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Compositions of theta correspondences

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Abstract

Theta correspondence θ over $\mathbb{R}$ is established by Howe (J. Amer. Math. Soc. 2 (1989) 535). In He (J. Funct. Anal. 199 (2003) 92), we prove that θ preserves unitarity under certain restrictions, generalizing the result of Li (Invent. Math. 97 (1989) 237). The goal of this paper is to elucidate the idea of constructing unitary representation through the propagation of theta correspondences. We show that under a natural condition on the sizes of the related dual pairs which can be predicted by the orbit method (J. Algebra 190 (1997) 518; Representation Theory of Lie Groups, Park City, 1998, pp. 179–238; The Orbit Correspondence for real and complex reductive dual pairs, preprint, 2001), one can compose theta correspondences to obtain unitary representations. We call this process quantum induction.

MSC: primary 22E45; 22E46

Keywords: ■; ■; ■

1. Introduction

An important problem in representation theory is the classification and construction of irreducible unitary representations. Let G be a reductive group and $\Pi(G)$ be its admissible dual. For an algebraic semisimple group G, the admissible dual $\Pi(G)$ is known mostly due to the works of Harish-Chandra, R. Langlands, and Knapp–Zuckerman (see [17,18]). Let $\Pi_u(G)$ be the set of equivalence classes of irreducible unitary representations of G, often called the unitary dual of G. The unitary dual of general linear groups is classified by Vogan [29]. The unitary

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dual of complex classical groups is classified by Barbasch [2]. Recently, Barbasch has classified all the spherical duals for split classical groups (see [3]). The unitary duals $\Pi_{\mathrm{u}}(O(p,q))$ and $\Pi_{\mathrm{u}}(Sp_{2n}(\mathbb{R}))$ are not known in general.

In [14], Howe constructs certain small unitary representations of the symplectic group using Mackey machine. Later, Jian-Shu Li generalizes Howe's construction of small unitary representations to all classical groups. In particular, Li defines a sesquilinear form $(,)_\pi$ that relates these constructions to the theta correspondence (see [11,20]). It then becomes clear to many people that some irreducible unitary representations can be constructed through the propagation of theta correspondences (see [15,21,28] and the references within them). So far, constructions can only be carried out for "complete small orbits" (see [21]). The purpose of this paper is to make it work for nilpotent orbits in general, for real orthogonal groups and symplectic groups.

Consider the group $O(p,q)$ and $Sp_{2n}(\mathbb{R})$. The theta correspondence with respect to

$$O(p,q)\to Sp_{2n}(\mathbb{R})$$

is formulated by Howe as a one-to-one correspondence

$$\theta(p,q;2n):\mathscr{R}(MO(p,q),\omega(p,q;2n))\to\mathscr{R}(MSp_{2n}(\mathbb{R}),\omega(p,q;2n)),$$

where $MO(p,q)$ and $MSp_{2n}(\mathbb{R})$ are some double coverings of $O(p,q)$ and $Sp_{2n}(\mathbb{R})$, respectively, and

$$\mathscr{R}(MO(p,q),\omega(p,q;2n))\subseteq\Pi(MO(p,q)),$$

$$\mathscr{R}(MSp_{2n}(\mathbb{R}),\omega(p,q;2n))\subseteq\Pi(MSp_{2n}(\mathbb{R}))$$

(see [13]). We denote the inverse of $\theta(p,q;2n)$ by $\theta(2n;p,q)$. For the sake of simplicity, we define

$$\theta(p,q;2n)(\pi)=0$$

if $\pi\notin\mathscr{R}(MO(p,q),\omega(p,q;2n))$. We define $\theta(p,q;2n)(0)=0$ and 0 can be regarded as the NULL representation.

For example, given an "increasing" string

$$O(p_1,q_1)\to Sp_{2n_1}(\mathbb{R})\to O(p_2,q_2)\to Sp_{2n_2}(\mathbb{R})\to\cdots\to Sp_{2n_m}(\mathbb{R})\to O(p_m,q_m),$$

$$p_1+q_1\equiv p_2+q_2\equiv\cdots\equiv p_m+q_m \pmod 2,$$

consider the propagation of theta correspondence along this string:

$$\theta(2n_m;p_m,q_m)\ldots\theta(2n_1;p_2,q_2)\theta(p_1,q_1;2n_1)(\pi).$$

Under some favorable conditions on $\pi\in\Pi_{\mathrm{u}}(O(p_1,q_1))$, one hopes to obtain a unitary representation in $\Pi_{\mathrm{u}}(O(p_m,q_m))$. In this paper, we supply a sufficient condition for

$$\theta(2n_m; p_m, q_m)\ldots\theta(2n_1; p_2, q_2)\theta(p_1, q_1; 2n_1)(\pi)$$

to be unitary. We denote the resulting representation of $MO(p_m, q_m)$ by

$$Q(p_1, q_1; 2n_1; p_2, q_2; 2n_2; \ldots; p_m, q_m)(\pi).$$

We call $Q(p_1, q_1; 2n_1; p_2, q_2; 2n_2; \ldots; p_m, q_m)$ quantum induction. In addition to the assumption that certain Hermitian forms do not vanish, we must also assume the matrix coefficients of π satisfy a mild growth condition.

Based on the work of Przebinda [26], we further determine the behavior of infinitesimal characters under quantum induction. In certain limit cases, the infinitesimal character under quantum induction behaves exactly in the same way as under parabolic induction. In fact, in some limit cases, quantum induced representations can be obtained from unitarity-preserving parabolic induction (see [10]). Finally, motivated by the works of Przebinda and his collaborators, we make a precise conjecture regarding the associated variety of the quantum induced representations (Conjecture 2).

There is one problem we did not address in this paper, namely, the nonvanishing of certain Hermitian forms $(,)_\pi$ with $\pi \in \Pi(Mp_{2n}(\mathbb{R}))$. In a forthcoming article [10], we partially address this problem and construct a set of special unipotent representations in the sense of Vogan [30].

2. Main results

2.1. Notations

In this paper, unless stated otherwise, all representations are regarded as Harish-Chandra modules. This should cause no problems since most representations in this paper will be admissible with respect to a reductive group. Thus unitary representations in this paper would mean unitrizable Harish-Chandra modules. "Matrix coefficients" of a representation π of a real reductive group G will refer to the K-finite matrix coefficients with respect to a maximal compact subgroup K. A vector v in an admissible representation π means that v is in the Harish-Chandra module of π which shall be evident within the context.

Let (G_1, G_2) be a reductive dual pair of type I (see [13,20]). The dual pairs in this paper will be considered as ordered. For example, the pair $(O(p,q), Sp_{2n}(\mathbb{R}))$ is considered different from the pair $(Sp_{2n}(\mathbb{R}), O(p,q))$. Unless stated otherwise, we will, in general, assume that the size of $G_1(V_1)$ is less or equal to the size of $G_2(V_2)$, i.e., $\dim_D(V_1) \leqslant \dim_D(V_2)$. Let (G_1, G_2) be a dual pair in the symplectic group Sp. Let Mp be the unique double covering of Sp. Let $\{1, \varepsilon\}$ be the preimage of the identity element in Sp. For a subgroup H of Sp, let MH be the preimage of H under the double covering. Whenever we use the notation MH, H is considered to be a subgroup of certain Sp which shall be evident within the context. Let $\omega(MG_1, MG_2)$

be a Schrödinger model of the oscillator representation of Mp equipped with a dual pair (MG_1, MG_2). The Harish-Chandra module of $\omega(MG_1, MG_2)$ consists of polynomials multiplied by the Gaussian function. Since the pair (G_1, G_2) is ordered, we use $\theta(MG_1, MG_2)$ to denote the theta correspondence from $\mathscr{R}(MG_1, \omega(MG_1, MG_2))$ to $\mathscr{R}(MG_2, \omega(MG_1, MG_2))$. We use $\mathbf{n}$ to denote the constant vector

$$(n, n, \ldots, n).$$

The dimension of $\mathbf{n}$ is determined within the context. Finally, we say a vector

$$x = (x_1, x_2, \ldots x_n) \prec 0$$

if

$$\sum_{j=1}^{k} x_j < 0 \quad \forall k \geqslant 1$$

and $x \preccurlyeq 0$ if

$$\sum_{j=1}^{k} x_j \leqslant 0 \quad \forall k \geqslant 1.$$

In this paper, the space of $m \times n$ matrices will be denoted by $M(m, n)$. The set of non-negative integers will be denoted by $\mathbb{N}$. For the group $O(p, q)$, we assume that $p \leqslant q$ unless stated otherwise. For a reductive group G, $\Pi(G)$, $\Pi_{\mathrm{u}}(G)$ will be the admissible dual and the unitary dual, respectively.

We extend the definition of matrix coefficients to the NULL representation. The matrix coefficients of the NULL representation is defined to be the zero function.

2.2. Theta correspondence in semistable range and unitary representations

Let $\pi \in \Pi(MG_1)$. Following [20], for every $u, v \in \pi$ and $\phi, \psi \in \omega(MG_1, MG_2)$, we formally define

$$(\phi \otimes v, \psi \otimes u)_\pi = \int_{MG_1} (\omega(MG_1, MG_2)(\tilde{g}_1)\phi, \psi)(u, \pi(\tilde{g}_1)v)\, d\tilde{g}_1. \tag{1}$$

Roughly speaking, if the functions

$$(\omega(MG_1, MG_2)(\tilde{g}_1)\phi, \psi)(u, \pi(\tilde{g}_1)v) \quad (\forall \phi, \psi \in \omega(MG_1, MG_2); \forall u, v \in \pi)$$

are in $L^1(MG_1)$ and $\pi(\varepsilon) = -1$, π is said to be in the semistable range of $\theta(MG_1, MG_2)$ (see [7]). We denote the semistable range of $\theta(MG_1, MG_2)$ by $\mathscr{R}_s(MG_1, MG_2)$.

Suppose from now on that $\pi\in\mathscr{R}_s(MG_1,MG_2)$. In [7], we showed that if $(,)_\pi$ does not vanish, then $(,)_\pi$ descends into a Hermitian form on $\theta(MG_1,MG_1)(\pi)$. For $\pi\in\mathscr{R}_s(MG_1,MG_1)$, we define

$$\theta_s(MG_1,MG_1)(\pi)=\begin{cases}\theta(MG_1,MG_2)(\pi) & \text{if } (,)_\pi\neq 0,\\ 0 & \text{if } (,)_\pi=0,\end{cases}\tag{2}$$

$\theta_s(MG_1,MG_2)(\pi)$ as a real vector space is just $\omega(MG_1,MG_2)\otimes\pi$ modulo the radical of $(,)_\pi$ (see [7,20]). The main object of study in this paper is θ_s.

If π is in $\mathscr{R}_s(MG_1,MG_2)$ but not in $\mathscr{R}(MG_1,\omega(MG_1,MG_2))$, our construction from [7] will result in a vanishing $(,)_\pi$. Thus $\theta_s(MG_1,MG_2)(\pi)$ "vanishes". In this case, $\theta_s=\theta$ trivially. The remaining question is whether $(,)_\pi\neq 0$ if $\pi\in\mathscr{R}(MG_1,\omega(MG_1,MG_2))$. Conjecturally, $\theta_s(MG_1,MG_1)$ should agree with the restriction of $\theta(MG_1,MG_1)$ on $\mathscr{R}_s(MG_1,MG_2)$ (see [7,19]).

For π a Hermitian representation, it can be easily shown that $(,)_\pi$ is an invariant Hermitian form on $\theta(MG_1,MG_2)(\pi)$ if $(,)_\pi$ does not vanish. This is a special case of Przebinda's result in [24]. For π unitary, we do not know whether $(,)_\pi$ must be positive semidefinite in general. Nevertheless, in [9], we have proved the semi-positivity of $(,)_\pi$ under certain condition on the leading exponents of π (see [16,32]). Fix a Cartan decomposition for $Sp_{2n}(\mathbb{R})$ and $O(p,q)$. Fix the standard basis of $\mathfrak{a}$ for $Sp_{2n}(\mathbb{R})$ and $O(p,q)$ (see 6.1). The leading exponents of an irreducible admissible representation are in the complex dual of the Lie algebra $\mathfrak{a}$ of A.

