, fixing the Lefschetz class. Weissauer proved [41, Thm. 2, p. 184] that the action of $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ on $\text{Pic}(M_N) \otimes \mathbb{C}$ factors over the abelian quotient $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$. Therefore we have a decomposition

$$\text{Pic}(M_N)_0 \otimes \mathbb{C} = \bigoplus_{\chi} \text{Pic}^{\chi}(M_N)_0$$

of isotypical spaces for the characters χ of $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$. By classfield theory these can be identified with idele class characters $\chi : \mathbb{A}^*/\mathbb{Q}^* \mathbb{R}_{>0}^* \rightarrow \mathbb{C}^*$. The space $\text{Pic}(M_N(\mathbb{C}))_0$ is the kernel of the canonical trace map

$$\text{Pic}(M_N(\mathbb{C})) \otimes \mathbb{C} \rightarrow \text{Pic}(M_1(\mathbb{C})) \otimes \mathbb{C}.$$

We can similarly define $\text{Pic}(M_N^0(\mathbb{C}))_0$. Evidently, for the inclusion of any connected component $M_N^0 \rightarrow M_N$, the restriction $\text{Pic}(M_N)_0 \otimes \mathbb{C} \rightarrow \text{Pic}(M_N^0)_0 \otimes \mathbb{C}$ is surjective. By choosing these inclusions compatibly we can define a map

$$\text{Pic}(M) := \varinjlim_N \text{Pic}(M_N)_0 \otimes \mathbb{C} \rightarrow \varinjlim_N \text{Pic}(M_N^0)_0 \otimes \mathbb{C} := \text{Pic}(M^0).$$

Lemma 5.6. *For the identity character 1, the map $\text{Pic}^1(M)_0 \rightarrow \text{Pic}(M^0)_0 \otimes \mathbb{C}$ is surjective.*

Proof. Let G_n be the kernel of the map $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. Let $\mathcal{M} \in \text{Pic}(M_N^0)_0$. Then there is an $N' \geq N$ with the property that $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{N'}))$ acts trivially on $i_{N,*}\mathcal{M}$, where $i_N : M_N^0 \rightarrow M_N$ is the inclusion (extension by 0 on all the other components). Let $f : M_{N'}^0 \rightarrow M_N^0$ be the canonical projection, and extend i_N to a map $i_{N'} : M_{N'}^0 \rightarrow M_{N'}$ which commutes with the projection $f : M_{N'} \rightarrow M_N$. The line bundle $\mathcal{M}' = i_{N',*}f^*\mathcal{M}$ is fixed by $G_{N'}$, and hence for any $g \in \text{Gal}(\mathbb{Q}(\zeta_{N'})/\mathbb{Q})$, $g^*\mathcal{M}'$ is well-defined. Then we define $\mathcal{N} \in \text{Pic}^1(M_{N'})_0$ by

$$\mathcal{N} = \sum_{g \in \text{Gal}(\mathbb{Q}(\zeta_{N'})/\mathbb{Q})} g^*\mathcal{M}'.$$

Moreover, since $\text{Gal}(\mathbb{Q}(\zeta_{N'})/\mathbb{Q})$ acts simply transitively on the components of $M_{N'}$, it follows that $i_{N'}^*\mathcal{N} = f^*\mathcal{M}$. This shows that after extension to N' the class $\mathcal{M} \in \text{Pic}(M_N^0)_0 \subset \text{Pic}(M^0)_0$ is in the image of $\text{Pic}^1(M_{N'})_0 \subset \text{Pic}^1(M)_0$. \square

Lemma 5.7. *Any element of $\text{Pic}(M^0)_0 \otimes \mathbb{C}$ is in the image (in the sense of section 5.4) of the Saito-Kurokawa lifts $\theta(\sigma)$ of holomorphic cusp forms σ of weight $5/2$.*

Proof. Weissauer showed more precisely that the elements of $\text{Pic}^X(M)_0$ are in the image of $\theta(\sigma, \psi) \otimes (\chi \circ \lambda)$. But lemma 5.6 shows that every element of $\text{Pic}(M^0)_0 \otimes \mathbb{C}$ is in the image of $\text{Pic}^1(M)_0$ and these are in the image of the Saito-Kurokawa lift. \square

6. STRUCTURE OF THE OSCILLATOR REPRESENTATION

This section follows [14], [22]. We let V be the \mathbb{R} -vector space with a quadratic form $(\ , \)$ of signature $(p, q) = (3, 2)$; $n = 5 = p + q$. Let W be the \mathbb{R} -vector space of dimension $m = 2$ with an alternating nondegenerate bilinear form $\langle \ , \ \rangle$. We describe a model for the infinitesimal oscillator representation $(\omega, \mathfrak{sp}(V \otimes W))$. The maximal compact subgroup of $\text{Sp}_{10} = \text{Sp}(V \otimes W) \sim \text{Sp}(10, \mathbb{R})$ is isomorphic to the unitary group U_5 . We let $\tilde{\text{Sp}}_{10} = \tilde{\text{Sp}}(10, \mathbb{R})$ be the metaplectic cover, and in general putting a tilde over an object in $\text{Sp}(10, \mathbb{R})$ denotes its inverse image in the metaplectic cover.

6.1. The space of \tilde{U}_5 -finite vectors in the Fock realization of ω is isomorphic to the space of polynomials $\mathcal{P}(\mathbb{C}^5)$ in five variables z_i , $i = 1, \dots, 5$. The action is given by

$$\begin{aligned} \omega(\mathfrak{sp}_{10} \otimes \mathbb{C}) &= \mathfrak{sp}^{(1,1)} \oplus \mathfrak{sp}^{(2,0)} \oplus \mathfrak{sp}^{(0,2)} \\ \mathfrak{sp}^{(1,1)} &= \text{span of } \left\{ \left(z_i \frac{\partial}{\partial z_j} + \frac{1}{2} \delta_i^j \right) \right\} \\ \mathfrak{sp}^{(2,0)} &= \text{span of } \{ z_i z_j \} \\ \mathfrak{sp}^{(0,2)} &= \text{span of } \left\{ \frac{\partial^2}{\partial z_i \partial z_j} \right\}. \end{aligned}$$

In the Cartan decomposition $\mathfrak{sp}_{10} = \mathfrak{u}_5 \oplus \mathfrak{q}$ we have

$$\omega(\mathfrak{u}_5 \otimes \mathbb{C}) = \mathfrak{sp}^{(1,1)}, \quad \omega(\mathfrak{q} \otimes \mathbb{C}) = \mathfrak{sp}^{(2,0)} \oplus \mathfrak{sp}^{(0,2)}.$$

6.2. We are interested in the reductive dual pair

$$(G, G') = (\tilde{O}(V) = \tilde{O}_{3,2}, \tilde{S}p(W) = \tilde{S}L_2(\mathbb{R}))$$

inside $\tilde{S}p(V \otimes W) = \tilde{S}p_{10}$, and especially the structure of $\mathcal{P}(\mathbb{C}^5)$ as a $(\mathfrak{g}, \tilde{K}) \times (\mathfrak{g}', \tilde{K}')$ -module, where $\mathfrak{g} = \text{Lie}(G) = \mathfrak{o}(V) = \mathfrak{o}_{3,2}$, $\mathfrak{g}' = \text{Lie}(G') = \mathfrak{sp}(W) = \mathfrak{sl}_2(\mathbb{R})$, $\tilde{K} = \text{O}(3) \times \text{O}(2)$ is the maximal compact subgroup of G , $\tilde{K}' = \text{SO}(2)$ is the maximal compact subgroup of G' . Following the convention in [22, p. 154] we number the variables z_i as z_α , $\alpha = 1, 2, 3$, and z_μ , $\mu = 4, 5$; generally indices α, β, \dots run from 1 to 3 and indices μ, ν, \dots run from 4 to 5. In this numbering group $\text{O}(3) \times \text{O}(2)$ acts so that $\text{O}(3)$ rotates the variables z_α and $\text{O}(2)$ rotates the variables z_μ .

6.3. Let

$$\mathcal{P} = \mathcal{P}(\mathbb{C}^5) = \bigoplus_{\sigma \in \mathcal{R}(\tilde{K}, \omega)} \mathcal{I}_\sigma$$

be the decomposition into \tilde{K} -isotypical components; the notation $\mathcal{R}(\tilde{K}, \omega)$ refers to the isomorphism classes of representations of \tilde{K} that occur in the oscillator representation. We recall the definition of harmonics. The Lie algebra $\mathfrak{k} = \mathfrak{o}_3 \times \mathfrak{o}_2$ of the maximal compact subgroup of $G = O(V)$ is a member of a dual reductive pair $(\mathfrak{k}, \mathfrak{l}')$. In this case, $\mathfrak{l}' = \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$. We can decompose

$$\mathfrak{l}' = \mathfrak{l}'^{(2,0)} \oplus \mathfrak{l}'^{(1,1)} \oplus \mathfrak{l}'^{(0,2)}, \quad \text{where } \mathfrak{l}'^{(i,j)} = \mathfrak{l}' \cap \mathfrak{sp}^{(i,j)}.$$

Then the harmonics are defined by

$$\begin{aligned}\mathcal{H}(K) &= \mathcal{H}(\tilde{K}) = \left\{ P \in \mathcal{P} : l(P) = 0 \text{ for all } l \in \mathfrak{l}'^{(0,2)} \right\} \\ \mathcal{H}(K)_\sigma &= \mathcal{H}(K) \cap \mathcal{I}_\sigma\end{aligned}$$

The crucial point for us the Howe's result [14, p.542]:

Theorem 6.4. *For each $\sigma \in \mathcal{R}(\tilde{K}, \omega)$:*

1. *The space $\mathcal{H}(K)_\sigma$ consists precisely of the polynomials of lowest degree in \mathcal{I}_σ ; these are homogeneous all of the same degree, $\deg(\sigma)$.*
- 2.

$$\mathcal{I}_\sigma = \mathcal{U}(\mathfrak{g}') \cdot \mathcal{H}(K)_\sigma = \mathcal{U}(\mathfrak{l}'^{(2,0)}) \cdot \mathcal{H}(K)_\sigma$$

where \mathcal{U} denotes the universal enveloping algebra of the respective Lie algebra.

