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ASSOCIATED VARIETIES AND HOWE'S N-SPECTRUM

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Abstract. Let G be a real semisimple group. There are two important invariants associated with the equivalence class of an irreducible unitary representation of G, namely, the associated variety of the annihilator in the universal enveloping algebra and Howe's N-spectrum where N is a nilpotent subgroup of G. The associated variety is defined in a purely algebraic way. The N-spectrum is defined analytically. In this paper, we prove some results about the relation between associated variety and N-associated variety (see Definition 1.2, Theorem. 0.1.) where N is a subgroup of G. We then relate N-associated variety with Howe's N-spectrum when N is Abelian (see Theorem. 0.2). This enables us to compute Howe's rank in terms of the associated variety (see Theorems 0.3, 0.4). The relationship between Howe's rank and the associated variety has been established by Huang-Li about the same time this paper was firstly written, using the result of Matomoto on Whittaker vectors. It can also be derived from works of Przebinda and Daszkiewicz-Kraśkiewicz-Przebinda. Our approach is independent and more self-contained. It does not involve Howe's correspondence in the stable range.

Introduction

0.1. **Associated Variety and $\mathcal{C}$-Associated Variety.** Let $\mathcal{D}$ be a noncommutative associative algebra over $\mathbb{C}$ with an identity. Suppose that $\mathcal{D}$ has a filtration $\{\mathcal{D}_i\}_{i\in\mathbb{Z}}$ such that

$$\mathcal{D}_i.\mathcal{D}_j \subseteq \mathcal{D}_{i+j}, \qquad [\mathcal{D}_i, \mathcal{D}_j] \subseteq \mathcal{D}_{i+j-1} \qquad (i, j \in \mathbb{Z}).$$

Let $gr(\mathcal{D}) = \oplus \mathcal{D}_{i+1}/\mathcal{D}_i$ be the associated graded algebra. Clearly, $gr(\mathcal{D})$ is a commutative algebra. Moreover, $gr(\mathcal{D})$ is a Poisson algebra ([Gabber]). Now suppose that $gr(\mathcal{D})$ is affine ([Ei]). Let $\mathrm{spec}(\mathcal{D})$ be the maximal spectrum of $gr(\mathcal{D})$. Let $\mathcal{J}$ be a left ideal of $\mathcal{D}$. Then $\{\mathcal{D}_j\}$ induces a filtration $\{\mathcal{D}_j \cap \mathcal{J}\}$ for $\mathcal{J}$. $gr(\mathcal{J})$ is an ideal of $gr(\mathcal{D})$. We define the associated variety of $\mathcal{J}$, $\mathcal{V}(\mathcal{J})$ to be the subvariety of maximal ideals of $gr(\mathcal{D})$ containing $gr(\mathcal{J})$.

Let $\mathcal{C}$ be a subalgebra of $\mathcal{D}$ with identity. Again $\mathcal{C}$ has an induced filtration $\{\mathcal{C} \cap \mathcal{D}_j\}$. There is a natural map $j : gr(\mathcal{C}) \to gr(\mathcal{D})$ which induces a map

$$j^* : \mathrm{spec}(\mathcal{D}) \to \mathrm{spec}(\mathcal{C}).$$

The first result we prove in this paper states that

$$j^*(\mathcal{V}(\mathcal{J})) \subseteq \mathcal{V}(\mathcal{J} \cap \mathcal{C}).$$

See Lemma 1.1.

Now let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. Let $\mathfrak{h}$ be a Lie subalgebra. Let $\mathcal{D} = U(\mathfrak{g})$ be the universal enveloping algebra equipped with the natural filtration. Then $gr(U(\mathfrak{g})) = S(\mathfrak{g})$. So $\mathrm{spec}(U(\mathfrak{g})) = \mathfrak{g}_{\mathbb{C}}^*$. Put $\mathcal{C} = U(\mathfrak{h})$. Then $\mathrm{spec}(U(\mathfrak{h})) = \mathfrak{h}_{\mathbb{C}}^*$ and the map j^* is the restriction map from $\mathfrak{g}_{\mathbb{C}}$ to $\mathfrak{h}_{\mathbb{C}}$. Let $\mathcal{J}$ be a left ideal of $U(\mathfrak{g})$. We call $\mathcal{V}(\mathcal{J} \cap \mathcal{C})$ the $\mathcal{C}$-associated variety of $\mathcal{J}$.