Theorem 2.2.1. *Suppose $p+q\leqslant 2n+1$. Let π be an irreducible unitary representation whose every leading exponent satisfies*

$$\Re(v)-\left(\mathbf{n}-\frac{\mathbf{p}+\mathbf{q}}{2}\right)+\rho(O(p,q))\preccurlyeq 0.\tag{3}$$

Then $(,)_\pi$ is positive semidefinite. Thus, $\theta_s(p,q;2n)(\pi)$ is either unitary or vanishes.

We denote the set of representations in $\Pi(MO(p,q))$ satisfying (3) by $\mathscr{R}_{ss}(p,q;2n)$. The set $\mathscr{R}_s(MO(p,q),MSp_{2n}(\mathbb{R}))$ is written as $\mathscr{R}_s(p,q;2n)$ in short.

Theorem 2.2.2. *Suppose $n<p\leqslant q$. Let π be an irreducible unitary representation whose every leading exponent satisfies*

$$\Re(v)-\left(\frac{\mathbf{p}+\mathbf{q}}{2}-\mathbf{n}-\mathbf{1}\right)+\rho(Sp_{2n}(\mathbb{R}))\preccurlyeq 0.\tag{4}$$

Then $(,)_\pi$ is positive semidefinite. Thus, either $\theta_s(p,q;2n)_s(\pi)$ is unitary or vanishes.

We denote the set of representations in $\Pi(MSp_{2n}(\mathbb{R}))$ satisfying (4) by $\mathscr{R}_{ss}(2n;p,q)$. The set $\mathscr{R}_s(MSp_{2n}(\mathbb{R}),MO(p,q))$ is written as $\mathscr{R}_s(2n;p,q)$ in short.

2.3. *Estimates on leading exponents and $L(p,n)$*

In this paper, we establish some estimates on the growth of the matrix coefficients of $\theta(p,q;2n)(\pi)$ and of $\theta(2n;p,q)(\pi)$ for π in $\mathscr{R}_s(p,q;2n)$ and $\mathscr{R}_s(2n;p,q)$, respectively. We achieve this by studying the decaying of the function

$$L(a,\phi)=\int_{b_1\geqslant b_2\geqslant\cdots\geqslant b_p\geqslant 1}\left(\prod_{i=1,j=1}^{n,p}(a_i^2+b_j^2)^{-\frac{1}{2}}\right)\phi(b_1,b_2,\ldots,b_p)\,db_1\,db_2\ldots db_p$$

as a function of $a\in\mathbb{R}^n$. In general, the decaying of $L(a,\phi)$ depends on the decaying of ϕ. In Section 5, we define a map $L(p,n)$ to describe this dependence. The map $L(p,n)$ is a continuous map from

$$C(p)=\{\lambda\prec 0\mid\lambda\in\mathbb{R}^p\}$$

to

$$C(n)=\{\mu\prec 0\mid\mu\in\mathbb{R}^n\}.$$

Its algorithm is developed in Section 5. For some special vectors in $C(p)$, $L(p,n)$ is just a reordering plus an augmentation or truncation. In this paper, we prove

Theorem 2.3.1. *Let $L(n,p)$ be defined as in Section 5. Let $a(g_2)$ be the middle term of the KA^+K decomposition of $g_2\in Sp_{2n}(\mathbb{R})$. Let $b(g_1)$ be the middle term of the KA^+K decomposition of $g_1\in O(p,q)$.*

1. *Suppose that $\pi\in\mathscr{R}_s(p,q;2n)$. Suppose $\lambda\prec-2\rho(O(p,q))+\mathbf{n}$ and for every leading exponent v of π, $\Re(v)\preccurlyeq\lambda$. Then the matrix coefficients of $\theta_s(p,q;2n)(\pi)$ are weakly bounded by*

$$a(g_2)^{L(p,n)(\lambda+2\rho(O(p,q))-\mathbf{n})-\frac{\mathbf{q-p}}{\mathbf{2}}}.$$

2. *Suppose that $\pi\in\mathscr{R}_s(2n;p,q)$. Suppose $\lambda\prec-2\rho(Sp_{2n}(\mathbb{R}))+\frac{\mathbf{p+q}}{2}$ and for every leading exponent v of π, $\Re(v)\preccurlyeq\lambda$. Then the matrix coefficients of $\theta_s(2n;p,q)(\pi)$ are weakly bounded by*

$$b(g_1)^{L(n,p)\left(\lambda+2\rho(Sp_{2n}(\mathbb{R}))-\frac{\mathbf{p+q}}{2}\right)}.$$

The definition of weakly boundedness is given in Section 3.

2.4. Quantum induction

The idea of composing two theta correspondences to obtain ‘‘new’’ representations has been known for years. For example, one can compose $\theta(p,q;2n)$ with $\theta(2n;p',q')$. The nature of $\theta(2n;p',q')\theta(p,q;2n)(\pi)$ seems to be inaccessible except for the cases of stable ranges. In this paper, we treat a somewhat more accessible object, namely,

$$\theta_s(2n;p',q')\theta_s(p,q;2n)(\pi).$$

Our construction is done through the studies of the Hermitian form $(,)_\pi$. Due to the unitarity theorems we proved in [9], under restrictions as specified in Eqs. (3) and (4), quantum induction preserves unitarity. Our main result can be stated as follows:

Theorem 2.4.1 (Main Theorem).

- *Suppose*
  1. $q' \geqslant p' > n$;
  2. $p'+q'-2n \geqslant 2n-(p+q)+2 \geqslant 1$;
  3. $p+q = p'+q' \pmod 2$.

  Let π be an irreducible unitary representation in $\mathscr{R}_{ss}(p,q;2n)$. Suppose that $(,)_\pi$ does not vanish. Then
  1. *$\theta_s(p,q;2n)(\pi)$ is unitary.*
  2. *$\theta_s(p,q;2n)(\pi) \in \mathscr{R}_{ss}(2n;p',q')$.*
  3. *$\theta_s(2n;p',q')\theta_s(p,q;2n)(\pi)$ is either an irreducible unitary representation or the NULL representation.*
- *Suppose*
  1. $2n'-p-q+2 \geqslant p+q-2n$;
  2. $n < p \leqslant q$.

  Let π be a unitary representation in $\mathscr{R}_{ss}(p,q;2n)$. Suppose $(,)_\pi$ does not vanish. Then
  1. *$\theta_s(2n;p,q)(\pi)$ is unitary.*
  2. *$\theta_s(2n;p,q)(\pi) \in \mathscr{R}_{ss}(p,q;2n')$.*
  3. *$\theta_s(p,q;2n')\theta_s(2n;p,q)(\pi)$ is either an irreducible unitary representation or the NULL representation.*

The purpose of assuming $\pi \in \mathscr{R}_{ss}$ is to guarantee the unitarity of $Q(*)(\pi)$. In fact, for any π, the condition on the sizes of related dual pairs can be computed easily to define nonunitary quantum induction. In general, the underlying Hilbert space of the induced representation is ‘‘invisible’’ under quantum induction except for certain limit cases where quantum induction becomes unitary parabolic induction (see Section 6 and [10]).

Conjecture 1. *Suppose π is a unitary representation in $\mathscr{R}_{ss}$.*

- *The quantum induction $Q(p,q;2n;p',q')(\pi)$ for $2n-p-q+2=p'+q'-2n$ can be obtained via unitarity-preserving parabolic induction and cohomological induction from π.*
- *The quantum induction $Q(2n;p,q;2n')(\pi)$ for $p+q-2n-2=2n'-p-q$ can be obtained as a subfactor via unitarity-preserving parabolic induction from π.*

For the cases $p+q=2n+1=p'+q'$ and $p+q=2n+1=2n'+1$, by a Theorem of Adams–Barbasch, Q is either the identity map or vanishes [1]. Our conjecture holds trivially, i.e., no induction is needed. For the case $p+q+p'+q'=4n+2$ and $p-p'=q-q'$, our result in Section 6 gives some indication that $Q(p,q;2n;p',q')(\pi)$ can be obtained from

$$Ind_{SO_0(p,q)GL_0(p'-p)N}^{SO_0(p',q')}(\pi\otimes 1).$$

Let me make one remark regarding the nonvanishing of $(,)_\pi$. In [8] we prove

Theorem 2.4.2 (He [8]). *Suppose $p+q\leqslant 2n+1$. Let $\pi\in\mathscr{R}_s(p,q;2n)$. Then at least one of*

$$(,)_\pi, (,)_{\pi\otimes det}$$

does not vanish.

For $\pi\in\mathscr{R}_s(2n;p,q)$, the nonvanishing of $(,)_\pi$ is hard to detect since it depends on p,q [1,6,22]. A result of Li says that $(,)_\pi$ does not vanish if $p,q\geqslant 2n$. We are not aware of any more general nonvanishing theorems.

Finally, concerning the associated varieties, Przebinda shows that the associated varieties behaves reasonably well under theta correspondence under certain strong hypothesis [25]. We conjecture that quantum induction induces an induction on associated varieties and wave front sets. The exact description of the associated variety under quantum induction can be predicted based on [5].

Conjecture 2.

- *Under the same assumptions from the main theorem, let π be a unitary representation in $\mathscr{R}_{ss}(p,q;2n)$. Let $\mathcal{O}_{\mathbf{d}}$ be the associated variety of π with $\mathbf{d}$ a partition (see [4], Chapter 5). Let $\mathcal{O}_{\mathbf{f}}$ be the associated variety of $Q(p,q;2n;p',q')(\pi)\neq 0$. Then $\mathbf{f}^t=(p'+q'-2n,2n-p-q,\mathbf{d}^t)$.*
- *Under the same assumptions from the main theorem, let π be a unitary representation in $\mathscr{R}_{ss}(2n;p,q)$. Let $\mathcal{O}_{\mathbf{d}}$ be the associated variety of π with $\mathbf{d}$ a partition. Let $\mathcal{O}_{\mathbf{f}}$ be the associated variety of $Q(2n;p,q;2n')(\pi)\neq 0$. Then $\mathbf{f}^t=(2n'-p-q,p+q-2n,\mathbf{d}^t)$.*

We remark that our situation is different from the situation treated in [25] with some overlaps. The description of the wave front set under quantum induction can be predicted based on [23].

3. Theta correspondence

Let $(O(p,q), Sp_{2n}(\mathbb{R}))$ be a reductive dual pair in $Sp_{2n(p+q)}(\mathbb{R})$. Let

$$j : Mp_{2n(p+q)}(\mathbb{R}) \rightarrow Sp_{2n(p+q)}(\mathbb{R})$$

be the double covering. Let $\{1, \varepsilon\} = j^{-1}(1)$. Let $MO(p,q) = j^{-1}(O(p,q))$ and $MSp_{2n}(\mathbb{R}) = j^{-1}(Sp_{2n}(\mathbb{R}))$. Fix a maximal compact subgroup U of $Sp_{2n(p+q)}(\mathbb{R})$ such that

$$U \cap Sp_{2n}(\mathbb{R}) \cong U(n), \quad U \cap O(p,q) \cong O(p) \times O(q).$$

Then MU is a maximal compact subgroup of $Mp_{2n(p+q)}(\mathbb{R})$. Let $\omega(p,q;2n)$ be the oscillator representation of $Mp_{2n(p+q)}(\mathbb{R})$. The representation $\omega(p,q;2n)$ or sometimes $\omega(2n;p,q)$ is regarded as an admissible representation of $Mp_{2n(p+q)}(\mathbb{R})$ equipped with a fixed dual pair $(O(p,q), Sp_{2n}(\mathbb{R}))$. Let $\mathscr{P}$ be the Harish-Chandra module. Then $\omega(p,q;n)$ can be restricted to $MO(p,q)$ and $MSp_{2n}(\mathbb{R})$. Howe's theorem states that there is a one-to-one correspondence

$$\theta(p,q;2n) : \mathscr{R}(MO(p,q), \omega(p,q;2n)) \rightarrow \mathscr{R}(MSp_{2n}(\mathbb{R}), \omega(p,q;2n)).$$

3.1. $MO(p,q)$ and $MSp_{2n}(\mathbb{R})$

The groups $MO(p,q)$ and $MSp_{2n}(\mathbb{R})$ are double covers of $O(p,q)$ and $Sp_{2n}(\mathbb{R})$. Depending on the parameter n, p and q, they may be quite different.