6.5. Let D be the Hermitian symmetric domain attached to the Lie group $\mathrm{SO}_0(3, 2)$. This is isomorphic to the Siegel half space \mathfrak{H}_2 via the isogeny $\mathrm{Sp}_4(\mathbb{R}) \rightarrow \mathrm{SO}_0(3, 2)$. The tangent bundle to D is the homogeneous vector bundle associated to the action of the maximal compact $K_0 = \mathrm{SO}(3) \times \mathrm{SO}(2)$ on $\mathfrak{p} := \mathfrak{g}/\mathfrak{k}$ via the adjoint representation. The complex structure is given by the action of the subgroup $\mathrm{SO}(2) \subset \mathrm{SO}(3) \times \mathrm{SO}(2)$, and we have a decomposition into Hodge types $\mathfrak{p}_\mathbb{C}^* = \mathfrak{p}^{(1,0)} \oplus \mathfrak{p}^{(0,1)}$, where

$$k(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in \mathrm{SO}(2)$$

acts as $\exp(i\theta)$ (resp. $\exp(-i\theta)$) on $\mathfrak{p}^{(1,0)}$ (resp. $\mathfrak{p}^{(0,1)}$). We are interested in the bundle of $(1, 1)$ -forms on D which is a subrepresentation

$$\wedge^{1,1} \mathfrak{p}^* \subset \wedge^2 \mathfrak{p}_\mathbb{C}^*;$$

in fact, as $\mathrm{SO}(3) \times \mathrm{SO}(2)$ -module, the $\mathrm{SO}(2)$ -factor acts trivially, and the $\mathrm{SO}(3)$ -module splits as $\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$ where, for each odd integer i , \mathbf{i} is the unique corresponding irreducible representation of $\mathrm{SO}(3)$. Thus, as $\mathrm{SO}(3) \times \mathrm{SO}(2)$ -module,

$$\wedge^{1,1} \mathfrak{p}^* \cong \mathbf{1} \otimes \mathbf{1} \oplus \mathbf{3} \otimes \mathbf{1} \oplus \mathbf{5} \otimes \mathbf{1}.$$

As these are self-dual, we have the same decomposition for $\wedge^{1,1} \mathfrak{p}$. The theta-lifting kernels relevant to us will define classes in $H^{1,1}(\mathfrak{g}, K; \mathcal{P}(\mathbb{C}^5))$, which is a subquotient of

$$\mathrm{Hom}_K(\wedge^{1,1} \mathfrak{p}, \mathcal{P}(\mathbb{C}^5)) = \mathrm{Hom}_K(\wedge^{1,1} \mathfrak{p}, \mathcal{I}_{\mathbf{1} \otimes \mathbf{1}}) \oplus \mathrm{Hom}_K(\wedge^{1,1} \mathfrak{p}, \mathcal{I}_{\mathbf{3} \otimes \mathbf{1}}) \oplus \mathrm{Hom}_K(\wedge^{1,1} \mathfrak{p}, \mathcal{I}_{\mathbf{5} \otimes \mathbf{1}})$$

In fact, only the first and last summand above will contribute to the Picard group, as we will see. Since these isotypical spaces are generated by their harmonics, we need to analyze those.

Proposition 6.6. 1. $\mathcal{H}(K)_{\mathbf{1} \otimes \mathbf{1}}$ is the one dimensional \mathbb{C} -vector space spanned by $1 \in \mathcal{P}(\mathbb{C}^5)$.

2. $\mathcal{H}(K)_{\mathbf{3} \otimes \mathbf{1}}$ is the three dimensional \mathbb{C} -vector space spanned by the $z_\alpha \in \mathcal{P}(\mathbb{C}^5)$.

3. $\mathcal{H}(K)_{\mathbf{5} \otimes \mathbf{1}}$ is the five dimensional \mathbb{C} -vector space consisting of quadratic forms

$$\sum_{\alpha, \beta=1}^3 c_{\alpha\beta} z_\alpha z_\beta, \quad c_{\alpha\beta} = c_{\beta\alpha} \in \mathbb{C}, \quad \text{with} \quad \sum_{\alpha=1}^3 c_{\alpha\alpha} = 0.$$

Proof. Define

$$\begin{aligned} X_\alpha &= -\frac{1}{2} \sum_{\alpha=1}^3 z_\alpha^2, & Y_\alpha &= \frac{1}{2} \sum_{\alpha=1}^3 \frac{\partial^2}{\partial z_\alpha^2}, & H_\alpha &= \frac{1}{2} \sum_{\alpha=1}^3 \left(z_\alpha \frac{\partial}{\partial z_\alpha} + \frac{\partial}{\partial z_\alpha} z_\alpha \right) \\ X_\mu &= -\frac{1}{2} \sum_{\mu=4}^5 z_\mu^2, & Y_\mu &= \frac{1}{2} \sum_{\mu=4}^5 \frac{\partial^2}{\partial z_\mu^2}, & H_\mu &= \frac{1}{2} \sum_{\mu=4}^5 \left(z_\mu \frac{\partial}{\partial z_\mu} + \frac{\partial}{\partial z_\mu} z_\mu \right) \end{aligned}$$

One verifies that each of these satisfies the relations for $\mathfrak{sl}_2(\mathbb{R})$: $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$. Evidently, $\mathfrak{l}'_\alpha = \text{span}\{H_\alpha, X_\alpha, Y_\alpha\}$ commutes with $\mathfrak{l}'_\mu = \text{span}\{H_\mu, X_\mu, Y_\mu\}$ and one checks that $\mathfrak{l}' = \mathfrak{l}'_\alpha \times \mathfrak{l}'_\mu$ commutes with the operators in $\omega(\mathfrak{k})$ where $\mathfrak{k} = \mathfrak{o}(V) = \mathfrak{o}_{3,2}$ is the maximal compact of $\mathfrak{o}(3,2)$: one must compute that all these operators H_i, X_i, Y_i commute with the operators $\omega(X_{\alpha\beta}), \omega(X_{\mu\nu})$ that appear in [22, Theorem 7.1, p. 155], which they do. This gives explicit formulas for the dual reductive pair $(\mathfrak{k}, \mathfrak{l}' = \mathfrak{l}'_\alpha \times \mathfrak{l}'_\mu)$.

The space $\mathfrak{l}'^{(0,2)}$ is spanned by the operators Y_α, Y_μ . Recalling that the first factor in $O_3 \times O_2$ acts by rotation in the variables z_α and the second factor acts on the variables z_μ , it is first of all clear that there are no constant or linear polynomials in $\mathcal{P}(\mathbb{C}^5)$ in the representation $\mathbf{5} \otimes \mathbf{1}$. It is also clear that the space of polynomials mentioned in the statement of the proposition are harmonic: they are annihilated by the operators Y_α, Y_μ , and do constitute a representation of type $\mathbf{5} \otimes \mathbf{1}$. This shows that $\deg(\mathbf{5} \otimes \mathbf{1}) = 2$, and there cannot be any other harmonic quadratic forms in the representation $\mathbf{5} \otimes \mathbf{1}$.

An Alternative Proof: It suffices to find the $SO(3) \times SO(2)$ -modules in $\mathcal{S}_{\mathbf{1} \otimes \mathbf{1}}$ and $\mathcal{S}_{\mathbf{5} \otimes \mathbf{1}}$ that are of lowest degrees. These modules, according to Howe, are unique, homogeneous and harmonic. The degree zero polynomials in $\mathcal{P}(\mathbb{C}^5)$, yield a trivial $SO(3) \times SO(2)$ -module. Therefore, $\mathcal{H}(K)_{\mathbf{1} \otimes \mathbf{1}}$ is the one dimensional \mathbb{C} -vector space spanned by $1 \in \mathcal{P}(\mathbb{C}^5)$. The degree 1 polynomials yield a standard representation $\mathbf{3} \otimes \mathbf{1}$ of $SO(3) \times SO(2)$. They are of lowest degree in $\mathcal{S}_{\mathbf{3} \otimes \mathbf{1}}$. Therefore $\mathcal{H}(K)_{\mathbf{3} \otimes \mathbf{1}}$ is the three dimensional \mathbb{C} -vector space spanned by the $z_\alpha \in \mathcal{P}(\mathbb{C}^5)$. The space of degree 2 polynomials is simply

$$S^2(\mathbb{C}^3 \oplus \mathbb{C}^2) \cong S^2(\mathbb{C}^3) \oplus \mathbb{C}^3 \otimes \mathbb{C}^2 \oplus S^2(\mathbb{C}^2).$$

The first summand decomposes into $\mathbf{5} \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathbf{1}$. The $\mathbf{5} \otimes \mathbf{1}$ summand is spanned by

$$\sum_{\alpha, \beta=1}^3 c_{\alpha\beta} z_\alpha z_\beta, \quad c_{\alpha\beta} = c_{\beta\alpha} \in \mathbb{C}, \quad \text{with} \quad \sum_{\alpha=1}^3 c_{\alpha\alpha} = 0.$$

Clearly, it is of the lowest degree in $\mathcal{S}_{\mathbf{5} \otimes \mathbf{1}}$. It must be equal to $\mathcal{H}(K)_{\mathbf{5} \otimes \mathbf{1}}$. □

6.7. Kudla and Millson define

$$\varphi^+ = \sum_{\alpha, \beta=1}^3 z_\alpha z_\beta \omega_{\alpha 4} \wedge \omega_{\beta 5}$$

This is an element of $\text{Hom}_K(\wedge^{1,1}\mathfrak{p}, \mathcal{P})$. It is clear that φ^+ induces a K -isomorphism from $\wedge^{1,1}\mathfrak{p}$, and $\mathcal{P}_{2,\alpha}$, the space of quadratic forms in the variables $z_\alpha, \alpha = 1, 2, 3$. As representation spaces these are $\mathbf{5} \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathbf{1}$. Thus φ^+ induces an isomorphism of isotypical spaces

$$(\wedge^{1,1}\mathfrak{p})_{\mathbf{5} \otimes \mathbf{1}} \xrightarrow{\sim} (\mathcal{P}_{2,\alpha})_{\mathbf{5} \otimes \mathbf{1}},$$

which are irreducible representations of K , the right-hand side being described in proposition 6.6. It is easily seen that $d\varphi^+ = 0$, so that we have a class $[\varphi^+] \in H^{1,1}(\mathfrak{g}, K; \mathcal{P})$.