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Let M be a $\mathfrak{g}$-module. Let N be a subspace of M. Let $\mathrm{Ann}_{U(\mathfrak{g})}(N)$ be the annihilator of N in $U(\mathfrak{g})$. Then $Ann_{U(\mathfrak{g})}(N)$ is a left ideal of $U(\mathfrak{g})$. By Lemma 1.1, we have

$$j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N))) \subseteq \mathcal{V}(\mathrm{Ann}_{U(\mathfrak{h})}(N)).$$

It is not known if the converse is true. But if $\mathfrak{g}$ is $\mathbb{Z}$-graded, we have the following.

Theorem 0.1. *Let $a \in \mathfrak{g}$ be such that $ad(a)$ is semisimple with real eigenvalues. Let $\mathfrak{h}$ be the highest eigenspace. Let $ad(a)|_{\mathfrak{h}} = \lambda I$ and suppose that $\lambda \geq 0$. Let M be a $\mathfrak{g}$-module. Let N be a subspace that is invariant under the action of a. Then we have*

$$cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N)))) = \mathcal{V}(\mathrm{Ann}_{U(\mathfrak{h})}(N)).$$

See the proof of Theorem 1.1 and Theorem 1.2. A similiar statement holds for $\mathfrak{h}$ the subspace with the lowest weight.

0.2. **Associated Variety and Support: Abelian Case.** Let G be a Lie group with a finite number of components. Let $(\pi, \mathcal{H})$ be a unitary representation of G. All Hilbert spaces in this paper are assumed to be separable. To apply the theory of associated varieties to unitary representations of G, we consider the annihilator. Let $\mathcal{H}^\infty$ be the space of smooth vectors. Clearly $U(\mathfrak{g})$ acts on $\mathcal{H}^\infty$. Define $\mathrm{Ann}_{U(\mathfrak{g})}(\pi)$ to be $\mathrm{Ann}_{U(\mathfrak{g})}(\mathcal{H}^\infty)$. In Theorem 1.3, we prove that $\mathcal{H}^\infty$ can be replaced by any dense subspace of $\mathcal{H}^\infty$. In particular, for G semisimple and K a maximal compact subgroup, a canonical choice is the space of smooth K-finite vectors. In addition, if $(\pi, \mathcal{H})$ is irreducible, then all K-finite vectors are smooth.

Next, let N be a connected Abelian group. The unitary dual of N can be identified with a subset of $i\mathfrak{n}^*$. Here $\mathfrak{n}^*$ is the space of real linear functionals of $\mathfrak{n}$. Let $(\pi, \mathcal{H})$ be a unitary representation of N. Then there is a projection valued measure $d\mu_\pi$ on $\hat{n}$ such that

$$\pi \cong \int_{\hat{N}} d\mu_\pi.$$

Define the support of π to be the complement of the maximal open set U with $\mu_\pi(U) = 0$. Regard $\mathrm{supp}(\pi)$ as a subset of $i\mathfrak{n}^*$. In this paper, we prove

Theorem 0.2. *Let π be a unitary representation of a connected Abelian group N. Then*

$$cl(\mathrm{supp}(\pi)) = \mathcal{V}(\mathrm{Ann}_{U(\mathfrak{n})}(\pi)).$$

Notice that $\mathrm{supp}(\pi) \subset i\mathfrak{n}^*$ and $cl(\mathrm{supp}(\pi))$ is in $\mathfrak{n}^*_{\mathbb{C}}$, the complexification of $\mathfrak{n}$. See Theorem 2.1 for the proof.

Corollary 0.1. *Let $(\pi, \mathcal{H})$ be a unitary representation of a connected Lie group G. Let N be a connected Abelian Lie subgroup of G. Suppose that there is a semisimple element $a \in \mathfrak{g}$ such that $ad(a)$ has only real eigenvalues and $\mathfrak{n}$ is the highest eigenspace of $ad(a)$. Suppose that the eigenvalue for $ad(a)|_{\mathfrak{n}}$ is nonnegative. Then*

$$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{n})}(\pi)) = cl(\mathrm{supp}(\pi|_N)) = cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$$

where $j : \mathfrak{g}^*_{\mathbb{C}} \to \mathfrak{n}^*_{\mathbb{C}}$ is the canonical projection.

0.3. **Unitary Representations, Howe's N-Specturm and Associated Variety.** We shall now apply our results to relate the associated variety to Howe's N-spectrum ([Howe1]). In particular, we can read Howe's rank from the associated variety.