Lemma 3.1.1. (1) *If $p+q$ is odd, then the double cover $MSp_{2n}(\mathbb{R})$ does not split. It is the metaplectic group $Mp_{2n}(\mathbb{R})$. The representations in $\mathscr{R}(Mp_{2n}(\mathbb{R}), \omega(p,q;2n))$ are genuine representation of $Mp_{2n}(\mathbb{R})$.*

(2) *If $p+q$ is even, then the double cover $MSp_{2n}(\mathbb{R})$ splits. It is the product of $Sp_{2n}(\mathbb{R})$ and $\{1, \varepsilon\}$. The representations in $\mathscr{R}(MSp_{2n}(\mathbb{R}), \omega(p,q;2n))$ can be identified with representations of $Sp_{2n}(\mathbb{R})$ by tensoring the nontrivial character of $\{1, \varepsilon\}$.*

(3) *In both cases, any representation in*

$$\mathscr{R}(MSp_{2n}(\mathbb{R}), \omega(p,q;2n))$$

can be identified with a representation of $Mp_{2n}(\mathbb{R})$. In the former case, a genuine representation, and in the latter case, a nongenuine representation.

We do not know the earliest reference. The details can be worked out easily and can be found in [1].

Lemma 3.1.2. (1) *As a group,*

$$MO(p,q) \cong \{(\xi, g) \mid g \in O(p,q), \xi^2 = det\, g^n\}$$

(2) *ξ is a character of $MO(p,q)$. Any representations in $\mathscr{R}(MO(p,q), \omega(p,q;2n))$ can be identified with representations of $O(p,q)$ by tensoring ξ.*

(3) *$MSO(p,q)$ can be identified as group product*

$$SO(p,q) \times \{1, \varepsilon\}.$$

(4) *If n is even, $MO(p,q) \cong O(p,q) \times \{1, \varepsilon\}$.*

The details can be found in [1] or [9]. We must keep in mind that for $p+q$ odd,

$$\mathscr{R}(MSp_{2n}(\mathbb{R}), \omega(p,q;2n)) \subset \Pi_{genuine}(Mp_{2n}(\mathbb{R}))$$

and for $p+q$ even

$$\mathscr{R}(MSp_{2n}(\mathbb{R}), \omega(p,q;2n)) \subset \Pi(Sp_{2n}(\mathbb{R})).$$

3.2. *Averaging integral $(,)_\pi$*

Let $O(p,q)$ be the orthogonal group preserving the symmetric form defined by

$$I_{p,q} = \begin{pmatrix} 0_p & 0 & I_p \\ 0 & I_{q-p} & 0 \\ I_p & 0 & 0_p \end{pmatrix}.$$

Fix a Cartan decomposition with

$$A = \{diag(a_1, a_2, \ldots, a_p, \overbrace{1, \ldots, 1}^{q-p}, a_1^{-1}, a_2^{-1}, \ldots, a_p^{-1}) \mid a_i > 0\}$$

and a positive Weyl chamber

$$A^+ = \{diag(a_1, a_2 \ldots, a_p, \overbrace{1, \ldots, 1}^{q-p}, a_1^{-1}, a_2^{-1}, \ldots, a_p^{-1}) \mid a_1 \geqslant a_2 \geqslant \cdots \geqslant a_p \geqslant 1\}.$$

The half sum of the positive restricted roots of $O(p,q)$

$$\rho(O(p,q)) = \overbrace{\left(\frac{p+q-2}{2}, \frac{p+q-4}{2}, \ldots, \frac{q-p}{2}\right)}^{p}.$$

Let $Sp_{2n}(\mathbb{R})$ be the symplectic group that preserves the skew-symmetric form defined by

$$W_n = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}.$$

Let K be the intersection of $Sp_{2n}(\mathbb{R})$ with the orthogonal group $O(2n)$ which preserves the Euclidean inner product on $\mathbb{R}^{2n}$. Let

$$A = \{a = diag(a_1, a_2, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}) \mid a_i > 0\},$$

$$A^+ = \{a = diag(a_1, a_2, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}) \mid a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 1\}.$$

The half sum of the positive restricted roots of $Sp_{2n}(\mathbb{R})$

$$\rho(Sp_{2n}(\mathbb{R})) = \overbrace{(n, n-1, \ldots, 1)}^{n}.$$

For each irreducible admissible representation of a semisimple group G of real rank r, there are number of r-dimensional complex vectors in $\mathfrak{a}^*$ called leading exponents attached to it. Leading exponents are the main data used to produce the Langlands classification (see [16,18]).

Definition 3.2.1. An irreducible representation π of $O(p,q)$ is said to be in the semistable range of $\theta(p,q;2n)$ if and only if each leading exponent v of π satisfies

$$\sum_{i=1}^{j} \Re(v_i) + (p+q-2i) - n < 0 \quad (\forall j \in [1,p]) \tag{5}$$

i.e.,

$$\Re(v) - \mathbf{n} + 2\rho(O(p,q)) \prec 0.$$

An irreducible representation π of $Mp_{2n}(\mathbb{R})$ is said to be in the semistable range of $\theta(2n;p,q)$ if and only if every leading exponent v of π satisfies

$$\sum_{i=1}^{k} \Re(v_i) - \frac{p+q}{2} + 2n + 2 - 2j < 0 \quad (\forall k \in [1,n]) \tag{6}$$

i.e.,

$$\Re(v) - \frac{\mathbf{p}+\mathbf{q}}{2} + 2\rho(Sp_{2n}(\mathbb{R})) \prec 0.$$

If W is a complex linear space, we use a superscript W^{c} to denote W equipped with the conjugate complex linear structure. Let $\pi \in \mathscr{R}_s(MG_1, MG_2)$. We define a complex linear pairing

$$(\mathscr{P}^{c} \otimes \pi, \mathscr{P} \otimes \pi^{c}) \to \mathbb{C}$$

as follows: for $\phi \in \mathscr{P}, \psi \in \mathscr{P}^{c}, v \in \pi^{c}, u \in \pi$,

$$(\phi \otimes v, \psi \otimes u)_{\pi} = \int_{MO(p,q)} (\phi, \omega(g)\psi)(\pi(g)u, v)\, dg.$$

If π is unitary, $(,)_{\pi}$ is an invariant Hermitian form with respect to the action of MG_2.

Theorem 3.2.1 (See He [7]). *Suppose* (π, V) *is a unitary representation in the semistable range of* $\theta(MG_1, MG_2)$. *Then* $(,)_{\pi}$ *is well-defined. Suppose* $\mathscr{R}_{\pi}$ *is the radical of* $(,)_{\pi}$ *with respect to* $\mathscr{P} \otimes V^{c}$. *If* $(,)_{\pi}$ *does not vanish, then*

- π *occurs in* $\mathscr{R}(MG_1, \omega(MG_1, MG_2))$;
- $\mathscr{P} \otimes V^{c}/\mathscr{R}_{\pi}$ *is irreducible*;
- $\mathscr{P} \otimes V^{c}/\mathscr{R}_{\pi}$ *is isomorphic to* $\theta(MG_1, MG_2)(\pi)$.
- $\theta_s(MG_1, MG_2)(\pi)$ *is a Hermitian representation of* MG_2.

Thus the Harish-Chandra module of $\theta_s(MG_1, MG_2)(\pi)$ can be defined as $\mathscr{P} \otimes V^{c}/\mathscr{R}_{\pi}$.

3.3. Oscillator representation

The oscillator representation, also known as the Segal–Shale–Weil representation, is a unitary representation of the metaplectic group *Mp*. The construction of the oscillator representation can be found in the papers of Segal [27], Shale [28] and Weil [33]. In this section, we give a basic estimate of the matrix coefficients of the oscillator representation. Proof of Theorem 3.3.1 can also be found in [12, Proposition 8.1].

Let $g \in Sp_{2n}(\mathbb{R})$. Let $a(g)$ be the midterm of the *KAK* decomposition of g such that $a \in A^{+}$. Let $H(g) = \log a(g)$. Then

$$H(g) = diag(H_1(g), H_2(g), \ldots, H_n(g), -H_1(g), \ldots, -H_n(g))$$

is in the Weyl chamber $\mathfrak{a}^{+}$.

Let $Mp_{2n}(\mathbb{R})$ be the double covering of $Sp_{2n}(\mathbb{R})$. The midterm of the KAK decomposition of $Mp_{2n}(\mathbb{R})$ remains the same. Let $(\omega_n, L^2(\mathbb{R}^n))$ be the Schrödinger model of the oscillator representation of $Mp_{2n}(\mathbb{R})$ as in [7]. Let

$$\mu(x) = \exp\left(-\tfrac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)\right)$$

be the Gaussian function. The Harish-Chandra module $\mathscr{P}_n$ are the polynomial functions multiplied by the Gaussian function as verified in [7]. We write

$$x^\alpha = \prod_1^n x_i^{\alpha_i}.$$

Harish-Chandra's theory says that the $Mp_{2n}(\mathbb{R})$ action on $\mathscr{P}_n$ can be controlled by the A action on fixed K-types of ω_n.

Theorem 3.3.1. *For any $a \in A$, we have*

$$(\omega_n(a)x^\alpha\mu(x), x^\beta\mu(x)) = c_{\alpha,\beta}\prod_{i=1}^n a_i^{\alpha_i+\frac{1}{2}}(1+a_i^2)^{-\frac{\alpha_i+\beta_i+1}{2}}.$$

In addition,

$$|(\omega_n(a)x^\alpha\mu(x), x^\beta\mu(x))| \leqslant c\prod_{i=1}^n (a_i + a_i^{-1})^{-\frac{1}{2}}.$$

In general, for every $\phi, \psi \in \mathscr{P}_n$, we have

$$|(\omega_n(g)\phi, \psi)| \leqslant c\prod_{i=1}^n (a_i(g) + a_i^{-1}(g))^{-\frac{1}{2}}.$$

The proof for the first statement can be found in [7]. We observe that

$$\begin{aligned} |(\omega_n(a)x^\alpha\mu(x), x^\beta\mu(x))| &= \left| c_{\alpha,\beta}\prod_{i=1}^n a_i^{\alpha_i+\frac{1}{2}}(1+a_i^2)^{-\frac{\alpha_i+\beta_i+1}{2}} \right| \\ &= \left| c_{\alpha,\beta}\prod_{i=1}^n (a_i + a_i^{-1})^{-\frac{1}{2}}(1+a_i^2)^{-\frac{\beta_i}{2}}(1+a_i^{-2})^{-\frac{\alpha_i}{2}} \right| \\ &\leqslant c_{\alpha,\beta}\prod_{i-1}^n (a_i + a_i^{-1})^{-\frac{1}{2}}. \end{aligned} \tag{7}$$

The second statement is proved. The third statement follows immediately from K-finiteness of ϕ and ψ.

The estimations on the right-hand side are invariant under Weyl group action, thus do not depend on the choices of the Weyl chamber $\mathfrak{a}^+$.

3.4. Growth of matrix coefficients

Definition 3.4.1. Suppose X is a Borel measure space equipped with a norm $||.||$ such that

- $||x|| \geqslant 0$ for all $x \in X$;
- the set $\{||x|| \leqslant r\}$ is compact.

Let $f(x)$ and $\phi(x)$ be continuous functions defined over X. Suppose $\phi(x)$ approaches 0 as $||x|| \to \infty$. A function $f(x)$ is said to be weakly bounded by the function $\phi(x)$ if there exists a $\delta_0 > 0$ such that for every $\delta_0 > \delta > 0$, there exists a $C > 0$ depending on δ such that

$$|f(x)| \leqslant C\phi(x)^{1-\delta} \quad (\forall x \in X).$$

The typical case is when $f(x)$ does not decay as fast as $\phi(x)$ but faster than $\phi(x)^{1-\delta}$.