6.8. Recall that the $U(\tilde{5})$ -finite vectors in the Fock model are the polynomials in $\mathcal{P}(\mathbb{C}^5)$. Given any (\mathfrak{g}, K) -module homomorphism $\mathcal{P}(\mathbb{C}^5) \rightarrow \mathcal{A}(\Gamma \backslash G)$ to the space of automorphic forms, $G = \mathrm{SO}(3, 2)$, we get a map

$$H^{1,1}(\mathfrak{g}, K; \mathcal{P}) \longrightarrow H^{1,1}(\mathfrak{g}, K; \mathcal{A}(\Gamma \backslash G)).$$

For our purposes these are given by theta lifting. Let $\sigma \subset \mathcal{A}(\Gamma' \backslash G')$, $G' = \tilde{\mathrm{SL}}_2(\mathbb{R})$, be the space of holomorphic cusp forms of weight $5/2$ belonging to an irreducible representation of G' . Given a linear functional $\Theta : \mathcal{P}(\mathbb{C}^5) \rightarrow \mathbb{C}$ with the property that $\Theta(\omega(\gamma', \gamma)\varphi) = \Theta(\varphi)$ for all $\varphi \in \mathcal{P}(\mathbb{C}^5)$, all $(\gamma', \gamma) \in \Gamma' \times \Gamma$, defining the theta kernel as $\theta_\varphi(g', g) = \Theta(\omega(g', g)\varphi)$, we define, for any $f \in \sigma$,

$$\theta_\varphi(f)(g) = \int_{\Gamma' \backslash G'} \theta_\varphi(g', g) f(g') dg',$$

and let $\theta_\varphi(\sigma) \subset \mathcal{A}(\Gamma \backslash G)$ be the space spanned by the $\theta_\varphi(f)$. Note that, for any fixed f , the map $\varphi \mapsto \theta_\varphi(f)$ is a (\mathfrak{g}, K) -module homomorphism $\mathcal{S}(\mathbb{C}^5)_{(K)} \rightarrow \mathcal{A}(\Gamma \backslash G)$. Hence, each $f \in \sigma$ defines a map

$$H^{1,1}(\mathfrak{g}, K; \mathcal{P}) \xrightarrow{\theta(f)} H^{1,1}(\mathfrak{g}, K; \mathcal{A}(\Gamma \backslash G)).$$

We define $\theta_{\varphi^+}(f) := \theta(f)([\varphi^+])$. Finally note that the symbols $\theta(f)$, etc., defined here are ambiguous in that they depend on the initial choice of functional Θ . In the theory of special cycles, functionals Θ are constructed by summing over subsets of the form $x + L \subset \mathbb{R}^5$ for rational vectors x and lattices L . By varying x and L we obtain all the special cycles in that theory. This is best formulated in adelic language. It will always be assumed that our functionals have this form. The more precise notation will be $\theta_\varphi^{x,L}(f)$, etc., but we will follow Kudla and Millson in simply writing $\theta_\varphi(f)$ when reference to the specific form of the kernel is not needed. We will need:

Lemma 6.9. *We have $\theta_{Z\varphi}(f) = \theta_\varphi(Z^*f)$ for the involution $Z \rightarrow Z^*$ induced by the map $g \rightarrow g^{-1}$ of G' .*

Proof: The map $Z \rightarrow Z^*$ is given by

$$X_1 X_2 \dots X_n \rightarrow (-1)^n X_n X_{n-1} \dots X_1,$$

and it is complex conjugate linear. Recall that $\theta : \varphi \in \mathcal{P} \rightarrow \theta_\varphi \in C^\infty(G', G)$ preserves the actions of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g}')$. $\forall Z \in \mathcal{U}(\mathfrak{g}')$, we obtain

$$\begin{aligned} \theta_{Z\varphi}(f) &= \int_{\Gamma' \backslash G'} \theta_{Z\varphi}(g', g) f(g') dg' \\ &= \int_{\Gamma' \backslash G'} Z(\theta_\varphi)(g', g) f(g') dg' \\ &= \int_{\Gamma' \backslash G'} \theta_\varphi(g', g) Z^* f(g') dg' \\ &= \theta_\varphi(Z^* f). \end{aligned}$$

6.10. According to Weissauer's theorems, any cohomology class $\xi \in H^{1,1}(X_\Gamma, \mathbb{C})_0$ in the complement to the Lefschetz class occurs in $H^{1,1}(\mathfrak{g}, K; \theta(\sigma))$ where $\theta(\sigma) \subset \mathcal{A}(\Gamma' \backslash G')$ is a theta-lifting (see section 6.8) belonging to a space σ of holomorphic cusp forms of weight $5/2$ for $G' = \tilde{\text{SL}}_2(\mathbb{R})$. We can lift this to an element $\xi \in \text{Hom}_K(\wedge^{1,1}\mathfrak{p}, \theta(\sigma))$, and without loss of generality we can assume that it factors as

$$\xi : \wedge^{1,1}\mathfrak{p} \longrightarrow (\wedge^{1,1}\mathfrak{p})_{\mathfrak{5} \otimes \mathfrak{1}} \xrightarrow{\xi_0} [\theta(\sigma)]_{\mathfrak{5} \otimes \mathfrak{1}}$$

where the first arrow is projection onto the isotypical component, and the second arrow is an injection of K -modules. These assertions follow from Vogan-Zuckerman theory [35]: any such class will factor through $H^{1,1}(\mathfrak{g}, K; A_q)$ for an inclusion of the cohomological representation $A_q \rightarrow \theta(\sigma)$. But the minimal K -type in A_q is $\mathfrak{5} \otimes \mathfrak{1}$, with multiplicity one. According to the isomorphism in section 6.7 each $\varphi \in \mathcal{P}(\mathbb{C}^5)_{2,\alpha} = \mathcal{H}(K)_{\mathfrak{5} \otimes \mathfrak{1}}$ is equal to $\varphi^+(v)$ for a unique $v \in (\wedge^{1,1}\mathfrak{p})_{\mathfrak{5} \otimes \mathfrak{1}}$. Thus, for any $v \in (\wedge^{1,1}\mathfrak{p})_{\mathfrak{5} \otimes \mathfrak{1}}$ we can write, applying Howe's main result theorem 6.6,

$$\begin{aligned} \xi_0(v) &= \sum \theta_{\varphi_j}^{x_j, L_j}(f_j), \quad \varphi_j \in \mathcal{P}(\mathbb{C}^5)_{\mathfrak{5} \otimes \mathfrak{1}}, f_j \in \sigma \\ &= \sum \theta_{Z_j \varphi_j^0}^{x_j, L_j}(f_j), \quad Z_j \in \mathcal{U}(\mathfrak{g}'), \varphi_j^0 \in \mathcal{H}(K)_{\mathfrak{5} \otimes \mathfrak{1}} \\ &= \sum \theta_{\varphi_j^0}^{x_j, L_j}(Z_j^* f_j), \quad \text{by lemma 6.9} \\ &= \sum \theta_{\varphi^+(v_j)}^{x_j, L_j}(Z_j^* f_j), \quad v_j \in (\wedge^{1,1}\mathfrak{p})_{\mathfrak{5} \otimes \mathfrak{1}} \\ &= \sum \theta_{\varphi^+}^{x_j, L_j}(g_j)(v_j), \quad g_j = Z_j^* f_j \in \sigma \end{aligned}$$

where the last line interprets $\theta_{\varphi^+}(g_j)$ as a K -morphism $(\wedge^{1,1}\mathfrak{p})_{\mathfrak{5} \otimes \mathfrak{1}} \rightarrow \theta(\sigma)$. This calculation shows that the image of the K -morphism ξ_0 is contained in the sum of the images of the K -morphisms $\theta_{\varphi^+}(g_j)$ and since the source of these maps is an irreducible K -representation, Schur's lemma implies that we must have $\xi_0 = \sum c_j \theta_{\varphi^+}(g_j)$ for some $c_j \in \mathbb{C}$ or in other words, we have proved:

Proposition 6.11. *Any cohomology class $\xi \in H^{1,1}(X_\Gamma)$ in the complement of the Lefschetz class can be written as a finite linear combination of $\theta_{\varphi^+}^{x, L}(f)$ where $f \in \sigma$, and where σ is an irreducible automorphic representation belonging to the holomorphic cusp forms of weight $5/2$ for $\tilde{\text{SL}}_2(\mathbb{R})$, and φ^+ is the special element of Kudla-Millson.*

7. MAIN THEOREM

We can now show that the Picard groups of Siegel modular threefolds are generated by special cycles by combining proposition 6.11 with theorem 9.7. In the notations there, we have $p = 3$, $q = 2$, $m = p + q = 5$, $n = 1$, so that we are in the range $n < m/4$. In this case, the special theta lifting is from the holomorphic cusp forms of weight $5/2$ for $\tilde{\text{SL}}_2(\mathbb{Q})$ to harmonic $(1, 1)$ -forms on the manifold X_Γ .

Theorem 7.1. *For any subgroup of finite index $\Gamma \subset \text{Sp}_4(\mathbb{Z})$, $\text{Pic}(X_\Gamma) \otimes \mathbb{Q}$ is spanned by the classes of the special cycles.*

Proof. We already know that $\text{Pic}(X_\Gamma) \otimes \mathbb{C} = H^{1,1}(X_\Gamma, \mathbb{C})$. We also know that $H^{1,1}(X_\Gamma, \mathbb{C}) = \mathbb{C} \cdot \eta \oplus H^{1,1}(X_\Gamma)_0$, where η is the Lefschetz class and $H^{1,1}(X_\Gamma, \mathbb{C})_0$ is its canonical complement.

In fact, η is in the span of the Humbert surfaces; this follows from Yamazaki's formula, [42, Lemma 7]

$$10\eta = 2[E] + N[D]$$

for the principal congruence subgroup $\Gamma(N)$. Here E is the divisor which is the sum of the Humbert surfaces of discriminant 1, and D is the sum of the boundary components. This formula holds on the toroidal Igusa compactification of X_N . Since X_N is the complement of the divisor D , this shows that a multiple of η is in the span of the special cycles. Thus it is enough to see that any class in the canonical complement $\text{Pic}(X_\Gamma)_0 \otimes \mathbb{Q}$ is in the span of the special cycles. Evidently the special cycles generate a subvector space. We know that $\text{Pic}(X_\Gamma)_0 \otimes \mathbb{C} = H^{1,1}(X_\Gamma)_0$ and by proposition 6.11 these are generated by the special theta lifts of Kudla-Millson $\theta_{\varphi_+}^{x,L}(f)$ for various lattices L . According to theorem 9.7 these are in the span of special cycles. \square

8. SPECIAL CYCLES

8.1. The theory of special cycles has its origin in the discovery, by Hirzebruch and Zagier, of special curves on Hilbert modular surfaces and the connection of these with modular forms: the intersection numbers of these special curves appear as Fourier coefficients of modular forms. This was vastly generalized by two groups: Kudla and Millson, [19], [20], [21], [22]; and Tong and Wang, [34], [37], [38]. The symmetric spaces in question are those associated to one of three classes of groups: $O(p, q)$, $U(p, q)$ and $Sp(p, q)$.

8.2. We will follow the notations of the papers of Kudla-Millson. We explain their results only in the orthogonal case.

$G = O(p, q)$,	$V =$ the standard module for G
$D \cong \text{SO}_0(p, q)/(\text{SO}(p) \times \text{SO}(q))$,	the symmetric space for G
$(,) : V \times V \rightarrow \mathbb{R}$	a symmetric bilinear form, $\text{sgn} = (p, q)$
$L \subset V$	a \mathbb{Z} -lattice with $(L, L) \subset \mathbb{Z}$
$\Gamma \subset G$	torsion free congruence subgroup
$G' = \text{Sp}(2n, \mathbb{R})$	$\tilde{G}' = \text{Mp}(2n, \mathbb{R})$

We assume that Γ preserves the lattice L . We can assume $p \geq q$. Here n is an integer with $1 \leq n \leq p$.