Let G be a connected classical Lie group, and K a maximal compact subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of G, and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ with complex coefficients. Let $(\pi, \mathcal{H})$ be a unitary representation of G. The classical way of studying $(\pi, \mathcal{H})$ is to analyze the associated $(\mathfrak{g}, K)$-module, obtained by taking the smooth K-finite vectors in $\mathcal{H}$. When a $(\mathfrak{g}, K)$-module satisfies a certain compatibility condition and is finitely generated, it will be called a *Harish-Chandra module* ([VO89]). Two irreducible unitary representations are isomorphic if and only if their Harish-Chandra modules are isomorphic as $U(\mathfrak{g})$-modules. In addition, $(\pi, \mathcal{H})$ is irreducible if and only if its Harish-Chandra module is an irreducible $U(\mathfrak{g})$-module. So problems concerning irreducible representations can often be reduced to problems concerning irreducible Harish-Chandra modules. The classification of all the irreducible Harish-Chandra modules of a linear connected semisimple group was carried out by Langlands, Knapp-Zuckerman ([LA], [KZ]). But Langlands' classification did not address the question of unitarity. In [Vogan], Vogan classified the unitary dual of general linear groups, i.e., classical groups of type II. We call the rest of the classical groups, classical groups of type I (see Definition 3.1). The unitary dual $\hat{G}$ for type I classical groups remains very much mysterious.

Let V be the Harish-Chandra module of an irreducible representation $(\pi, \mathcal{H})$. A well-known theorem of Borho-Brylinski-Joseph stated that the associated variety $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(V))$ is the closure of a single coadjoint nilpotent orbit. Thus one may focus on the classification of all the unitarizable Harish-Chandra modules associated with a fixed nilpotent orbit. This problem is quite difficult to solve, but not hopeless. The rich structure of the nilpotent orbits provides a lot of information about the unitary representation. Progress has been made in classifying unitary representations with a fixed associated variety (see for example [HL]).

Let H be a type I subgroup of G ([Di], [Wallach]). From the direct integral theory, the restriction of π to H yields a projection-valued measure $\mu_H(\pi)$ on $\hat{H}$, i.e.,

$$\mathcal{H} = \int_{s \in \hat{H}} \mathcal{H}_s \hat{\otimes} V_s \, d\mu_{\pi|_H}(s) \qquad ((\pi_s, \mathcal{H}_s) \in \hat{H}) \tag{1}$$

where H acts trivially on V_s. $\dim(V_s)$ is often called the multiplicity function of $\pi|_H$. It is defined almost everywhere. R. Howe called the projection-valued measure $\mu_H(\pi)$, and the unitary equivalence class it defines, the H-spectrum of π [Howe1]. When $\hat{H}$ is well-understood, the H-spectrum of π should shed some lights on the structure of the representation $(\pi, \mathcal{H})$. We shall point out that all classical Lie groups and nilpotent Lie groups are Lie groups of Type I. Lie groups of type I is not to be confused with type I classical groups, which refer to classical groups that preserve a nondegenerate sesquilinear form (see Definition 3.1).

In [Howe1], Howe studied the case where $G = Sp_{2n}(\mathbb{R})$ and H is the (Abelian) nilradical N_n of the Siegel parabolic subgroup P_n. In this case, $\hat{N}_n$ can be regarded as the space of real symmetric bilinear forms. In particular, Howe defined the notion of N_n-rank for a unitary representation π, to be the highest rank of the support of $\mu_{N_n}(\pi)$ regarded as symmetric bilinear forms. Later, Howe's ZN_k-rank was extended to all the type I classical groups by J.-S. Li([Li]), to all the type II classical groups by R. Scaramuzzi [Sc], to exceptional groups by H. Salmasian [S]. This approach of studying ZN_k-spectrum has lead to the classification of the "small" unitary representations for type I classical groups [Li].

A natural way to relate $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))$ to Howe's H-spectrum is to relate the H-associated variety, $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{h})}(\pi))$, to the H-spectrum. More precisely, one may study the Lie algebra action of $\mathfrak{h}$ (as skew-adjoint differential operators) in the framework of direct integral theory. In general, this is not an easy task since the direct integral theory is an L^2-theory. Nevertheless, for an Abelian group H, our result is sharp, that is, $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{h})}(\pi))$ is the Zariski closure of the support of the H-spectrum of π.