Let π be an irreducible representation of a reductive group G. Let K be a maximal compact subgroup of G. We adopt the notation from Chapter VIII in [16]. We equip G with a norm

$$g \to ||\log(a(g))|| = (\log a(g), \log a(g))^{\frac{1}{2}},$$

where $(,)$ is a real $\mathfrak{g}$-invariant symmetric form whose restriction on $\mathfrak{a}$ is positive definite.

Example. An irreducible representation π of a reductive group G is tempered if and only if its matrix coefficients are weakly bounded by

$$a(g)^{-\rho},$$

where ρ is the half sum of positive restricted roots and $a(g)$ is the mid term of the KAK decomposition with $a(g)$ in the positive Weyl chamber A^+ (see [16]).

Theorem 3.4.1. *Let π be an irreducible unitary representation of G. Let $\lambda \prec 0$. The following are equivalent*:

1. *Every leading exponent v of π has $\Re(v) \preccurlyeq \lambda$.*
2. *There is an integer $q \geqslant 0$ such that every K-finite matrix coefficient is bounded by a multiple of $(1 + ||\log a(g)||)^q \exp(\lambda(\log a(g)))$.*

3. *Every K-finite matrix coefficient $\phi(g)$ of π is bounded by $Ca(g)^{\lambda+\delta}$ for any $\delta \succ 0$.*
4. *Every K-finite matrix coefficient of π is weakly bounded by $a(g)^{\lambda}$.*

See Chapter VIII.8, 13 [16] or Chapter 4.3 [32] for details. The first three statements are equivalent without assuming the unitarity of π and $\lambda \prec 0$.

4. Twisted integral

Let $A^+ = \{a_1 \geqslant a_2 \geqslant \cdots \geqslant 1\}$. In this section, we will study the following integrals:

$$L(a, \lambda) = \int_{B^+} \prod_{i=1}^{p} \left(\prod_{k=1}^{n} (a_k^2 + b_i^2)^{-\frac{1}{2}} \right) b_i^{\lambda_i} \, db_i$$

and

$$L(a, \phi) = \int_{b_1 \geqslant b_2 \geqslant \cdots \geqslant b_p \geqslant 1} \prod_{i,j} (a_i^2 + b_j^2)^{-\frac{1}{2}} \phi(b_1, b_2, \ldots, b_p) \, db_1 \, db_2 \ldots db_p.$$

The domain of a will always be A^+ unless stated otherwise. We are interested in the growth of $L(a, \phi)$ as a goes to infinity. Variables and parameters are assumed to be real in this section.

4.1. Single variable case $a \geqslant 1$

Lemma 4.1.1. *Suppose that $a \geqslant 1$. The integral*

$$L(a, \lambda) = \int_{b \geqslant 1} (a^2 + b^2)^{-\frac{1}{2}} b^{\lambda} \, db$$

converges if and only if $\lambda < 0$. In addition, $L(a, \lambda)$ is weakly bounded by a^{λ} if $-1 \leqslant \lambda < 0$ and is bounded by a multiple of a^{-1} if $\lambda < -1$.

Proof. From classical analysis, the integral

$$\int_{b \geqslant 1} b^{-1+\lambda} \, db$$

converges if and only if $\lambda < 0$. For a fixed a and any $b > 1$, $b^2 \leqslant a^2 + b^2 \leqslant (1 + a^2) b^2$. Hence

$$\int_{b\geqslant 1} b^{-1}b^{\lambda}\,db \geqslant \int_{b\geqslant 1}(a^2+b^2)^{-\frac{1}{2}}b^{\lambda}\,db \geqslant \int_{b\geqslant 1}(1+a^2)^{-\frac{1}{2}}b^{-1}b^{\lambda}\,db.$$

Hence, $L(a,\lambda)$ converges if and only if $\lambda<0$.

For $a\geqslant 1$,

$$\begin{aligned} L(a,\lambda) &= \int_{b\geqslant 1}(a^2+b^2)^{-\frac{1}{2}}b^{\lambda}\,db \\ &= \int_{ab\geqslant 1}(a^2+a^2b^2)^{-\frac{1}{2}}a^{\lambda+1}b^{\lambda}\,db \\ &= a^{\lambda}\int_{b\geqslant a^{-1}}(1+b^2)^{-\frac{1}{2}}b^{\lambda}\,db \\ &= a^{\lambda}\int_{b\geqslant 1}(1+b^2)^{-\frac{1}{2}}b^{\lambda}\,db + a^{\lambda}\int_{a^{-1}}^{1}(1+b^2)^{-\frac{1}{2}}b^{\lambda}\,db \qquad (8) \end{aligned}$$

For $a\geqslant 1$ and $a^{-1}\leqslant b\leqslant 1$ and $\lambda\neq -1$,

$$\frac{1}{\sqrt{2}}b^{\lambda}\leqslant(1+b^2)^{-\frac{1}{2}}b^{\lambda}\leqslant b^{\lambda}.$$

Taking $\int_{a^{-1}}^{1}\,db$, we obtain

$$\frac{1}{\sqrt{2}(\lambda+1)}(a^{\lambda}-a^{-1})\leqslant a^{\lambda}\int_{a^{-1}}^{1}(1+b^2)^{-\frac{1}{2}}b^{\lambda}\,db\leqslant\frac{1}{\lambda+1}(a^{\lambda}-a^{-1}).$$

Therefore, for $-1<\lambda<0$, $L(a,\lambda)$ is bounded by a multiple of a^{λ}; for $\lambda<-1$, $L(a,\lambda)$ is bounded by a multiple of a^{-1}. For $\lambda=-1$,

$$\frac{1}{\sqrt{2}}a^{-1}\ln a\leqslant a^{-1}\int_{a^{-1}}^{1}(1+b^2)^{-\frac{1}{2}}b^{-1}\,db\leqslant a^{-1}\ln a.$$

Therefore, $L(a,-1)$ is weakly bounded by a^{-1}. □

Lemma 4.1.2. *Suppose $\lambda_0<0$. Suppose $f(a)$ is weakly bounded by a^{λ} for any $0>\lambda>\lambda_0$. Then $f(a)$ is weakly bounded by a^{λ_0}.*

Combining these two lemmas, we obtain

Theorem 4.1.1. *Suppose that $a\geqslant 1$. Suppose $\phi(b)$ is weakly bounded by b^{λ} for some $\lambda<0$. Then the integral*

$$L(a,\phi(b)) = \int_{b\geqslant 1}(a^2+b^2)^{-\frac{1}{2}}\phi(b)\,db$$

converges. In addition, $L(a,\phi)$ *is weakly bounded by* a^{λ} *if* $-1\leqslant\lambda$ *and is bounded by a multiple of* a^{-1} *if* $\lambda<-1$.

In conclusion, the growth rate of $L(a,\phi(b))$ is a "truncation" of the growth rate of $\phi(b)$.

4.2. Multivariate b

Let $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_p)$. Let $B^+=\{b_1\geqslant b_2\geqslant\cdots\geqslant b_p\geqslant 1\}$. Let us consider

$$L(a,\lambda)=\int_{B^+}\prod_{i=1}^{p}(a^2+b_i^2)^{-\frac{1}{2}}b_i^{\lambda_i}\,db_i.$$

First, we observe that

$$a^2+b_i^2\geqslant a^{2\eta_i}b_i^{2-2\eta_i}$$

for any $\eta_i\in[0,1]$. The η_i is to be determined later. We obtain

$$\begin{aligned}L(a,\lambda)&\leqslant\int_{B^+}\prod_{i=1}^{p}a^{-\eta_i}b_i^{-1+\eta_i+\lambda_i}\,db_i\\&=a^{\sum_{i=1}^{p}-\eta_i}\int_{B^+}\prod_{i=1}^{p}b_i^{-1+\eta_i+\lambda_i}\,db_i.\end{aligned}\tag{9}$$

Secondly, we change the coordinates and let

$$r_i=\frac{b_i}{b_{i+1}}\quad(i=1,\ldots,p-1),$$

$$r_p=b_p.$$

Then

$$b_i=\prod_{j=i}^{p}r_j\quad(i=1,\ldots,p).$$

In addition, B^+ is transformed into $[1,\infty)^p$. The differential

$$\prod_{i=1}^{p}db_i=\prod_{i=1}^{p}\left(\prod_{j=i}^{p}r_j\right)\frac{dr_i}{r_i}=\prod_{i=1}^{p}b_i\frac{dr_i}{r_i}.$$

We obtain

$$\begin{aligned}
L(a,\lambda) &\leqslant a^{-\sum_{i=1}^{p}\eta_i} \int_{[1,\infty)^p} \prod_{i=1}^{p} b_i^{\eta_i+\lambda_i} \frac{dr_i}{r_i} \\
&= a^{-\sum_{i=1}^{p}\eta_i} \int_{[1,\infty)^p} \prod_{i=1}^{p} \left(\prod_{j=i}^{p} r_j^{\eta_i+\lambda_i} \right) \frac{dr_i}{r_i} \\
&= a^{-\sum_{i=1}^{p}\eta_i} \int_{[1,\infty)^p} \prod_{j=1}^{p} r_j^{\sum_{i=1}^{j}\eta_i+\lambda_i} \frac{dr_j}{r_j}. \qquad (10)
\end{aligned}$$

This integral converges if

$$\sum_{i=1}^{j} \eta_i + \lambda_i < 0 \quad (\forall j).$$

Theorem 4.2.1. *Suppose* $a \geqslant 1$. *If* $\lambda \prec 0$, *then* $L(a,\lambda)$ *converges. Furthermore,* $L(a,\lambda)$ *is bounded by a multiple of*

$$a^{\sum_{i=1}^{p}\eta_i}$$

with any η_i *satisfying the condition*

$$\left\{ 0 \leqslant \eta_j \leqslant 1, \sum_{i=1}^{j} \eta_i + \sum_{i=1}^{j} \lambda_i < 0 \quad (j = 1, \dots, p) \right\}.$$

The condition

$$\sum_{i=1}^{j} \eta_i + \sum_{i=1}^{j} \lambda_i < 0 \quad (j = 1, \dots, p)$$

can be restated as $\eta + \lambda \prec 0$. Combined with Lemma 4.1.2, we have

Theorem 4.2.2. *Suppose* $\phi(b_1, b_2, \dots, b_p)$ *on* B^+ *is weakly bounded by* b^λ *for some* $\lambda \prec 0$. *Then the function*

$$L(a,\phi) = \int_{B^+} \left(\prod_{i=1}^{p} (a^2 + b_i^2)^{-\frac{1}{2}} \right) \phi(b)\, db_1 \dots db_p$$

is weakly bounded by $a^{-\mu}$ *with*

$$\mu = \max\left\{ \sum_{i=1}^{p} \eta_i \,|\, 0 \leqslant \eta_j \leqslant 1, \lambda + \eta \preccurlyeq 0 \right\}.$$

We point out the second ingredient needed to carry out estimations on $L(a, \phi)$, namely, the coordinate transform from b to r.

4.3. *Multivariate* $a \in [1, \infty)^n$

This case is more complicated since the function $L(a, \phi)$ is no longer of single variable. Our result here is weaker than the results for single variable a.