$G_0 = \text{SO}_0(p, q)$ is the connected component of the identity in G .

It is also the set of elements of spinorial norm 1. We can identify the symmetric space D with $\text{Gr}_q^-(V)$, the subspace of the Grassmannian of q -planes Z such that $Z \mid (,)$ is negative definite. Let $V_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$. Define $X_\Gamma = \Gamma \backslash D$, a smooth manifold of dimension $a = pq$. In case $q = 2$ this carries a canonical complex structure, and is the set of \mathbb{C} -points of a Shimura variety; these are studied in detail in [18] in the anisotropic case. The relevant case for us is when $(p, q) = (3, 2)$. Then there is an isogeny $\text{Sp}_4(\mathbb{R}) \sim \text{SO}_0(3, 2)$ and D is isomorphic with the Siegel space of genus 2. See sections 10, and 11 for the dictionary to go between the symplectic and orthogonal viewpoints.

8.3. We are now going to define certain cycles on X_Γ . Let $U_\mathbb{Q} \subset V$ be an oriented subspace such that $(\ , \) \upharpoonright U$ is nondegenerate. Here we use the convention that suppression of an index such as \mathbb{Q} means the \mathbb{R} -span of the corresponding object; here $U = U_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{R}$.

Then we have a decomposition $V = U \oplus U^\perp$. We define

$$D_U = \{Z \in D : Z = Z \cap U + Z \cap U^\perp\} \subset D$$

and let G_U be the stabilizer of U in G and G_U^0 the connected component of the identity. Put $\Gamma_U = \Gamma \cap G_U$ and $\Gamma_U^0 = \Gamma \cap G_U^0$. Define $C_U = \Gamma_U^0 \backslash D_U$. The natural map $\pi : C_U \rightarrow \Gamma \backslash D$ is proper, and thus the pair (C_U, π) is a locally finite singular cycle in X_Γ . This will be orientable if Γ is a subgroup of $\mathrm{SO}_0(p, q)$, which will always be the case in our examples, since the Γ we work with will be the images of corresponding subgroups of $\mathrm{Sp}(4, \mathbb{R})$, which is the spin covering of $\mathrm{SO}_0(3, 2)$.

8.4. Given subspace U as above we define an involution $\tau = \tau_U$ via

$$\tau = \begin{cases} +1 & \text{on } U \\ -1 & \text{on } U^\perp \end{cases}.$$

Then G_U is the centralizer of τ and $D_U = \{Z : \tau Z = Z\}$. If $(\ , \) \upharpoonright U$ has signature (r, s) then $G_U \cong \mathrm{O}(r, s) \times \mathrm{O}(p-r, q-s)$ and D_U has codimension $ps + qr - 2rs$ in D . Most important for us is the case where the plane U is positive definite. In these cases

$$D_U = \{Z \in D : Z \subset U^\perp\}$$

and clearly $n = \dim U \leq p$. In general, when $q = 2$, the cycles C_U are algebraic cycles if U is positive definite; if U is not positive-definite, they are totally geodesic (singular) submanifolds of D . From now on, we only consider positive definite q -planes.

8.5. In our case, $(p, q) = (3, 2)$, and via the isomorphism $D \cong \mathfrak{H}_2$, D_U is an embedded copy of $\mathfrak{H}_1 \times \mathfrak{H}_1$, resp. \mathfrak{H}_1 , resp. a point, corresponding to $\dim U = 1$, resp. 2, resp. 3. When $\dim U = 1$, D_U is a Humbert surface (see section 11). For a good discussion of Humbert surfaces, see [7, Ch. IX].

8.6. We now define certain linear combinations of the cycles C_U . Let n be an integer in the range 1 to p . Let $X = \{x_1, \dots, x_n\} \subset V^n$; such an X is called an n -frame. Let (X, X) be the symmetric matrix given by $(X, X)_{ij} = (x_i, x_j)$. We consider the orbit $\mathcal{O} = G \cdot X$ in V^n . We call the orbit *nonsingular* if $\mathrm{rank}(X, X) = n$ and *nondegenerate* if $\mathrm{rank}(X, X) = \dim_\mathbb{R} \mathrm{span} X$. The zero orbit is nondegenerate but singular. It is known that the orbit \mathcal{O} is closed if and only if it is nondegenerate. If \mathcal{O} is a closed orbit, then by a theorem of Borel, $\mathcal{O} \cap L^n$ consists of a finite number of Γ -orbits. Letting Y_1, \dots, Y_l be a set of representatives of these orbits and $U_j = \mathrm{span} Y_j$ we define

$$C_\mathcal{O} = \sum_{j=1}^l C_{U_j}.$$

Given any symmetric $n \times n$ matrix β , define

$$\mathcal{Q}_\beta = \{X \in V^n : (X, X) = \beta\}.$$

If $X \in \mathcal{Q}_\beta$ then $\mathcal{O} = G \cdot X \subset \mathcal{Q}_\beta$. In case β is positive definite, G acts transitively on \mathcal{Q}_β , thus $\mathcal{O} = \mathcal{Q}_\beta$ and we may write C_β for $C_\mathcal{O}$. If β is positive semidefinite of rank $t < n$, \mathcal{Q}_β contains a unique closed orbit defined by

$$\mathcal{Q}_\beta^\circ = \{X \in \mathcal{Q}_\beta : \dim \mathrm{span} X = t\}$$

and it is known that G acts transitively on \mathcal{Q}_β^c . If β is positive semidefinite, we let C_β denote $\mathcal{C}_\mathcal{O}$ for this unique closed orbit.

In this paper, $(p, q) = (3, 2)$, $n = 1$, so that β is a positive rational number. Classically one referred to β as the discriminant of the Humbert surface.

8.7. As Kudla-Millson observe, these cycles are often zero for trivial reasons. By their conventions on orientations the frames (x_1, \dots, x_n) and $(-x_1, x_2, \dots, x_n)$ could both occur and give cancelling contributions. Therefore they fix a $h \in L^n$ and an integer $N \geq 1$ such that $\gamma \in \Gamma$ implies $\gamma \equiv 1 \pmod{N}$, and replace $\mathcal{Q}_\beta \cap L^n$ in the above definition with $\mathcal{Q}_\beta \cap (h + NL^n)$. Taking Γ -orbit representatives in this set then defines cycles $C_{\beta, h, N}$ as above. Note that Kudla-Millson simplify this notation to C_β in all their subsequent work.

9. THETA CORRESPONDENCE

9.1. Given a dual reductive pair (G, G') in the sense of Howe [12], the theta correspondence is a mapping between automorphic forms on G and automorphic forms on G' , and conversely. With $(G, G') = (\mathrm{O}(V), \mathrm{Sp}(2n, \mathbb{R}))$ the general set-up is as follows: Let $\mathcal{S}(V^n)$ be the Schwartz space of C^∞ complex-valued functions all of whose derivatives decrease rapidly to 0 at infinity. This carries a canonical action ω of $G \times \tilde{G}'$, where $\tilde{G}' \rightarrow G'$ is the metaplectic double cover. The first factor acts via:

$$\omega(g)\varphi(X) = \alpha(g)\varphi(g^{-1}X).$$

where $\alpha : G \rightarrow \mathbb{C}^*$ is the n th power of spinor norm. The second factor acts via the Weil, or oscillator, representation. Given any $\Gamma \times \tilde{\Gamma}'$ -invariant distribution ($\tilde{\Gamma}'$ the inverse image of Γ' in \tilde{G}') $\Theta : \mathcal{S}(V^n) \rightarrow \mathbb{C}$, and any $\varphi \in \mathcal{S}(V^n)$ one can form the kernel $\theta_\varphi(g, g') = \Theta(\omega(g)\omega(g')\varphi)$ then one defines

$$\begin{aligned} \theta_\varphi(f)(g') &= \int_{\Gamma \backslash G} f(g)\theta_\varphi(g, g')dg \\ \theta_\varphi(f')(g) &= \int_{\tilde{\Gamma}' \backslash \tilde{G}'} f'(g')\theta_\varphi(g, g')dg'. \end{aligned}$$

The most important case for us will be the distributions given by summing over a lattice in V^n . For instance, we can define, for $h \in L^n$, $N \in \mathbb{Z}$,

$$\Theta_{h, N}(\varphi) = \sum_{\substack{X \in L^n \\ X \equiv h \pmod{N}}} \varphi(X)$$

From now on we assume our distribution has this form. If f (resp. f') is a cusp form on G (resp. G') then $\theta_\varphi(f)$ (resp. $\theta_\varphi(f')$) will be an automorphic form on G' (resp. G) provided that φ is $K \times \tilde{K}'$ -finite where K (resp. \tilde{K}') is a maximal compact subgroup of G (resp. G').

9.2. In practice, φ will transform according to specific representations σ, σ' of K, \tilde{K}' . These representations define homogeneous vector bundles E_σ (resp. $E_{\sigma'}$) on the symmetric space D (resp. \mathfrak{H}_n), and we may interpret θ_φ as defining linear operators between spaces of sections:

$$\Gamma(X_\Gamma, E_\sigma) \rightarrow \Gamma(\Gamma' \backslash \mathfrak{H}_n, E_{\sigma'}), \quad \Gamma(\Gamma' \backslash \mathfrak{H}_n, E_{\sigma'}) \rightarrow \Gamma(X_\Gamma, E_\sigma)$$

The crucial case for us is when σ defines the bundle of differential forms of degree nq on D and σ' defines the line bundle \mathcal{L}_m whose holomorphic sections are the Siegel cusp forms of weight $m/2 = (p+q)/2$. It is a nontrivial fact that there exists a kernel $\theta_{\varphi+}$ which gives rise to a linear map

$$\Lambda : S_{m/2}(\Gamma') \rightarrow \mathbf{H}^{nq}(M_\Gamma)$$

where the left side above is the space of holomorphic cusp forms of weight $m/2$ on a congruence subgroup $\Gamma' \subset \tilde{G}' = \mathrm{Mp}(2n, \mathbb{R})$, and the right hand side is the space of closed harmonic nq forms on X_Γ . The element φ^+ is then a Schwartz function with values in differential forms on D .

9.3. Kudla and Millson present this construction in the following way. Let

$$\theta : H_{\mathrm{ct}}^*(G, \mathcal{S}(V^n)) \rightarrow H^*(X_\Gamma, \mathbb{C})$$

where the left hand side above is continuous cohomology, be the composite

$$H_{\mathrm{ct}}^*(G, \mathcal{S}(V^n)) \xrightarrow{\text{restriction}} H^*(\Gamma, \mathcal{S}(V^n)) \xrightarrow{\Theta} H^*(\Gamma, \mathbb{C}) = H^*(X_\Gamma, \mathbb{C}).$$

Here $\Theta : \mathcal{S}(V^n) \rightarrow \mathbb{C}$ is any Γ -invariant distribution. By the van Est theorem, the continuous cohomology is computed from the complex of G -invariant differential forms

$$C^i = (A^i(D) \otimes \mathcal{S}(V^n))^{G_0}$$

so that we can identify a continuous cohomology class with the class $[\varphi]$ of a closed differential form φ on D with values in $\mathcal{S}(V^n)$. Another way of understanding θ is the following: If Θ is a Γ -invariant distribution then $g \mapsto \theta(\omega(g)\varphi)$ is in $C^\infty(\Gamma \backslash G)$, for any $[\varphi] \in H_{\mathrm{ct}}^*(G, \mathcal{S}(V^n))$. Utilizing well-known isomorphisms (see [1]), θ is the composite

$$H_{\mathrm{ct}}^*(G, \mathcal{S}(V^n)) = H^*(\mathfrak{g}, \mathbb{K}, \mathcal{S}(V^n)) \xrightarrow{\varphi \mapsto \theta(\omega(g)\varphi)} H^*(\mathfrak{g}, \mathbb{K}, C^\infty(\Gamma \backslash G)) = H^*(X_\Gamma, \mathbb{C})$$

Kudla and Millson construct a pairing

$$((,)) : H_c^i(X_\Gamma, \mathbb{C}) \times H_{\mathrm{ct}}^{a-i}(G, \mathcal{S}(V^n)) \longrightarrow C^\infty(\tilde{G}'), \quad a = \dim X_\Gamma$$

via

$$\theta_\varphi(\eta)(g') := ((\eta, \varphi))(g') = \int_{X_\Gamma} \eta \wedge \theta((\omega(g')\varphi)$$

where η is represented by a closed compactly supported i -form and $\theta((\omega(g')\varphi)$ by a closed $(a-i)$ -form. One of the main results of [22] is that, with suitable restriction on φ , the image of this is in the holomorphic sections in $\Gamma(\mathcal{L}_m)$ the space of Siegel modular forms of weight $m/2$ on \tilde{G}' . The relevant φ define classes in an space they denote

$$H_{\mathrm{ct}}^{a-i}(G, \mathcal{S}(V^n))_{\chi_m}^q$$

whose precise definition can be found in the introduction of [22]. Thus they obtain a pairing:

$$((,)) : H_c^i(M_\Gamma, \mathbb{C}) \times H_{\mathrm{ct}}^{a-i}(G, \mathcal{S}(V^n))_{\chi_m}^q \longrightarrow \Gamma(\mathcal{L}_m).$$

Kudla and Millson construct a canonical element $\varphi^+ = \varphi_{nq}^+ \in H_{\mathrm{ct}}^{nq}(G, \mathcal{S}(V^n))_{\chi_m}^q$ such that, for any $\eta \in H_c^i(X_\Gamma, \mathbb{C})$, the Fourier coefficients of the Siegel modular form $\theta_\varphi(\eta)(\tau)$ are essentially given by the periods of η over the special cycles. Recall that the Fourier expansion is given by ($\tau = u + iv$)

$$\theta_{\varphi^+}(u + iv) = \sum_{\beta \in \mathcal{L}} a_\beta(v) \exp(2\pi i \mathrm{Tr}(\beta u))$$

the sum ranging over a lattice \mathcal{L} in the space of symmetric matrices of size n with \mathbb{Q} -coefficients.

They prove:

Theorem 9.4. (i) *The induced pairing*

$$((\ , \)) : \mathbb{H}_c^i(X_\Gamma, \mathbb{C}) \times \mathbb{H}_{\text{ct}}^{\mathfrak{a}-i}(G, \mathcal{S}(V^n))_{\chi_m}^q \longrightarrow \Gamma(\mathcal{L}_m)$$

takes values in the holomorphic sections.

(ii) *If $\eta \in \mathbb{H}_c^i(X_\Gamma, \mathbb{C})$ and $\varphi \in \mathbb{H}_{\text{ct}}^{\mathfrak{a}-i}(G, \mathcal{S}(V^n))_{\chi_m}^q$ then all the Fourier coefficients a_β of $\theta_\varphi(\eta)(g')$ are zero except the positive semi-definite ones. Suppose further that φ takes values in $\mathbf{S}(V^n)$, the polynomial Fock space (see [22, Intro.] for the definition) then these Fourier coefficients are expressible in terms of periods over the special cycles C_β . For the canonical class $\varphi^+ = \varphi_{nq}^+$, $i = nq$, and for positive definite β one has*

$$a_\beta(\theta_{\varphi^+}(\eta))(v) = e^{-2\pi\text{Tr}(\beta v)} \int_{C_\beta} \eta$$

9.5. Recall the pairing

$$((\ , \)) : \mathbb{H}_c^i(M_\Gamma, \mathbb{C}) \times \mathbb{H}_{\text{ct}}^{\mathfrak{a}-i}(G, \mathcal{S}(V))_{\chi_m}^q \longrightarrow \Gamma(\mathcal{L}_m).$$

The line bundle \mathcal{L}_m is on the space $\Gamma' \backslash \mathfrak{H}_n$ for some congruence subgroup Γ' . The holomorphic sections of this bundle is the space $S_{m/2}(\Gamma')$ of Siegel cusp forms of weight $m/2$. Let $f \in S_{m/2}(\Gamma')$ be such a cusp form. For any $\varphi \in \mathbb{H}_{\text{ct}}^{nq}(G, \mathcal{S}(V))_{\chi_m}^q$ we get a linear functional

$$\eta \mapsto \int_{\Gamma' \backslash G'} \theta_\varphi(\eta)(g') \bar{f}(g') dg' : \mathbb{H}_c^{(p-n)q}(X_\Gamma, \mathbb{C}) \rightarrow \mathbb{C}, \quad \theta_\varphi(\eta) := ((\eta, \varphi))$$

which is essentially the Petersson inner product. By the perfect pairing given by Poincaré duality,

$$\mathbb{H}^{nq}(M_\Gamma) \times \mathbb{H}_c^{(p-n)q}(M_\Gamma) \longrightarrow \mathbb{C}$$

this linear form is identified with a class $\theta_\varphi(f) \in \mathbb{H}^{nq}(X_\Gamma, \mathbb{C})$. By construction:

$$[\theta_\varphi(\eta), f] := \int_{\Gamma' \backslash G'} \theta_\varphi(\eta) \bar{f} = \int_{X_\Gamma} \eta \wedge \theta_\varphi(f) := (\eta, \theta_\varphi(f)).$$

The map $f \mapsto$ class of $\theta_\varphi(f)$ is the theta lifting $\Lambda_\varphi : S_{m/2}(\Gamma') \rightarrow \mathbb{H}^{nq}(M_\Gamma)$. We denote this simply by Λ for the canonical $\varphi = \varphi = \varphi_{nq}^+$.

9.6. Let $\mathbf{H}_\theta^\perp \subset \mathbb{H}_c^{(p-n)q}(X_\Gamma)$ be the subspace of all classes of closed compactly supported $(p-n)q$ forms that are orthogonal under the pairing $[\ , \]$ to the image of $S_{m/2}(\Gamma')$ under Λ . Let $\mathbf{H}_{\text{cycle}}^\perp \subset \mathbb{H}_c^{(p-n)q}(X_\Gamma)$ be the space of all classes of closed compactly supported $(p-n)q$ forms that have period 0 over all the special cycles C_β with $\beta > 0$ (positive-definite). Let $\mathbf{H}_\theta, \mathbf{H}_{\text{cycle}}$ be the subspaces of $\mathbb{H}^{nq}(X_\Gamma)$ defined by these by Poincaré duality. They prove:

Theorem 9.7. [21, Theorem 4.2] *If $n < m/4$ then $\mathbf{H}_\theta = \mathbf{H}_{\text{cycle}}$.*

Thus by Poincaré duality, in the case of finite volume but noncompact quotients the subspace of $\mathbb{H}^{nq}(X_\Gamma)$ spanned by the duals of the special cycles coincides with the space of theta lifts. In the above theorem the special lifting kernel φ^+ is used. It is important to realize that there is also an initial choice of a theta distribution of the type

$$\Theta_{h,N}(\varphi) = \sum_{\substack{X \in L^n \\ X \equiv h \pmod{N}}} \varphi(X).$$

This choice appears both in the definition of the kernel θ_{φ^+} and in the definition of the special cycles C_β . This distribution is invariant under the arithmetic subgroup $\Gamma \times \Gamma'$. The

above isomorphism should more properly be written as $\mathbf{H}_\theta^{h,N} = \mathbf{H}_{\text{cycle}}^{h,N}$. In our application, $(p, q) = (3, 2)$, $n = 1$, $m = p + q = 5$, so we are in this stable range. Eventually we take the limit over all (h, N) .

10. THE ISOGENY $\text{Sp}(4, \mathbb{R}) \sim \text{SO}(3, 2)$

10.1. Let $V_{\mathbb{Q}} = \mathbb{Q}^4$ with standard basis e_i , $i = 1, \dots, 4$ and let Ψ be the alternating bilinear form

$$\langle x, y \rangle = \langle x, y \rangle_{\Psi} = {}^t x \Psi y, \text{ where } \Psi = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$$

The group of symplectic similitudes is defined as:

$$\text{GSp}(V_{\mathbb{Q}}, \Psi) = \text{GSp}(\Psi) = \text{GSp}(4) = \{g \in \text{Mat}(4, 4; \mathbb{Q}) \mid {}^t g \Psi g = \eta(g) \Psi\}$$

where $\eta(g) \in \mathbb{Q}^*$, called the multiplier of g . The map $g \rightarrow \eta(g)$ is a homomorphism $\eta : \text{GSp}(V_{\mathbb{Q}}, \Psi) \rightarrow \mathbb{Q}^*$ whose kernel is the symplectic group $\text{Sp}(V_{\mathbb{Q}}, \Psi)$. We can regard $\text{GSp}(\Psi)$ as defining a group scheme over \mathbb{Z} , whose points in any ring R , denoted $\text{GSp}(\Psi, R)$ or $\text{GSp}(4, R)$, is defined by the same formulas as above, but with matrix entries in R , with $\eta(g) \in R^*$ being a unit. The same definitions can be given for any nondegenerate alternating bilinear form, but recall that (over a field) these are all equivalent by a change of coordinates.