First we consider

$$L(a, \lambda) = \int_{B^+} \prod_{i=1}^{p} \left(\prod_{k=1}^{n} (a_k^2 + b_i^2)^{-\frac{1}{2}} \right) b_i^{\lambda_i} \, db_i$$

We again set the parameters $\eta_{k,i}$ to be in $[0, 1]$. We have

$$a_k^2 + b_i^2 \geqslant a_k^{2\eta_{k,i}} b_i^{2-2\eta_{k,i}}.$$

Therefore, we obtain

$$\begin{aligned} L(a, \lambda) &\leqslant \int_{B^+} \prod_{i=1}^{p} \left(\prod_{k=1}^{n} a_k^{-\eta_{k,i}} b_i^{-1+\eta_{k,i}} \right) b_i^{\lambda_i} \, db_i \\ &= \prod_{k=1}^{n} a_k^{-\sum_{i=1}^{p} \eta_{k,i}} \int_{B^+} \prod_{i=1}^{p} b_i^{\lambda_i - n + \sum\limits_{k=1}^{n} \eta_{k,i}} \, db_i. \end{aligned} \tag{11}$$

Now we change the coordinates b into r. We obtain

$$\begin{aligned} L(a, \lambda) &\leqslant \prod_{k=1}^{n} a_k^{-\sum_{i=1}^{p} \eta_{k,i}} \int_{[1,\infty)^p} \prod_{i=1}^{p} \left(\prod_{j=i}^{p} r_j^{\lambda_i - n + \sum_{k=1}^{n} \eta_{k,i}} \prod_{j=i}^{p} r_j \right) \frac{dr_i}{r_i} \\ &= \prod_{k=1}^{n} a_k^{-\sum_{i=1}^{p} \eta_{k,i}} \int_{[1,\infty)^p} \prod_{i=1}^{p} \left(\prod_{j=i}^{p} r_j^{\lambda_i - n + 1 + \sum_{k=1}^{n} \eta_{k,i}} \right) \frac{dr_i}{r_i} \\ &= \prod_{k=1}^{n} a_k^{-\sum_{i=1}^{p} \eta_{k,i}} \int_{[1,\infty)^p} \prod_{j=1}^{p} r_j^{\sum_{i=1}^{j} (\lambda_i - n + 1 + \sum_{k=1}^{n} \eta_{k,i})} \frac{dr_j}{r_j}. \end{aligned} \tag{12}$$

This integral converges if

$$\sum_{i=1}^{j}\left(\lambda_i - n + 1 + \sum_{k=1}^{n} \eta_{k,i}\right) < 0 \quad (\forall 1 \leqslant j \leqslant p).$$

Since $\eta_{k,i} \in [0, 1]$, we obtain the following theorem.

Theorem 4.3.1. *Suppose $a \in [1, \infty)^p$. The integral $L(a, \lambda)$ converges if*

$$\sum_{i=1}^{j} \lambda_i - n + 1 < 0$$

for every integer $1 \leqslant j \leqslant p$. In this situation $L(a, \lambda)$ is bounded by a multiple of

$$a^{-\mu} = \prod_{k=1}^{n} a_k^{-\mu_k},$$

where $\mu_k = \sum_{i=1}^{p} \eta_{k,i}$ and $\{\eta_{k,i}\}$ satisfy

$$\eta_{k,i} \in [0, 1] \quad \forall k, i,$$

$$\sum_{i=1}^{j}\left(\lambda_i - n + 1 + \sum_{k=1}^{n} \eta_{k,i}\right) < 0 \quad \forall j. \tag{13}$$

Similarly, we obtain

Theorem 4.3.2. *Suppose $a \in [1, \infty)^p$. Suppose $\phi(b)b^{-\mathbf{n}+1}$ on B^+ is bounded by b^{λ} with $\lambda \prec 0$. Then the integral $L(a, \phi)$ converges. Furthermore, $L(a, \phi)$ is bounded by a multiple of*

$$a^{-\mu} = \prod_{k=1}^{n} a_k^{-\mu_k},$$

where $\mu_k = \sum_{i=1}^{p} \eta_{k,i}$ and $\{\eta_{k,i}\}$ satisfy

$$\eta_{k,i} \in [0, 1] \quad \forall k, i,$$

$$\sum_{i=1}^{j}\left(\lambda_i + \sum_{k=1}^{n} \eta_{k,i}\right) < 0 \quad \forall j. \tag{14}$$

5. Algorithm and examples

Suppose $\lambda \prec 0$. We are interested in finding the "maximal" η where

$$\mu_k = \sum_{i=1}^{p} \eta_{k,i}$$

with $\eta_{k,i}$ satisfying

$$\eta_{k,i} \in [0,1] \quad \forall k, i,$$

$$\sum_{i=1}^{j} \left(\lambda_i + \sum_{k=1}^{n} \eta_{k,i} \right) < 0 \quad \forall j. \tag{15}$$

5.1. *A Theorem for $a \in [1, \infty)^n$*

Write (15) as

$$\sum_{i=1}^{j} \left(\sum_{k=1}^{n} \eta_{k,i} \right) < -\sum_{i=1}^{j} \lambda_i \quad \forall j. \tag{16}$$

First of all, since $\eta_{k,i} \geqslant 0$, the sequence

$$\left\{ \sum_{i=1}^{j} \sum_{k=1}^{n} \eta_{k,i} \,|\, j \in [1,p] \right\}$$

is increasing. However, the sequence

$$\left\{ -\sum_{i=1}^{j} \lambda_i \,|\, j \in [1,p] \right\}$$

might not be increasing. Therefore, there are redundancies in inequalities (16). Let j_1 be the greatest index such that

$$\sum_{i=1}^{j_1} -\lambda_i = \min\left\{ -\sum_{i=1}^{j} \lambda_i \,|\, j \in [1,p] \right\}.$$

Then we consider $j \geqslant j_1$. Let j_2 be the greatest number such that

$$\sum_{i=1}^{j_2} -\lambda_i = \min\left\{ -\sum_{i=1}^{j} \lambda_i \,|\, j \in [j_1,p] \right\}.$$

If $j_2 = j_1$, we stop. Otherwise, we can continue on and define a sequence

$$j_0 = 0 < j_1 < j_2 < j_3 < \cdots \leqslant p$$

with

$$0 < \sum_{i=1}^{j_1} -\lambda_i < \sum_{i=1}^{j_1} -\lambda_i < \cdots < \sum_{i=1}^{p} -\lambda_i. \tag{17}$$

Our problem is equivalent to finding $\{\eta_{k,i}\}$ such that

$$\eta_{k,i} \in [0,1] \quad \forall k, i,$$

$$\sum_{i=1}^{j_s} \left(\lambda_i + \sum_{k=1}^{n} \eta_{k,i} \right) < 0 \quad (\forall j_s).$$

Once we determine the sequence

$$j_0 = 0 < j_1 < j_2 < j_3 < \cdots \leqslant p,$$

we assign numbers in $[0,1]$ to $\eta_{k,i}$ for $j_{s-1} < i \leqslant j_s$ such that

$$\sum_{i=1}^{j_s} \sum_{k=1}^{n} \eta_{k,i} < -\sum_{i=1}^{j_s} \lambda_i. \tag{18}$$

Theorem 5.1.1. *Suppose $a \in [1, \infty)^p$. Suppose $\phi(b) b^{-\mathbf{n}+\mathbf{1}}$ on B^+ is bounded weakly by b^{λ} with $\lambda \prec 0$. Then the integral $L(a, \phi)$ converges. Furthermore, $L(a, \phi)$ is weakly bounded by*

$$a^{-\mu} = \prod_{k=1}^{n} a_k^{-\mu_k},$$

where $\mu_k = \sum_{i=1}^{p} \eta_{k,i}$ and for each $j_s > 0$, $\{\eta_{k,i} \in [0,1]\}$ satisfy one of the following

1.

$$\sum_{i=1}^{j_s} \left(\lambda_i + \sum_{k=1}^{n} \eta_{k,i} \right) = 0; \tag{19}$$

2.

$$\sum_{i=1}^{j_s}\left(\lambda_i+\sum_{k=1}^{n}\eta_{k,i}\right)<0; \quad \textit{and} \quad \eta_{k,i}=1 \quad \forall k\in[1,n], i\in[j_{s-1}+1,j_s]. \tag{20}$$

Proof. It suffices to show that for any $0<t<1$, $t\eta_{k,i}$ satisfies the conditions in Theorem 4.3.2. Apparently, we have

$$t\eta_{k,i}\in[0,1] \quad (\forall i,k)$$

and

$$\sum_{i=1}^{j_s}\left(\lambda_i+\sum_{k=1}^{n}\eta_{k,i}\right)\leqslant 0.$$

From (17), for every $s\geqslant 1$,

$$\sum_{i=1}^{j_s}\left(\lambda_i+\sum_{k=1}^{n}t\eta_{k,i}\right)\leqslant(1-t)\sum_{i=1}^{j_s}\lambda_i<0.$$

We have shown that (14) holds for $j=j_s$. For $j_{s-1}+1\leqslant j\leqslant j_s$, since $\eta_{k,i}\geqslant 0$,

$$\begin{aligned}\sum_{i=1}^{j}\sum_{k=1}^{n}t\eta_{k,i}&\leqslant\sum_{i=1}^{j_s}\sum_{k=1}^{n}t\eta_{k,i}\\&<-\sum_{i=1}^{j_s}\lambda_i\\&\leqslant-\sum_{i=1}^{j}\lambda_i.\end{aligned} \tag{21}$$

Thus, (14) holds for all $1\leqslant j\leqslant p$. By Theorem 4.3.2, $L(a,\phi)$ is bounded by $a^{-t\mu}$ with $\mu_k=\sum_{i=1}^{p}\eta_{k,i}$. Hence, $L(a,\phi)$ is weakly bounded by $a^{-\mu}$. $\square$

5.2. $L(p,n)$ and Algorithm for $a\in A^+$

Theorem 5.1.1 only assumes $a\in[1,\infty)^n$. Suppose from now on

$$a\in A^+=\{a_1\geqslant a_2\geqslant\cdots\geqslant a_n\geqslant 1\}.$$

In order to gain a better control over $L(a,\phi)$, we just need to assign numbers to $\eta_{1,i}$

to make μ_1 as big as possible, then assign numbers to $\eta_{2,i}$ to make μ_2 as big as possible and so on. The only requirement is either (19) or (20). Our algorithm can be stated as follows.

Definition 5.2.1. Fix j_s and assume that $\{\eta_{k,i} \mid i \leqslant j_{s-1}\}$ are known. We assign numbers between 0 and 1 to $\eta_{k,i}$ for $j_{s-1} < i \leqslant j_s$ in the following way. If (20) holds, assign $\eta_{k,i} = 1$ for all k and all $j_{s-1} + 1 \leqslant i \leqslant j_s$. We are done. If (19) holds, we choose $\{\eta_{1,i} \mid j_{s-1} + 1 \leqslant i \leqslant j_s\}$ satisfying (19) and maximizing $\sum_{i=j_{s-1}+1}^{j_s} \eta_{1,i}$. The order of assigning numbers to $\{\eta_{1,i}\}$ for $j_{s-1} < i \leqslant j_s$ is not of our concern. Update (19). If (19) is trivial, we assign zero to the rest of $\{\eta_{k,i} \mid j_{s-1} + 1 \leqslant i \leqslant j_s\}$ and stop. If not, choose $\{\eta_{2,i} \mid j_{s-1} + 1 \leqslant i \leqslant j_s\}$ satisfying (19) and maximizing $\sum_{i=j_{s-1}+1}^{j_s} \eta_{2,i}$. Update (19) and repeat this process. We do this for each j_s until we reach $i = p$. Finally, we compute

$$\mu_k = \sum_{i=1}^{p} \eta_{k,i} \quad (1 \leqslant k \leqslant n)$$

and obtain a unique μ. Write

$$L(p,n)(\lambda) = -\mu.$$

The domain of $L(p,n)$ are apparently p-dimensional real vectors such that

$$\lambda \prec 0.$$

The range of $L(p,n)$ are n-dimensional real vectors such that

$$\mu \prec 0.$$

$L(p,n)$, in general, does not produce the precise information for the Langlands parameters under theta correspondence; but for a special class of representations, $L(p,n)$ will be precise. Now, Theorem 5.1.1 can be restated as follows.

Theorem 5.2.1. *Suppose $a \in A^+$. Suppose $\phi(b)b^{-\mathbf{n}+\mathbf{1}}$ on B^+ is bounded weakly by b^λ with $\lambda \prec 0$. Then the integral $L(a,\lambda)$ converges. Furthermore, $L(a,\lambda)$ is weakly bounded by a^μ for $\mu = L(p,n)(\lambda)$.*

5.3. Examples

Now let us compute a few examples. Suppose $p \leqslant n$.