10.2. Let $V_{\mathbb{Z}}$ be the free \mathbb{Z} -module with basis e_1, e_2, e_3, e_4 . We have a symmetric bilinear form

$$b : \bigwedge^2 V_{\mathbb{Z}} \times \bigwedge^2 V_{\mathbb{Z}} \rightarrow \bigwedge^4 V_{\mathbb{Z}} \xrightarrow{\det} \mathbb{Z}$$

where the first arrow is wedge product and the second is the isomorphism defined by $\det(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$. Clearly, the natural action of $\text{GL}(4)$ on $\bigwedge^2 V$ preserves this quadratic form up to scalars, namely the determinant homomorphism. The subgroup $\text{GSp}(4)$ stabilizes the line spanned by $\psi = e_1 \wedge e_3 + e_2 \wedge e_4$ and thus we have a representation on the orthogonal complement relative to b :

$$\alpha : \text{GSp}(4) \rightarrow \text{GO}_0(\psi^\perp, b \mid \psi^\perp) := \text{GO}_0(b_\psi),$$

where the group $\text{GO}_0(b_\psi)$ is the connected component of the group of orthogonal similitudes. Restricting this to the subgroup $\text{Sp}(4)$, one knows:

Proposition 10.3. *α induces an isomorphism*

$$\alpha : \text{Sp}(4)/\{\pm 1\} \rightarrow \text{SO}_0(b_\psi)$$

One checks that the 5-dimensional quadratic form b_ψ has signature $(3, 2)$.

10.4. Let $\text{Skew}(4, \mathbb{Q})$ be the space of skew-symmetric matrices of size 4 with entries in \mathbb{Q} . There is a natural action of $\text{GL}(4, \mathbb{Q})$ on this space by $M \rightarrow g.M.{}^t g$. Let

$$\Psi = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \in \text{Skew}(4, \mathbb{Q}).$$

The stabilizer of Ψ for this action is $\text{Sp}(4, \mathbb{Q})$. One can check that the symmetric bilinear form on $\text{Skew}(4, \mathbb{Q})$ defined by

$$b_0(M, N) := \frac{1}{2} \text{Tr}(M \Psi N \Psi) - \frac{1}{4} \text{Tr}(M \Psi) \text{Tr}(N \Psi)$$

is invariant under all $M \rightarrow g.M.^t g$ for $g \in \mathrm{Sp}(4, \mathbb{Q})$. It is also \mathbb{Z} -valued on $\mathrm{Skew}(4, \mathbb{Z})$. The space

$$\Psi^\perp := \{M \in \mathrm{Skew}(4, \mathbb{Q}) : b_0(M, \Psi) = 0\}$$

is 5-dimensional and invariant under $\mathrm{Sp}(4, \mathbb{Q})$. We therefore obtain a morphism of algebraic groups $\mathrm{Sp}(4) \rightarrow \mathrm{O}(\Psi^\perp)$. This necessarily lands in the connected component $\mathrm{SO}_0(\Psi^\perp)$ since $\mathrm{Sp}(4)$ is connected. It is well-known that this is an isogeny with kernel ± 1 . The signature of the form on Ψ^\perp is $(3, 2)$.

10.5. Given $\eta = \sum_{i < j} r_{ij} e_i \wedge e_j \in \bigwedge^2 V_{\mathbb{Q}}$, we can associate the skew-symmetric matrix $R_\eta = R = (r_{ij}) \in \mathrm{Skew}(4, \mathbb{Q})$, $r_{ij} = -r_{ji}$. This assignment sets up a $\mathrm{GL}(4, \mathbb{Q})$ -equivariant isomorphism

$$\bigwedge^2 V \cong \mathrm{Skew}(4),$$

with the action of $g \in \mathrm{GL}(4, \mathbb{Q})$ given on the skew-symmetric matrices as

$$R \rightarrow gR.^t g.$$

The form ψ above maps to Ψ . Under this isomorphism the form b in section 10.2 goes over into the form b_0 of section 10.4, ie., $b(\xi, \eta) = b_0(R_\xi, R_\eta)$. In coordinates,

$$b_0(M, N) = m_{12}n_{34} + m_{34}n_{12} + m_{23}n_{14} + m_{14}n_{23} - m_{24}n_{13} - m_{13}n_{24}.$$

Ψ^\perp is defined by $m_{13} + m_{24} = 0$, ψ^\perp is defined by $r_{13} + r_{24} = 0$ and these bilinear forms restrict to this subspace as:

$$b_0(M, N) = 2m_{13}n_{13} + m_{12}n_{34} + m_{34}n_{12} + m_{14}n_{23} + m_{23}n_{14}.$$

10.6. Here is a geometric description of this. Consider, for any ring R , the module $V_R = R^4$ with the standard alternating form Ψ . Viewed this way, V is the affine space \mathbf{A}^4 regarded as a scheme over the integers. It is known that the Grassmannian of 2-planes through the origin $\mathbf{Gr}_2(V) = \mathbf{Gr}(2, 4)$, or equivalently the Grassmannian of lines in \mathbf{P}^3 , $\mathbf{Gr}_1(\mathbf{P}(V)) = \mathbf{PGr}(1, 3)$ is canonically imbedded

$$\mathbf{Gr}_2(V) = \mathbf{Gr}(2, 4) \longrightarrow \mathbf{P} \left(\bigwedge^2 V \right) = \mathbf{P}^5$$

whose image is a quadric. In coordinates (x_1, x_2, x_3, x_4) on \mathbf{A}^4 , a plane $h = x \wedge y$ is mapped to the vector of 2 by 2 minors

$$p_{ij}(h) = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

The quadric is given by Plücker's relation

$$Q(p) = p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$$

The alternating form Ψ is given by the formula $p_{13} + p_{24}$, therefore a 2-plane through the origin \mathbf{A}^4 is isotropic for this skew-form iff $p_{13}(h) + p_{24}(h) = 0$. In other words, the variety of isotropic lines in \mathbf{P}^3 is given by these two equations. We can think of this scheme $\mathbf{Gr}_2(V, \Psi)$ as the 3-dimensional quadric in $[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}]$ -space given by

$$q(p) = p_{13}^2 + p_{12}p_{34} + p_{14}p_{23} = 0$$

This variety is the dual symmetric space of the group $G = \mathrm{Sp}(\Psi)$ and is thus isomorphic with G/Q where Q is the parabolic subgroup stabilizing any isotropic 2-plane. In any ring where 2 is a unit this is easily seen to be equivalent to the form $u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2$, of signature $(3, 2)$.

10.7. Now $\mathrm{GSp}(V, \Psi)$ operates on V fixing Ψ up to scalar multiple, hence it acts on $\wedge^2 V$ fixing the subvarieties $Q(p) = 0$, $p_{13} + p_{24} = 0$, and hence fixing the variety $q(p) = 0$. We therefore obtain a homomorphism from the symplectic similitudes to the orthogonal similitudes of q , but as $\mathrm{GSp}(\Psi)$ is connected, we get a map

$$\mathrm{GSp}(V, \Psi) \longrightarrow \mathrm{GO}_0(q)$$

which is an isogeny. The quadratic form q is essentially $b_{0,\psi}$ above. More precisely, consider the wedge product

$$\bigwedge^2 V \times \bigwedge^2 V \longrightarrow \bigwedge^4 V \simeq \mathbf{G}_a$$

where the last isomorphism is gotten by the determinant. Choose any basis $\{e_i\}$ of $V_{\mathbb{Z}} = \mathbb{Z}^4$. Then it is easily seen that the quadratic form associated to this pairing is

$$2Q(a) = 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})$$

Consider $\psi = e_1 \wedge e_3 + e_2 \wedge e_4$. Then the orthogonal $U = \psi^\perp$ carries the induced form $2(b^2 - ac - de)$ in the basis

$$\{f_1, f_2, f_3, f_4, f_5\} = \{e_1 \wedge e_2, e_1 \wedge e_3 - e_2 \wedge e_4, -e_3 \wedge e_4, -e_1 \wedge e_4, e_2 \wedge e_3\}$$

For our purposes it is better to work with the form on the dual lattice spanned by $\{f_1, (f_2)/2, f_3, f_4, f_5\}$, or rather twice it, given in this basis as

$$\Delta(a, b, c, d, e) = b^2 - 4ac - 4de$$

This is the quadratic form that appears in the theory of Humbert surfaces.

11. SYMMETRIC SPACES AND HUMBERT SURFACES

11.1. Symplectic viewpoint.

11.1.1. The underlying analytic space A of a complex abelian variety over \mathbb{C} of dimension n is a quotient $A = \mathbb{C}^n/L$ where $L \subset \mathbb{C}^n$ is a lattice. Thus $L \otimes \mathbb{R} := V = \mathbb{C}^n$ has a complex structure J ($J^2 = -1$). We have canonically $L = H^1(A, \mathbb{Z})$, and $H^s(A, \mathbb{Z}) = \mathrm{Hom}(\wedge^s L, \mathbb{Z})$ for all s . There is a Riemann form for A , or polarization, ie., an alternating bilinear form $\psi : L \times L \rightarrow \mathbb{Z}$ such that

1. $\psi(Ju, Jv) = \psi(u, v)$ for all $u, v \in V$.
2. $\psi(v, Jv) > 0$ for all $0 \neq v \in V$.

The first condition for a polarization is that $\psi \in H^2(A, \mathbb{Z}) \cap H^{1,1}(A)$ in the Hodge structure on cohomology. Recall that the existence of a complex structure on V is equivalent to the existence of a Hodge structure with

$$V_{\mathbb{C}} = V \otimes \mathbb{C} = V^{-1,0} \oplus V^{0,-1}$$

where $V^{-1,0}$ (resp. $V^{0,-1}$) is the $+i$ (resp. $-i$) eigenspace for J . As is well known, the cohomology of A then has a \mathbb{Z} -Hodge structure, with $H^1(A, \mathbb{R})$ being the dual \check{V} . There is a canonical isomorphism

$$A = H_1(A, \mathbb{Z}) \backslash H_1(A, \mathbb{C})/F^0 = H_1(A, \mathbb{Z}) \backslash V^{-1,0}.$$

The holomorphic tangent space at 0 is canonically identified $T_0A = H_1(A, \mathbb{C})/F^0 = V^{-1,0}$. In coordinates z_1, \dots, z_n we get a basis $\partial/\partial z_1, \dots, \partial/\partial z_n$ and thus a dual basis dz_1, \dots, dz_n of $T_0^*A = H^0(A, \Omega_{A/\mathbb{C}}) = \check{V}^{1,0} = F^0(H^1(A, \mathbb{C}))$. The natural map $L = H_1(A, \mathbb{Z}) \rightarrow H^1(A, \mathbb{R}) \cong \text{Hom}_{\mathbb{C}}(H^0(A, \Omega_{A/\mathbb{C}}), \mathbb{C})$ sends γ to the functional $\omega \mapsto \int_{\gamma} \omega$, so that L is the lattice of periods.

$$\begin{array}{ccc} L = H_1(A, \mathbb{Z}) & & \\ \downarrow & & \\ V = H_1(A, \mathbb{R}) & \xrightarrow{\sim} & \text{Hom}_{\mathbb{C}}(F^0 H^1, \mathbb{C}) \\ \downarrow & & \uparrow \\ V_{\mathbb{C}} = H_1(A, \mathbb{C}) & \longrightarrow & \text{Hom}_{\mathbb{C}}(H^1(A, \mathbb{C}), \mathbb{C}) \end{array}$$