Example 1. For

$$\lambda = (-\tfrac{1}{2}, -\tfrac{3}{2}, \ldots, -p + \tfrac{1}{2}),$$

$$L(p,n)(\lambda) = (-p + \tfrac{1}{2}, -p + 1 + \tfrac{1}{2}, \ldots, -\tfrac{1}{2}, 0, \ldots, 0).$$

Example 2. For

$$\lambda = (-1, -2, \ldots, -p),$$

$$L(p,n)(\lambda) = (-p, -p + 1, \ldots, -1, 0, \ldots, 0).$$

Example 3. For

$$\lambda = (-\tfrac{1}{2}, -\tfrac{3}{2}, \ldots, -n + \tfrac{1}{2}),$$

$$L(n,p)(\lambda) = (-n + \tfrac{1}{2}, -n + \tfrac{3}{2}, \ldots, -n - \tfrac{1}{2} + p).$$

Example 4. For

$$\lambda = (-1, -2, \ldots, -n),$$

$$L(n,p)(\lambda) = (-n, -n + 1, \ldots, -n + p - 1).$$

6. Dual pair $(O(p,q), Sp_{2n}(\mathbb{R}))$ and estimates on $\theta_s(\pi)$

Let $O(p,q)$ be the orthogonal group preserving the symmetric form defined by

$$I_{p,q} = \begin{pmatrix} 0_p & 0 & I_p \\ 0 & I_{q-p} & 0 \\ I_p & 0 & 0_p \end{pmatrix}$$

and $Sp_{2n}(\mathbb{R})$ be the standard symplectic group. We define a symplectic form on $V = M(p+q, 2n)$ by

$$\Omega(v_1, v_2) = Trace(v_1 W v_2^t I_{p,q}) \quad (\forall v_1, v_2 \in V).$$

Now as a dual pair in $Sp(V,\Omega)$, $O(p,q)$ acts by left multiplication and $Sp_{2n}(\mathbb{R})$ acts by (inverse) right multiplication. We denote both actions on $M(p+q,2n)$ by m.

6.1. *The dual pair representation $\omega(p,q;2n)$*

Let $x_{i,j}$ be the entries in first n columns of $v \in V$ and $y_{i,j}$ be the entries in the second n columns of v. Let

$$X = \{v \in V \mid y_{i,j} = 0\}, \quad Y = \{v \in V \mid x_{i,j} = 0\}.$$

Then X and Y are both Lagrangian subspaces of (V,Ω). We realize the Schrödinger model of $Mp(V,\Omega)$ on $L^2(X)$. Let $\mathscr{P}(p,q;2n)$ be the Harish-Chandra module. We call the admissible representation

$$(\omega(p,q;2n), \mathscr{P}(p,q;2n))$$

the dual pair representation.

Now let $b = diag(b_1, b_2, \ldots b_p, 1, \ldots, 1, b_1^{-1}, \ldots, b_p^{-1})$. Let

$$B^+ = \{b \mid b_1 \geqslant b_2 \geqslant \cdots \geqslant b_p \geqslant 1\} \subseteq O(p,q).$$

Let $a = diag(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}, a_1, \ldots, a_n)$. Let

$$A^+ = \{a \mid a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 1\} \subseteq Sp_{2n}(\mathbb{R}).$$

For $1 \leqslant j \leqslant n$, let

$$m(b)e_{i,j} = \begin{cases} b_i e_{i,j}, & i = 1, \ldots, p, \\ e_{i,j}, & i = p+1, \ldots, q, \\ b_i^{-1} e_{i,j}, & i = q+1, \ldots, p+q, \end{cases}$$

$$m(a)e_{i,j} = a_j e_{i,j} \quad (i = 1, \ldots, p+q)$$

These formulae indicate that the embedding m of A and B into $GL(X)$ are simply the left multiplication and the (inverse) right multiplication. In fact,

$$m(ab)e_{i,j} = \begin{cases} b_i a_j e_{i,j}, & i = 1, \ldots, p, \\ a_j e_{i,j}, & i = p+1, \ldots, q, \\ b_i^{-1} a_j e_{i,j}, & i = q+1, \ldots, p+q. \end{cases}$$

Let $b(g_1)$ be the middle term of KAK decomposition of g_1 with $b(g_1) \in B^+$. Let $a(g_2)$ be the middle term of KAK decomposition of g_2 with $a(g_2) \in A^+$. Observe that

$$(b_i a_j + b_i^{-1} a_j^{-1})(b_i^{-1} a_j + b_i a_j^{-1}) = (b_i^2 + b_i^{-2} + a_j^2 + a_j^{-2}).$$

From Theorem 3.3.1, we obtain

Theorem 6.1.1. *For any* $\phi, \psi \in \mathscr{P}(p, q; 2n)$,

$$\begin{aligned} &|(\omega(p,q;2n)(m(ab))\phi, \psi)| \\ &\leqslant C \prod_{i=1}^{p} \prod_{j=1}^{n} (b_i^2 + b_i^{-2} + a_j^2 + a_j^{-2})^{-\frac{1}{2}} \prod_{j=1}^{n} (a_j + a_j^{-1})^{-\frac{q-p}{2}}. \end{aligned}$$

Furthermore, this estimate holds for $m(g_1 g_2)$ *by substituting* $b(g_1)$ *and* $a(g_2)$ *into the right-hand side.*

We denote

$$\prod_{i=1}^{p} \prod_{j=1}^{n} (b_i^2 + b_i^{-2} + a_j^2 + a_j^{-2})^{-\frac{1}{2}}$$

by $H(a, b)$.

6.2. *Growth control on* $\theta_s(p, q; 2n)(\pi)$

Let (π, V) be an irreducible Harish-Chandra module in $\mathscr{R}_s(p, q; 2n)$. We are interested in the following integral:

$$\int_{MO(p,q)} (\omega(p,q;2n)(g_1 g_2)\phi, \psi)(v, \pi(g_1)u)\, dg_1 \quad (u, v \in V; \psi, \psi \in \mathscr{P}(p, q; 2n)).$$

Our goal is to control the growth of this integral as a function on $MSp_{2n}(\mathbb{R})$. From Theorems 6.1.1 and 3.4.1, we may as well consider

$$\int_{B^+} \prod_{j=1}^{n} (a_j + a_j^{-1})^{-\frac{q-p}{2}} H(a, b) b^{\lambda} b^{2\rho_1} \prod_{i=1}^{p} \frac{db_i}{b_i}. \tag{22}$$

Here ρ_1 is the half sum of the restricted positive roots of $O(p, q)$:

$$\rho_1 = \left(\frac{p+q-2}{2}, \frac{p+q-4}{2}, \ldots, \frac{q-p}{2}\right)$$

and $(\pi(g_1)u, v)$ is bounded by a multiple of $b(g_1)^{\lambda}$. We observe that

$$\prod_{j=1}^{n}(a_j+a_j^{-1})^{-\frac{\mathbf{q}-\mathbf{p}}{2}}\int_{B^+}H(a,b)b^{\lambda}b^{2\rho_1}\prod_{i=1}^{p}\frac{db_i}{b_i}\leqslant Ca(g_2)^{-\frac{\mathbf{q}-\mathbf{p}}{2}}L(a,\lambda+2\rho_1-\mathbf{1}).$$

From Theorem 5.2.1, we obtain

Lemma 6.2.1. *Let* $\pi\in\mathscr{R}_s(p,q;2n)$. *Suppose K-finite matrix coefficients of* π *are bounded by some* $Cb(g_1)^{\lambda}$ *with*

$$\lambda+2\rho(O(p,q))-\mathbf{n}\prec 0.$$

Then the matrix coefficients of $\theta_s(p,q;2n)(\pi)$ *are weakly bounded by*

$$a(g_2)^{L(p,n)(\lambda+2\rho(O(p,q))-\mathbf{n})-\frac{\mathbf{q}-\mathbf{p}}{\mathbf{2}}}.$$

Recall that $\pi\in\mathscr{R}_{ss}(p,q;2n)$ if and only if

$$\Re(v)-\left(\mathbf{n}-\frac{\mathbf{p}+\mathbf{q}}{2}\right)+\rho(O(p,q))\preccurlyeq 0$$

for every leading exponent v of π. Take

$$\lambda=\mathbf{n}-\frac{\mathbf{p}+\mathbf{q}}{2}-\rho(O(p,q))+(\delta,0,\ldots,0)$$

with δ a small positive number. Then matrix coefficients of π are bounded by multiples of $b(g_1)^{\lambda}$:

$$\begin{aligned}
&L(p,n)(\lambda+2\rho(O(p,q))-\mathbf{n})\\
&\quad=L(p,n)\Big(-\frac{\mathbf{p}+\mathbf{q}}{2}+\rho(O(p,q))+(\delta,0,\ldots,0)\Big)\\
&\quad=L(p,n)(-1+\delta,-2,\ldots,-p)\\
&\quad=\begin{cases}(-p+\delta,-p+1,\ldots,-1,0,\ldots,0), & n\geqslant p,\\ (-p+\delta,-p+1,\ldots,-p+n-1), & n<p.\end{cases}
\end{aligned}\tag{23}$$

From Lemma 4.1.2, we obtain the following theorem:

Theorem 6.2.1. *Suppose that* $\pi\in\mathscr{R}_{ss}(p,q;2n)$. *Then the matrix coefficients of* $\theta_s(p,q;2n)(\pi)$ *are weakly bounded by*

$$a(g_2)^{\left(-\frac{p+q}{2},-\frac{p+q-2}{2},\ldots,-\frac{q-p}{2},\ldots,-\frac{q-p}{2}\right)}\quad(\textit{if}\ \ n\geqslant p),$$

$$a(g_2)^{\left(-\frac{p+q}{2},-\frac{p+q-2}{2},\ldots,-\frac{p+q-2n+2}{2}\right)} \quad (\textit{if } \; n<p).$$

6.3. *Growth control on $\theta(2n;p,q)_s(\pi)$*

Let (π, V) be an irreducible Harish-Chandra module in $\mathscr{R}_s(2n;p,q)$. We are interested in the following integral:

$$\int_{MSp_{2n}(\mathbb{R})} (\omega(p,q;2n)(g_1g_2)\phi,\psi)(v,\pi(g_2)u)\, dg_2 \quad (u,v\in V;\phi,\psi\in\omega(p,q;2n)).$$

Our goal is to control the growth of this integral as a function on $MO(p,q)$. From Theorems 6.1.1 and 8.47 in [16], it suffices to consider

$$\int_{A^+} H(a,b)a^{\lambda}a^{2\rho_2}\prod_{j=1}^{n}(a_j+a_j^{-1})^{-\frac{q-p}{2}}\frac{da_j}{a_j}. \tag{24}$$

Here ρ_2 is the half sum of the restricted positive roots of $Sp_{2n}(\mathbb{R})$:

$$\rho_2 = (n, n-1, \ldots, 1)$$

and $(\pi(g_2)u, v)$ is bounded by a multiple of $a(g_2)^{\lambda}$. Apparently, the integral (24) can be controlled by $CL(a, \lambda - \frac{\mathbf{q}-\mathbf{p}}{\mathbf{2}} - \mathbf{1} + 2\rho_2)$. From Theorem 5.2.1, we obtain

Lemma 6.3.1. *Suppose that $\pi\in\mathscr{R}_s(2n;p,q)$, i.e., the matrix coefficients of π are bounded by multiples of $a(g_2)^{\lambda}$ for some*

$$\lambda + 2\rho_2 - \frac{\mathbf{p}+\mathbf{q}}{2} \prec 0.$$

Then the matrix coefficients of $\theta_s(2n;p,q)(\pi)$ are weakly bounded by

$$b(g_1)^{L(n,p)\left(\lambda+2\rho_2-\frac{\mathbf{p}+\mathbf{q}}{2}\right)}.$$

Recall that the representation π is in $\mathscr{R}_{ss}(2n;p,q)$ if and only if

$$\Re(v) + \mathbf{n} + \mathbf{1} + \rho_2 - \frac{\mathbf{p}+\mathbf{q}}{2} \preccurlyeq 0$$

for every leading exponent v of π. Now let

$$\lambda = -\mathbf{n} - \mathbf{1} - \rho_2 + \frac{\mathbf{p}+\mathbf{q}}{2} + (\delta, 0, \ldots, 0),$$

where δ is a small positive number. Then the matrix coefficients of π are bounded by multiples of $a(g_2)^{\lambda}$ and

$$\lambda + 2\rho_2 - \frac{\mathbf{p}+\mathbf{q}}{2} = -\mathbf{n} - \mathbf{1} + \rho_2 + \delta = (-1+\delta, -2, \ldots, -n).$$

Therefore

$$L(n,p)\Big(\lambda + 2\rho_2 - \frac{p+q}{2}\Big) = (-n+\delta, -n+1, \ldots, -1, 0, \ldots, 0) \quad (p>n),$$

$$L(n,p)\Big(\lambda + 2\rho_2 - \frac{p+q}{2}\Big) = (-n+\delta, -n+1, \ldots, -n+p-1) \quad (p\leqslant n).$$

From Lemma 4.1.2, we obtain

Theorem 6.3.1. *Suppose that* π *is in* $\mathscr{R}_{ss}(2n; p, q)$. *Then matrix coefficients of* $\theta(2n; p, q)_s(\pi)$ *is weakly bounded by*

$$b(g_1)^{(-n,-n+1,\ldots,-1,0,\ldots,0)} \quad (p>n),$$

$$b(g_1)^{(-n,-n+1,\ldots,-n+p-1)} \quad (p\leqslant n).$$

6.4. Applications to unitary representations

We may now combine our results from [9] with the results we established in the previous two sections. Let us start with a unitary representation in $\mathscr{R}_{ss}(p, q; 2n)$.

Theorem 6.4.1. *Suppose* $p+q\leqslant 2n+1$. *Suppose* π *is a unitary representation in* $\mathscr{R}_{ss}(p, q; 2n)$ *and* $(,)_\pi$ *is nonvanishing. Then* $\theta_s(p, q; 2n)(\pi)$ *is unitary. Furthermore, the matrix coefficients of* $\theta(p, q; 2n)(\pi)$ *is weakly bounded by*

$$a(g_2)^{\left(\overbrace{-\frac{p+q}{2}, -\frac{p+q-2}{2}, \ldots, -\frac{q-p}{2}-1}^{p}, \overbrace{-\frac{q-p}{2}, \ldots, -\frac{q-p}{2}}^{n-p}\right)}.$$

In [9], we have proved that for $p+q$ odd we can loose our restrictions from $\mathscr{R}_{ss}(p,q;2n)$ a little bit and unitarity still holds for $\theta_s(p,q;2n)(\pi)$. The precise statement can be stated as follows.

Theorem 6.4.2. *Suppose $p+q \leqslant 2n+1$ and $p+q$ is odd. Suppose π is a unitary representation in $\mathscr{R}_s(p,q;2n)$ such that each leading exponent v of π satisfies*

$$\Re(v) - \left(\mathbf{n} - \frac{\mathbf{p}+\mathbf{q}-\mathbf{1}}{\mathbf{2}}\right) + \rho(O(p,q)) \preccurlyeq 0.$$

If $(,)_\pi$ is nonvanishing, then $\theta_s(p,q;2n)(\pi)$ is unitary. Furthermore, the matrix coefficients of $\theta_s(p,q;2n)(\pi)$ is weakly bounded by

$$a(g_2)^{\left(\overbrace{-\frac{p+q-1}{2}, -\frac{p+q-3}{2}, \ldots, -\frac{q-p+1}{2}}^{p}, \overbrace{-\frac{q-p}{2} \ldots, -\frac{q-p}{2}}^{n-p}\right)}.$$

Similarly, we obtain the following theorem regarding $\theta_s(2n;p,q)(\pi)$.

Theorem 6.4.3. *Suppose that $n < p \leqslant q$. Suppose that π is a unitary representation in $\mathscr{R}_{ss}(2n;p,q)$. If $(,)_\pi$ is nonvanishing, then $\theta_s(2n;p,q)(\pi)$ is unitary. Furthermore, the matrix coefficients of $\theta_s(2n;p,q)(\pi)$ are weakly bounded by*

$$b(g_1)^{\left(\overbrace{-n, -n+1, \ldots, -1}^{n}, \overbrace{0, \ldots, 0}^{p-n}\right)}.$$

7. The idea of quantum induction

In this section, we will define quantum induction first. Then we compute the infinitesimal characters of quantum induced representations. Finally, we give some indication how the limit of quantum induction will become parabolic induction.

7.1. Quantum induction on orthogonal group

Consider the composition of $\theta_s(p,q;2n)$ with $\theta_s(2n;p',q')$. Suppose $\pi \in \mathscr{R}_{ss}(p,q;2n)$ and $p+q \leqslant 2n+1$. If $(,)_\pi$ is nonvanishing, then $\theta_s(p,q;2n)(\pi)$ is unitary and its leading exponents satisfy

$$\Re(v) \preccurlyeq \left(\overbrace{-\frac{p+q}{2}, -\frac{p+q-2}{2}, \ldots, -\frac{q-p+2}{2}}^{p}, \overbrace{-\frac{q-p}{2}, \ldots -\frac{q-p}{2}}^{n-p} \right).$$

The representation $\theta_s(p,q;2n)(\pi)$ is in $\mathscr{R}_{ss}(2n;p',q')$ if

$$\left(\overbrace{-\frac{p+q}{2}, -\frac{p+q-2}{2}, \ldots, -\frac{q-p+2}{2}}^{p}, \overbrace{-\frac{q-p}{2}, \ldots -\frac{q-p}{2}}^{n-p} \right) + (\mathbf{n}+\mathbf{1}) + \rho(Sp_{2n}(\mathbb{R})) - \frac{\mathbf{p}'+\mathbf{q}'}{2} \preccurlyeq 0.$$

This is true if and only if

$$-\frac{p+q}{2} + n + 1 + n - \frac{p'+q'}{2} \leqslant 0.$$

We obtain

Theorem 7.1.1. *Suppose*

$$q' \geqslant p' > n,$$

$$p' + q' - 2n \geqslant 2n - (p+q) + 2 \geqslant 1,$$

$$p + q = p' + q' \pmod 2.$$

Let π be an irreducible unitary representation in $\mathscr{R}_{ss}(p,q;2n)$. Suppose that $(,)_\pi$ does not vanish. Then $\theta_s(p,q;2n)(\pi)$ is unitary and

$$\theta_s(p,q;2n)(\pi) \in \mathscr{R}_{ss}(2n;p',q').$$

Furthermore, $\theta_s(2n;p',q')\theta_s(p,q;2n)(\pi)$ is either a unitary representation or the NULL representation.

Definition 7.1.1. Let π be a unitary representation in $\mathscr{R}_{ss}(p,q;2n)$. Suppose that

$$q' \geqslant p' > n,$$

$$p' + q' - 2n \geqslant 2n - (p+q) + 2 \geqslant 1,$$

$$p + q = p' + q' \pmod 2.$$

We call

$$Q(p,q;2n;p',q'):\pi \to \theta_s(2n;p',q')\theta_s(p,q;2n)(\pi)$$

the (one-step) quantum induction.

If one of $(,)_\pi$ and $(,)_{\theta(p,q;2n)(\pi)}$ vanishes, we define our quantum induction $Q(p,q;2n;p',q')(\pi)$ to be the NULL representation.

7.2. Quantum induction on symplectic group

Next, we consider the composition of $\theta_s(2n;p,q)$ with $\theta_s(p,q;2n')$. Suppose $n<p\leqslant q$. Let π be a unitary representation in $\mathscr{R}_{ss}(p,q;2n)$. Suppose $(,)_\pi$ is not vanishing. Then the leading exponents of $\theta(2n;p,q)$ satisfy

$$\Re(v) \preccurlyeq (-n,-n+1,\ldots,-1,0,\ldots,0).$$

Therefore, $\theta(2n;p,q)$ is in $\mathscr{R}_{ss}(MO(p,q),\omega(p,q;2n'))$ if

$$(-n,-n+1,\ldots,-1,0,\ldots,0)-\mathbf{n}'+\frac{\mathbf{p}+\mathbf{q}}{2}+\rho(O(p,q))\preccurlyeq 0.$$

This is true if and only if

$$-n-n'+p+q-1\leqslant 0.$$

Theorem 7.2.1. *Suppose $2n'-p-q\geqslant p+q-2n-2$ and $n<p\leqslant q$. Suppose π is a unitary representation in $\mathscr{R}_{ss}(2n;p,q)$. If $(,)_\pi$ does not vanish, then $\theta_s(2n;p,q)(\pi)$ is unitary and it is in $\mathscr{R}_{ss}(p,q;2n')$. Furthermore, $\theta_s(p,q;2n')\theta_s(2n;p,q)(\pi)$ is a unitary representation or the NULL representation.*

Definition 7.2.1. Let p,q,n,n' be nonnegative integers such that

$$n<p\leqslant q,$$

$$p+q-2n-2\leqslant 2n'-p-q.$$

Let π be a unitary representation in $\mathscr{R}_{ss}(2n;p,q)$. We call

$$Q(2n;p,q;2n'):\pi \to \theta_s(p,q;2n')\theta_s(2n;p,q)(\pi)$$

the (one-step) quantum induction.

If one of $(,)_\pi$ and $(,)_{\theta_s(2n;p,q)(\pi)}$ vanishes, we define our quantum induction $Q(2n;p,q;2n')(\pi)$ to be 0. Thus the domain of our quantum induction is $\mathscr{R}_{ss}(2n;p,q)$.

7.3. *Quantum inductions*

We can further define two-step quantum induction and so on. The general quantum induction

$$Q(p_1, q_1; 2n_1; p_2, q_2; 2n_2; \ldots)(\pi)$$

is defined as the composition of θ_s, under the following conditions:

1. *Initial conditions*: $p_1 + q_1 \leqslant 2n_1 + 1$.
   π is a unitary representation in $\mathscr{R}_{ss}(p_1, q_2; 2n_1)$, i.e., its leading exponents satisfy

$$\Re(v) - \mathbf{n_1} + \frac{\mathbf{p_1 + q_1}}{2} + \rho(O(p_1, q_1)) \preccurlyeq 0.$$

2. *Inductive conditions*: $\forall j$,

$$n_j < p_{j+1} \leqslant q_{j+1},$$

$$p_{j+1} + q_{j+1} - 2n_j \leqslant 2n_{j+1} - p_{j+1} - q_{j+1} + 2,$$

$$2n_j - p_j - q_j + 2 \leqslant p_{j+1} + q_{j+1} - 2n_j,$$

$$p_j + q_j \equiv p_{j+1} + q_{j+1} \pmod 2.$$

Theorem 7.3.1. *The representation*

$$Q(p_1, q_1; 2n_1; p_2, q_2; 2n_2; \ldots)(\pi)$$

is either an irreducible unitary representation or the NULL representation.

The general quantum induction

$$Q(2n_1; p_1, q_1; 2n_2; p_2, q_2; 2n_3; \ldots)(\pi)$$

is defined as the composition of θ_s under the following conditions:

1. *Initial conditions*: $n_1 < p_1 \leqslant q_1$.
   π is a unitary representation in $\mathscr{R}_{ss}(2n_1; p_1, q_1)$, i.e., its leading exponents satisfy

$$\Re(v) - \frac{\mathbf{p_1 + q_1}}{2} + \mathbf{n} + \mathbf{1} + \rho(Sp_{2n_1}(\mathbb{R})) \preccurlyeq 0.$$

2. *Inductive conditions*: $\forall j$,

$$n_j < p_j \leqslant q_j,$$

$$p_j + q_j - 2n_j \leqslant 2n_{j+1} - p_j - q_j + 2,$$

$$2n_{j+1} - p_j - q_j + 2 \leqslant p_{j+1} + q_{j+1} - 2n_{j+1},$$

$$p_j + q_j \equiv p_{j+1} + q_{j+1} \pmod 2.$$

Theorem 7.3.2. *The representation*

$$Q(2n_1; p_1, q_1; 2n_2; p_2, q_2; 2n_3; \ldots)(\pi)$$

is either an irreducible unitary representation or the NULL representation.