11.1.2. Let ψ be a principal polarization, i.e., ψ induces an isomorphism $L \rightarrow \check{L} = \text{Hom}(L, \mathbb{Z})$. We may then find a basis e_1, \dots, e_{2n} of L such that $\psi(e_i, e_j) = 0$ unless $|i - j| = n$, and $\psi(e_i, e_{n+i}) = 1$, for all $i = 1, \dots, n$. It is also well-known that we may find a basis $\omega_1, \dots, \omega_n$ for $H^0(A, \Omega_{A/\mathbb{C}})$ such that the $n \times 2n$ period matrix $\int_{e_j} \omega_i$ has the shape $(\tau, 1_n)$ for some $\tau \in \mathfrak{H}_n$. Since we have $\omega_i = \sum_{j=1}^{2n} (\int_{e_j} \omega_i) \check{e}_j$, once we fix the symplectic basis e_i we can regard the row span F_{τ} of the matrix $(\tau, 1_n)$ as the subspace

$$F_{\tau} = H^0(A, \Omega_{A/\mathbb{C}}) \subset H^1(A, \mathbb{C}) = \mathbb{C}^n.$$

Thus there is an isomorphism

$$\tau \mapsto F_{\tau} : \mathfrak{H}_n \simeq \text{Gr}_n^+(\check{V}_{\mathbb{C}}),$$

where $\text{Gr}_n^+(\check{V}_{\mathbb{C}})$ is the Grassmannian of n -dimensional complex subspaces $F \subset \check{V}_{\mathbb{C}}$ such that

- a) F is $\check{\psi}$ -isotropic, i.e., $\check{\psi}(x, y) = 0$ for all $x, y \in F$.
- b) $-i\check{\psi}(x, \bar{x}) > 0$ for all $0 \neq x \in F$, where \bar{x} denotes conjugation relative to $V_{\mathbb{R}}$.

Here, $\check{\psi}$ is the dual alternating form on \check{V} . Then F_{τ} is $F^0 H^1(A_{\tau}, \mathbb{C})$ for the Hodge structure on the principally polarized abelian variety $A_{\tau} = \mathbb{C}^n / L_{\tau}$, where $L_{\tau} = \mathbb{Z}^n + \mathbb{Z}^n \tau$.

11.1.3. Now we specialize the preceding to the case $n = 2$. Let $T_{\mathbb{Z}} = \bigwedge^2 \check{V}_{\mathbb{Z}}$, Then $\tau \in \mathfrak{H}_2$ defines a Hodge structure of dimension 6

$$(T_{\mathbb{C}}, T_{\mathbb{Z}}, F_{\tau}) \text{ of type } (2, 0) + (1, 1) + (0, 2).$$

such that $T_{\mathbb{Z}} = H^2(A_{\tau}, \mathbb{Z})$. Note that a real form $\eta \in \bigwedge^2 \check{V}_{\mathbb{R}}$ has type $(1, 1)$ if and only if $\eta(u, v) = 0$ for all $u, v \in F_{\tau}$. Since F_{τ} is generated by the rows of $(\tau, 1)$ we see that a real form η is of type $(1, 1)$ if and only if (notation as in section 10.5)

$$(\tau, 1) R_{\eta} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0.$$

Since the Néron-Severi group of A_{τ} is isomorphic with $H^2(A_{\tau}, \mathbb{Z}) \cap H^{1,1}(A_{\tau})$ we see

$$\text{NS}(A_{\tau}) \cong \{R \in \text{Skew}(4, \mathbb{Z}) : (\tau, 1) R \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0\}.$$

Now let $\text{Skew}(4, R)_0 := \Psi^{\perp} \subset \text{Skew}(4, R)$, for any commutative ring $R \subset \mathbb{C}$, where the orthogonal is taken with respect to the inner product in section 10.4.

We define, for any $X \in \text{Skew}(4, \mathbb{R})_0$

$$\mathfrak{H}_2 \supset \mathfrak{H}_X := \{\tau \in \mathfrak{H}_2 : (\tau, 1)X \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0\}.$$

Clearly $\mathfrak{H}_X = \mathfrak{H}_{tX}$ for any $t \in \mathbb{R}^*$. It can be shown that this set is nonempty if and only if $\Delta(X) > 0$, in which case it is isomorphic with $\mathfrak{H}_1 \times \mathfrak{H}_1$. When the entries of X are integers without common divisor, we call this the Humbert surface associated to X , provided it is nonempty, and we call $\Delta(X)$ the discriminant of the Humbert surface. It is a positive integer congruent to 0 or 1 modulo 4.

Dually we define, for any $\tau \in \mathfrak{H}_2$, and any subring $A \subset \mathbb{C}$,

$$T_A^{1,1}(\tau) = \{X \in \text{Skew}(4, A) : (\tau, 1)X \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0\}.$$

11.2. Orthogonal viewpoint.

11.2.1. The symmetric space for the group $\text{SO}(3, 2)$ is

$$\text{Pos}_{3,2} = \{Z \in M_{5,3}(\mathbb{R}) : {}^t M I_{3,2} M > 0\}, \quad I_{3,2} = \begin{pmatrix} 1_3 & 0 \\ 0 & -1_2 \end{pmatrix}$$

More generally, replacing the split form $I_{3,2}$ by any real symmetric B of signature $(3, 2)$, the condition is that the real symmetric 3 by 3 matrix ${}^t M B M$ is positive definite. More intrinsically, fix a 5-dimensional real vector space $T_{\mathbb{R}}$ with a symmetric bilinear form b of signature $(3, 2)$. Then the symmetric space is the open subset of the Grassmannian of 3-planes in $T_{\mathbb{R}}$:

$$\text{Gr}_3^+(T_{\mathbb{R}}) := \{U \subset T_{\mathbb{R}} : \dim U = 3, b|_U > 0\}$$

By choosing an orthonormal basis we can identify $T_{\mathbb{R}} = \mathbb{R}^5$, and $\text{Pos}_{3,2}$ with $\text{Gr}_3^+(T_{\mathbb{R}})$ by assigning to M the subspace of \mathbb{R}^5 spanned by the rows of M .

11.2.2. For any commutative ring R let $V_R = R^4$ with basis $\underline{e} = \{e_1, e_2, e_3, e_4\}$. Let $W = \text{Hom}(V, \mathbf{G}_a)$ be the dual with dual coordinates \check{e}_i . Let $\psi \in \wedge^2(W) = \text{Hom}(\wedge^2(V), \mathbf{G}_a)$ be the alternating bilinear form whose matrix in the coordinates \underline{e} is

$$\Psi = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}.$$

Each point $\tau \in \mathfrak{H}_2$ determines a Hodge structure of type $(1, 0), (0, 1)$ on W , ie.,

$$W^\tau = (W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1} \supset W_{\mathbb{R}} \supset W_{\mathbb{Z}}.)$$

Concretely, $W^{0,1} = F_\tau^0(W_{\mathbb{C}})$ is the span of the rows of the matrix $(\tau, 1_2)$. This is the canonical Hodge structure on $\text{H}^1(A_\tau)$ for the 2-dimensional abelian variety $A_\tau = \mathbb{C}^2 / \mathbb{Z}^2 \tau + \mathbb{Z}^2$. The form ψ can be identified to an element of $\text{H}^2(A_\tau, \mathbb{Z}) \cap \text{H}^{1,1}(A_\tau)$, and is a principal polarization.

11.2.3. Each $\tau \in \mathfrak{H}_2$ thus gives a Hodge structure on any tensor space of W . In particular, consider $\wedge^2(W)$. Since $\psi \in \wedge^2(W)$ is of type $(1, 1)$ for this Hodge structure, and since the bilinear form of section 10.2 is a morphism of Hodge structures, the orthogonal space $T := \psi^\perp \subset \wedge^2(W)$ carries a \mathbb{Z} -Hodge structure. We have seen that the bilinear form denoted b_ψ on $T_{\mathbb{R}}$ has signature $(3, 2)$. Thus, any $\tau \in \mathfrak{H}_2$ gives a Hodge structure

$$T^\tau = (T_{\mathbb{C}} = T^{2,0} \oplus T^{1,1} \oplus T^{0,2} \supset T_{\mathbb{R}} \supset T_{\mathbb{Z}})$$

The space $T^{1,1}$ is the complexification of a real 3-dimensional subspace $Z_\tau \subset T_{\mathbb{R}}$ and it is known that $b_\psi|_{Z_\tau}$ is positive-definite.

Proposition 11.3. *The map $\tau \rightarrow Z_\tau$ sets up an isomorphism*

$$\mathfrak{H}_2 \simeq \mathrm{Gr}_3^+(T_{\mathbb{R}})$$

This map is equivariant, via the isogeny $\rho : \mathrm{Sp}_4(\mathbb{R}) \rightarrow \mathrm{SO}_0(b_\psi) = \mathrm{SO}_0(3, 2) : Z_{g\tau} = \rho(g)Z_\tau$.

Note that we have a canonical equivariant isomorphism $\mathrm{Gr}_3^+(T_{\mathbb{R}}) = \mathrm{Gr}_2^-(T_{\mathbb{R}})$ where the right-hand side is the Grassmannian on 2-planes Z' in $T_{\mathbb{R}}$ such that $b_\psi \upharpoonright Z'$ is negative-definite: let $Z' = Z^\perp$.

11.3.1. In section 8 the letter V represents a 5-dimensional real vector space with a quadratic form $(\ , \)$ of signature $(3, 2)$, which corresponds to $T_{\mathbb{R}}$ here, and the lattice L in that section corresponds to $T_{\mathbb{Z}}$. Also in the notations of that section, $D = \mathrm{Gr}_2^-(T_{\mathbb{R}})$. For any $x \in L = T_{\mathbb{Z}}$ with $(x, x) > 0$, let $U = \mathbb{R}x \subset V = T_{\mathbb{R}}$. Under the natural identification $x \mapsto X : T_{\mathbb{Z}} = \mathrm{Skew}(4, \mathbb{Z})_0$ (see sections 10.4, 11.1.3), the Humbert surface \mathfrak{H}_X maps to the special locus D_U of section 8. This is because:

$$\begin{aligned} Z' \in D_U &\iff Z' \subset U^\perp \\ &\iff U \subset (Z')^\perp := Z \\ &\iff U \subset Z_\tau \text{ for a unique } \tau \in \mathfrak{H}_2 \\ &\iff x \text{ is of type } (1, 1) \text{ in the Hodge structure } T^\tau \\ &\iff (\tau, 1)X \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0 \\ &\iff \tau \in \mathfrak{H}_X. \end{aligned}$$

12. COHOMOLOGICAL UNITARY REPRESENTATIONS

This section records the basic facts relevant to us from Vogan-Zuckerman theory. These results are well-known in that they have appeared in print on multiple occasions, but with no proofs. We do not provide complete proofs either, but at least some more detail. It is convenient for us to work with the orthogonal as opposed to the symplectic viewpoint. So $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{R}) = \mathfrak{so}(3, 2)$.