Our inductive conditions are natural within the frame work of orbit method (see [6,23,25,31]). The nonvanishing of θ_s has been studied in [6,8]. It can be assumed as a working hypothesis in the framework of quantum induction. Notice that Q is defined as a composition of θ_s. Thus, it is not known that Q is exactly the composition of theta correspondences over $\mathbb{R}$. This problem hinges on one earlier problem mentioned by Li [19]:

Is $(,)_\pi$ nonvanishing if $\pi \in \mathscr{R}(MG_1, MG_2) \cap \mathscr{R}_s(MG_1, MG_2)$?

Our result in [7] which is derived from Howe's results in [13] confirms the converse: π is in $\mathscr{R}(MG_1, MG_2)$ if $(,)_\pi$ does not vanish.

Therefore, if $Q(*)(\pi) \neq 0$, $Q(*)$ is the composition of θ.

7.4. *Infinitesimal characters*

Infinitesimal characters under theta correspondence were studied by Przebinda [26]. We denote the infinitesimal character of an irreducible representation π by $\mathscr{I}(\pi)$. Przebinda's result can be stated as follows.

Theorem 7.4.1 (Przebinda). 1. *Suppose* $p+q<2n+1$. *Then*

$$\mathscr{I}(\theta(p,q;2n)(\pi)) = \mathscr{I}(\pi) \oplus \left(n - \frac{p+q}{2}, n - \frac{p+q}{2} - 1, \ldots, 1 + \left[\frac{p+q}{2}\right] - \frac{p+q}{2}\right).$$

2. *Suppose* $2n+1<p+q$. *Then*

$$\mathscr{I}(\theta(2n;p,q)(\pi)) = \mathscr{I}(\pi) \oplus \left(\frac{p+q}{2} - n - 1, \frac{p+q}{2} - n - 2, \ldots, \frac{p+q}{2} - \left[\frac{p+q}{2}\right]\right).$$

3. *Suppose* $p+q=2n$ *or* $p+q=2n+1$. *Then* $\mathscr{I}(\theta(p,q;2n)(\pi)) = \mathscr{I}(\pi)$.

Now we can compute the infinitesimal character under quantum induction.

Corollary 7.4.1. *Suppose* $Q(*)(\pi)\neq 0$.

1. *If* $p+q$ *is even, then*

$$\mathscr{I}(Q(2n;p,q;2n')(\pi))=\mathscr{I}(\pi)\oplus\left(\frac{p+q}{2}-n-1,\frac{p+q}{2}-n-2,\ldots,0\right)$$
$$\oplus\left(n'-\frac{p+q}{2},n'-\frac{p+q}{2}-1,\ldots,1\right).$$

2. *If* $p+q$ *is odd, then*

$$\mathscr{I}(Q(2n;p,q;2n')(\pi))=\mathscr{I}(\pi)\oplus\left(\frac{p+q}{2}-n-1,\frac{p+q}{2}-n-2,\ldots,\frac{1}{2}\right)$$
$$\oplus\left(n'-\frac{p+q}{2},n'-\frac{p+q}{2}-1,\ldots,\frac{1}{2}\right).$$

3. *If* $p+q$ *is even, then*

$$\mathscr{I}(Q(p,q;2n;p',q')(\pi))=\mathscr{I}(\pi)\oplus\left(n-\frac{p+q}{2},n-\frac{p+q}{2}-1,\ldots,1\right)$$
$$\oplus\left(\frac{p'+q'}{2}-n-1,\frac{p'+q'}{2}-n-2,\ldots,0\right).$$

4. *If* $p+q$ *is odd, then*

$$\mathscr{I}(Q(p,q;2n;p',q')(\pi))=\mathscr{I}(\pi)\oplus\left(n-\frac{p+q}{2},n-\frac{p+q}{2}-1,\ldots,\frac{1}{2}\right)$$
$$\oplus\left(\frac{p'+q'}{2}-n-1,\frac{p'+q'}{2}-n-2,\ldots,\frac{1}{2}\right).$$

We shall now take a look at some "limit" cases under quantum induction.

Example I. $p+q+p'+q'=4n+2$.
In this case,

$$n-\frac{p+q}{2}=\frac{p'+q'}{2}-n-1.$$

Therefore,

$$\mathscr{I}(Q(p,q;2n;p',q')(\pi)) = \mathscr{I}(\pi) \oplus \overbrace{\left(n-\tfrac{p+q}{2}, n-\tfrac{p+q}{2}-1, \ldots, 1+\tfrac{p+q}{2}-n, \tfrac{p+q}{2}-n\right)}^{2n-p-q+1}.$$

Example II. $2n-p-q+2 = p'+q'-2n$ and $p-p' = q-q'$.

Notice first that

$$p'-p+q'-q = (p'+q')-(p+q) = 4n+2-2(p+q).$$

Therefore

$$\frac{p'-p}{2} = \frac{p'-p+q'+q}{4} = n-\frac{p+q}{2}+\frac{1}{2}.$$

Recall from Proposition 8.22 [16]

$$\begin{aligned}
&\mathscr{I}(Ind^{SO_0(p',q')}_{SO_0(p,q)GL_0(p'-p)N}(\pi\otimes 1))\\
&= \mathscr{I}(\pi\otimes 1)\\
&= \mathscr{I}(\pi)\oplus\left(\frac{p'-p-1}{2}, \frac{p'-p-3}{2}, \ldots, -\frac{p'-p-3}{2}, -\frac{p'-p-1}{2}\right)\\
&= \mathscr{I}(\pi)\oplus\left(n-\frac{p+q}{2}, n-\frac{p+q}{2}-1, \ldots, 1+\frac{p+q}{2}-n, \frac{p+q}{2}-n\right)\\
&= \mathscr{I}(Q(p,q;2n;p',q')(\pi)).
\end{aligned} \tag{25}$$

This suggests that $Q(p,q;2n;p',q')(\pi)$ as a representation of $SO_0(p,q)$ can be decomposed as direct sum of some parabolically induced unitary representation (see Conjecture I).

Example III. $n+n'+1 = p+q$.

In this case,

$$\frac{p+q}{2}-n-1 = n'-\frac{p+q}{2},$$

$$\frac{n'-n-1}{2} = \frac{p+q}{2}-n-1.$$

From Proposition 8.22 [16] and the corollary,

$$\begin{aligned}
&\mathscr{I}(Ind_{Sp_{2n}(\mathbb{R})GL(n'-n)N}^{Sp_{2n'}(\mathbb{R})}(\pi\otimes 1))\\
&\quad=\mathscr{I}(\pi)\oplus\left(\frac{n'-n-1}{2},\frac{n'-n-3}{2},\ldots,-\frac{n'-n-3}{2},-\frac{n'-n-1}{2}\right)\\
&\quad=\mathscr{I}(\pi)\oplus\left(\frac{p+q}{2}-n-1,\frac{p+q}{2}-n-2,\ldots,-\frac{p+q}{2}+n+2,-\frac{p+q}{2}+n+1\right)\\
&\quad=\mathscr{I}(Q(2n;p',q';2n')(\pi)). \qquad (26)
\end{aligned}$$

This suggests that $Q(2n;p,q;2n')(\pi)$ can be obtained as subfactors of certain parabolic induced representation. We prove this connection in [10].

Let me make some final remarks concerning the definition of quantum induction Q. Notice that $Q(p,q;2n;p',q')(\pi)$ contains distributions of the following form:

$$\begin{aligned}
&\int_{MSp_{2n}(\mathbb{R})}\omega(p',q';2n)(g_1)\phi_1\otimes\int_{MO(p,q)}\omega(p,q;2n)^c(g_1g_2)\phi_2\otimes\pi(g_2)v\,dg_2\,dg_1\\
&\quad=\int_{MO(p,q)}\omega(p,q;2n)^c(g_2)\left[\int_{MSp_{2n}(\mathbb{R})}\omega(p'+q,q'+p;2n)(g_1)(\phi_1\otimes\phi_2)\,dg_1\right]\\
&\qquad\otimes\pi(g_2)v. \qquad (27)
\end{aligned}$$

Our discussions in this paper guaranteed absolute integrability of this integral. Notice that the vectors in $[*]$ are in $\theta(2n;p'+q,q+p')(1)$.

Definition 7.4.1. Suppose $p'+q\geqslant 2n$, $q'+p\geqslant 2n$ and $p+q+p'+q'$ is even. Consider the dual pair $(O(p'+q,q'+p),Sp_{2n}(\mathbb{R}))$. This is a dual pair in the stable range [14,20]. Then $\theta(2n;p'+q,q'+p)(1)$ is an unitary representation of $MO(p'+q,q'+p)$ (see [20,34]). Let $O(p,q)$ and $O(p',q')$ be embedded diagonally in $O(p'+q,q'+p)$. Let $\pi\in\Pi(MO(p,q))$. Formally define a Hermitian form $(,)$ on $\theta(2n;p'+q,q'+p)(1)\otimes\pi$ by integrating the matrix coefficients of $\theta(2n;p'+q,q'+p)(1)$ against the matrix coefficients of π over $MO(p,q)$ as in (1). Suppose that $(,)$ converges. Define $\mathscr{Q}(p,q;2n;p',q')(\pi)$ to be $\theta(2n;p'+q,q'+p)(1)\otimes\pi$ modulo the radical of $(,)$. $\mathscr{Q}(p,q;2n;p',q')(\pi)$ is thus a representation of $MO(p',q')$.

One must assume that $p'+q'\equiv p+q\pmod 2$. Otherwise, $\theta(2n;p'+q,q'+p)(1)=0$. $\mathscr{Q}$ can be regarded as a more general definition of quantum induction. It is no longer clear that $\mathscr{Q}$ preserves unitarity.

Theorem 7.4.2. *Under the assumptions from Theorem* 7.1.1,

$$\mathscr{Q}(p,q;2n;p',q')(\pi)\cong Q(p,q;2n;p',q')(\pi).$$

Similarly, one can define nonunitary quantum induction $\mathscr{Q}(2n;p,q;2n')(\pi)$.

Definition 7.4.2. Suppose that $p+q \leqslant n+n'+1$. Consider the dual pair $(O(p,q), Sp_{2n+2n'}(\mathbb{R}))$. Then $\theta(p,q;2n+2n')(1)$ is a unitary representation of $MSp_{2n+2n'}(\mathbb{R})$ (see [14,20,24]). Let $\pi \in \Pi(MSp_{2n}(\mathbb{R}))$. Formally, define a Hermitian form $(,)$ on $\theta(p,q;2n+2n')(1) \otimes \pi$ by integrating the matrix coefficients of $\theta(p,q;2n+2n')(1)$ against the matrix coefficients of π as in (1). Suppose that $(,)$ converges. Define $\mathcal{Q}(2n;p,q;2n')(\pi)$ to be $\theta(p,q;2n+2n')(1) \otimes \pi$ modulo the radical of $(,)$. $\mathcal{Q}(2n;p,q;2n')(\pi)$ is a representation of $MSp_{2n}(\mathbb{R})$.

For $p+q$ odd, the MSp in this definition are metaplectic groups. For $p+q$ even, the MSp in this definition split (see Lemma 3.1.1).

Theorem 7.4.3. *Under the assumptions from Theorem* 7.2.1,

$$\mathcal{Q}(2n;p,q;2n')(\pi) \cong Q(2n;p,q;2n')(\pi).$$

There is a good chance that $\mathcal{Q}(*)(\pi)$ will be irreducible.

Quantum induction fits well with the general philosophy of induction. On the one hand, similar to parabolic induced representation $Ind_P^G \tau$ whose vectors are in

$$Hom_P(C_c^\infty(G), \tau),$$

quantum induced $\mathcal{Q}(p,q;2n;p',q')(\pi)$ lies in

$$Hom_{\mathfrak{o}(p,q),O(p)\times O(q)}(\theta(2n;p'+q,q'+p)(1),\pi).$$

On the other hand, $Ind_P^G \tau$ has a nice geometric description. It consists of sections of the vector bundle

$$G \times_P \tau \to G/P.$$

In contrast, quantum induction does not possess this kind of classical interpretation except for some limit case.

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