12.1.

$$\mathfrak{so}(3, 2) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A = -{}^t A \in \mathrm{M}_{3,3}(\mathbb{R}), D = -{}^t D \in \mathrm{M}_{2,2}(\mathbb{R}), B \in \mathrm{M}_{3,2}(\mathbb{R}), C = {}^t B \right\}$$

The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ has $\mathfrak{k} = \{B = C = 0\} = \mathfrak{so}(3) \times \mathfrak{so}(2)$, and

$$\mathfrak{p} = \{A = D = 0\} \cong \mathrm{M}_{3,2}(\mathbb{R}) \quad \text{via} \quad \begin{pmatrix} 0 & B \\ {}^t B & 0 \end{pmatrix} \mapsto B.$$

The action of $(A, D) \in \mathrm{O}(3) \times \mathrm{O}(2)$ by conjugation on $\mathfrak{p} = \mathfrak{g}/\mathfrak{k} = \mathrm{M}_{3,2}(\mathbb{R})$ is

$$X \mapsto AXD^{-1}.$$

The complex structure on $\mathfrak{p} = \mathrm{M}_{3,2}(\mathbb{R})$ is given by right multiplication by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{so that } \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-, \mathfrak{p}^\pm = \pm i \text{ eigenspace of } J.$$

12.2. A compact maximal torus for \mathfrak{g} is

$$\mathfrak{t} = \left\{ \begin{pmatrix} x_1 J & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 J \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\} = \{[x_1, x_2]\}.$$

The roots are

$$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}) = \{\pm\alpha, \pm\beta, \pm\alpha \pm \beta\}$$

where $\alpha([x_1, x_2]) = i x_1$, $\beta([x_1, x_2]) = i x_2$. We have

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha}$$

$$\mathfrak{p}^{+} = \mathfrak{g}_{\beta} \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{-\alpha+\beta}$$

$$\mathfrak{p}^{-} = \mathfrak{g}_{-\beta} \oplus \mathfrak{g}_{\alpha-\beta} \oplus \mathfrak{g}_{-\alpha-\beta}$$

Writing $\mathfrak{g}_{\gamma} = \mathbb{C}X_{\gamma}$, we have:

$$X_{\alpha} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ -1 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad X_{\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & -1 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$X_{\alpha+\beta} = \begin{pmatrix} 0 & 0 & 0 & -i & 1 \\ 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 & 0 \\ -i & 1 & 0 & 0 & 0 \\ 1 & i & 0 & 0 & 0 \end{pmatrix} \quad X_{\alpha-\beta} = \begin{pmatrix} 0 & 0 & 0 & i & 1 \\ 0 & 0 & 0 & -1 & i \\ 0 & 0 & 0 & 0 & 0 \\ i & -1 & 0 & 0 & 0 \\ 1 & i & 0 & 0 & 0 \end{pmatrix}$$

The map $i \mapsto -i$ sends root/spaces $\gamma \rightarrow -\gamma$.

12.3. The unitary representations with nonzero (\mathfrak{g}, K) -cohomology are of the form $A_{\mathfrak{q}}$ for θ -stable parabolic subalgebras $\mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$ (more generally $A_{\mathfrak{q}}(\lambda)$ for coefficients in a local system). These parabolics can be taken up to K -conjugation. Each such \mathfrak{q} can be constructed by choosing a $x \in \mathfrak{t}$ and defining

\mathfrak{q} = sum of the nonnegative eigenspaces of $\text{ad}(x)$.

\mathfrak{l} = sum of the zero eigenspaces of $\text{ad}(x)$.

\mathfrak{u} = sum of the positive eigenspaces of $\text{ad}(x)$.

Then we have a Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Also, if $R^{\pm} = \dim(\mathfrak{u} \cap \mathfrak{p}^{\pm})$ and $p - R^{+} = q - R^{-} = j \geq 0$

$$H^{p,q}(\mathfrak{g}, K; \mathbb{C}) = \text{Hom}_{L \cap K}(\wedge^{2j}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}).$$

If $\mathfrak{f} \subset \mathfrak{q}$ is any subspace stable under $\text{ad}(\mathfrak{t})$, we let $\rho(\mathfrak{f})$ be as usual half the sum of the roots of \mathfrak{t} occurring in \mathfrak{f} . Then for a θ -stable parabolic \mathfrak{q} it is known that if a representation of \mathfrak{k} with highest weight $\delta \in \Delta^{+}(\mathfrak{k}, \mathfrak{t})$ occurs in $A_{\mathfrak{q}}$, then

$$\delta = 2\rho(\mathfrak{u} \cap \mathfrak{p}) + \sum_{\gamma \in \Delta(\mathfrak{u} \cap \mathfrak{p})} n_{\gamma} \gamma,$$

for integers $n_\gamma \geq 0$, and the representation of K with highest weight $2\rho(\mathfrak{u} \cap \mathfrak{p})$ exists and occurs in $A_{\mathfrak{q}}$ (K is the connected Lie group with Lie algebra \mathfrak{k}).

12.4. The cohomological representations are gotten by choosing, respectively $x \in \mathfrak{it}$ as follows (see [8, pp. 91-92]):

$x = [0, 0]$, $L \cong \mathrm{SO}_0(3, 2)$, nonzero in bidegrees (j, j) for $0 \leq j \leq 3$.

$x = [-ix_1, 0]$, $x_1 > 0$, $L \cong S^1 \times \mathrm{SO}_0(1, 2)$, nonzero in bidegrees (j, j) for $1 \leq j \leq 2$.

$x = [-i|x_2|, ix_2]$, $x_2 \neq 0$, $L \cong \mathrm{U}(1, 1)$, nz in bideg $(2, 0), (3, 1)$, if $x_2 < 0$; $(0, 2), (1, 3)$ if $x_2 > 0$.

$x = [-ix_1, ix_2]$, $x_1 > |x_2| \neq 0$, $L \cong S^1 \times \mathrm{U}(0, 1)$, nz in bideg $(2, 1)$, if $x_2 < 0$; $(1, 2)$, if $x_2 > 0$.

$x = [-ix_1, ix_2]$, $|x_2| > x_1 > 0$, $L \cong S^1 \times \mathrm{U}(0, 1)$, nz in bideg $(3, 0)$, if $x_2 < 0$; $(0, 3)$, if $x_2 > 0$.

A complete list even with nontrivial local coefficients can be found in [33].

12.5. If we choose $x = [-ix_1, 0] \in \mathfrak{it}$ with $x_1 > 0$, we find

$$\mathfrak{l} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_{\beta} \oplus \mathfrak{g}_{-\beta}, \quad \mathfrak{u} = \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\beta+\alpha} \oplus \mathfrak{g}_{\beta+\alpha}, \quad R^{\pm} = \pm 1.$$

The $K = \mathrm{SO}(3) \times \mathrm{SO}(2)$ -representation $\mu(\mathfrak{q})$ with highest weight $2\rho(\mathfrak{u} \cap \mathfrak{p}) = 2\alpha$ is the tensor product $\mathbf{5} \otimes \mathbf{1}$ of the irreducible 5-dimensional representation of $\mathrm{SO}(3)$ with the trivial representation of $\mathrm{SO}(2)$. Since this representation occurs with multiplicity one in $\wedge^{1,1}\mathfrak{p}$ we see that

$$H^{1,1}(\mathfrak{g}, K; A_{\mathfrak{q}}) = \mathrm{Hom}_K(\wedge^{1,1}\mathfrak{p}, \mu(\mathfrak{q})) = \mathrm{Hom}_{K \cap L}(\wedge^0(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}) = \mathbb{C}$$

is one-dimensional, and also that $\mu(\mathfrak{q})$ occurs with multiplicity one in $A_{\mathfrak{q}}$.

12.6. Let $\pi_{\frac{5}{2}}$ be the discrete series representation of $\tilde{S}L_2(\mathbb{R})$ of lowest weight $\frac{5}{2}$. Now we must show that the representation $\theta(\pi_{\frac{5}{2}})$ has the following properties:

- i. The minimal K -type is the 5 dimensional representation $\mathbf{5} \otimes \mathbf{1}$;
- ii. The infinitesimal character is equal to the infinitesimal character of the trivial representation, $(\frac{3}{2}, \frac{1}{2})$.

Then by the theorem of Vogan-Zuckerman, $\theta(\pi_{\frac{5}{2}})$ is the $A_{\mathfrak{q}}$ for which $H^{1,1}(\mathfrak{g}, K; A_{\mathfrak{q}})$ is non-vanishing. This is a direct consequence of a theorem of Jian-Shu Li ([27]).

As a matter of fact, this can be seen by the following observations. First, back to Prop. 6.6, the compact group $\tilde{U}(1) \subseteq \tilde{S}L_2(\mathbb{R})$ acts on the constant functions by $x^{\frac{1}{2}}$, ($x \in \tilde{U}(1)$). It acts on the first three variables by $x^{\frac{3}{2}}$, ($x \in \tilde{U}(1)$). Consequently, it acts on polynomials of degree 2 on the first 3 variables by $x^{\frac{5}{2}}$, ($x \in \tilde{U}(1)$). In addition, the polynomials of degree 2 on the first 3 variables give the first occurrence of representations of weight $\frac{5}{2}$ for $\tilde{U}(1)$ in \mathcal{P} . So the constituent $\mathbf{5} \otimes \mathbf{1}$ consists of the joint harmonics for $\tilde{U}(1)$ and for $O(3) \times O(2)$. By Howe's theorem, $\theta(\pi_{\frac{5}{2}})$ contains a unique $O(3) \times O(2)$ -type $\mathbf{5} \otimes \mathbf{1}$. So $\mathbf{5} \otimes \mathbf{1}$ is the $O(3) \times O(2)$ -type that is of minimal degree $\theta(\pi_{\frac{5}{2}})$ in the sense of Howe. As we have seen from the proof of Prop. 6.6, the $O(3) \times O(2)$ -types of smaller degrees, $\mathbf{3} \otimes \mathbf{1}$ or $\mathbf{1} \otimes \mathbf{1}$, must not occur in $\theta(\pi_{\frac{5}{2}})$. Therefore, $\mathbf{5} \otimes \mathbf{1}$ is also the minimal $O(3) \times O(2)$ -type of $\theta(\pi_{\frac{5}{2}})$ in the sense of Vogan.

Secondly, the infinitesimal character of $\pi_{\frac{5}{2}}$ is $(\frac{3}{2})$, under the Harish-Chandra homomorphism. By a theorem of Przebinda [32], the infinitesimal character of $\theta(\pi_{\frac{5}{2}})$ can be obtained from $(\frac{3}{2})$ by adding an entry $\frac{1}{2}$. So the infinitesimal character of $\theta(\pi_{\frac{5}{2}})$ is exactly $(\frac{3}{2}, \frac{1}{2})$.

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