Let G be a type I classical group . Let P_k be a maximal parabolic subgroup of G, and N_k its nilradical. Let ZN_k be the center of N_k. Since ZN_k is a connected and simply connected Abelian group, $\widehat{ZN_k}$ can be regarded as the purely imaginary dual of $\mathfrak{zn}_k$. Let $j^* : \mathfrak{g}^*_{\mathbb{C}} \rightarrow \mathfrak{zn}_{k\mathbb{C}}^*$ be the canonical projection from the complex dual of $\mathfrak{g}$ to the complex dual of $\mathfrak{zn}_k$. Our results immediately implies the following

Theorem 0.3. *$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{zn}_k)}(\pi))$ is the Zariski closure of $j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi)))$. It is also the Zariski closure of* $\mathrm{supp}(\mu_{ZN_k}(\pi))$.

See Theorem 3.1 and Theorem 2.1.

0.4. **Howe's Rank and Associated Variety.** Finally, We compute Howe's ZN_k-rank for an irreducible unitary representation of a type I classical group, namely, $U(p,q)$, $O_{p,q}$, $O^*(2n)$, $O(n,\mathbb{C})$, $Sp_{2n}(\mathbb{R})$, $Sp(n,\mathbb{C})$ and $Sp(p,q)$ in terms of the associated variety. Since $\mathfrak{g}_{\mathbb{C}}$ can always be represented by a standard matrix Lie algebra, we define the rank of a subset of $\mathfrak{g}_{\mathbb{C}}$ to be the maximal rank of its elements.

Theorem 0.4. *(see also* [HL]*) Let (π, H) be an irreducible unitary representation of a type I classical group G. Then*

1. *for $G = Sp_{2n}(\mathbb{R}), U(p,q)$, ZN_k-rank of (π,H) equals* $\min(k, \mathrm{rank}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$*;*
2. *for $G = O_{p,q}$, ZN_k-rank of (π,H) equals* $\min(k, \mathrm{rank}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$ *if k is even,* $\min(k-1, \mathrm{rank}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$ *if k is odd;*
3. *for $G = O^*(2n), Sp(p,q)$, ZN_k-rank of (π,H) equals* $\min(k, \frac{1}{2}\mathrm{rank}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$*;*
4. *for $G = Sp(n,\mathbb{C})$, ZN_k-rank of (π,H) equals* $\min(k, \frac{1}{2}\mathrm{rank}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$*;*
5. *for $G = O(n,\mathbb{C})$, ZN_k-rank of (π,H) equals* $\min(k, \frac{1}{2}\mathrm{rank}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$ *when k is even, and* $\min(k-1, \frac{1}{2}\mathrm{rank}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$ *when k is odd.*

For (4) and (5), one can substitute $\frac{1}{2}\mathrm{rank}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi)))$ by $\mathrm{rank}(\mathrm{WF}(\pi))$. I should remark that essentially the same statement was proved by Hang and Li when k is the real rank of G ([HL]). This theorem can also be derived from the results of Przebinda and Daszkiewicz-Kraśkiewicz-Przebinda ([Pr], [DKP]). These two approaches involve Howe's correspondence in stable range ([Howe], [Li]). Our approach is independent and more self-contained.

The following is an outline of the paper. In Section 1, we study the associated variety of a left ideal of a special type of filtered noncommutative algebra. We investigate the relationship between the associated variety of M and the H-associated variety of M where M is a $U(\mathfrak{g})$ module. In Section 2, we study the Lie algebra action under the framework of direct integral for Abelian Lie groups. We show that for a unitary representation of a connected Abelian Lie group G, the associated variety of the annihilator is the Zariski closure of the support of its spectral measure. In Section 3, we present the structure theory of parabolic subgroups for a type I classical group. In Section 4, we compute the ZN_k-rank using associated varieties.

After I finished this work, it was pointed out by Vogan that there should be a real version of Theorem 0.1, namely, there must be a strong connection between the wave front set of π and the wave front set of π restricted to certain subgroups. Let $\mathrm{WF}(\pi)$ be the wave front set of a representation π of a Lie group G in the sense of Howe [Howe2]. From [Howe2] proposition 2.1, it is easy to see that

$$\mathrm{WF}(\pi|_{ZN_k}) = \mathrm{supp}(\pi|_{ZN_k})$$

since $\mathrm{supp}(\pi|_{ZN_k})$ is conic. On the other hand, it is well-known that the associated variety is the Zariski closure of the wave front set, i.e.,

$$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi)) = cl(\mathrm{WF}(\pi))$$

From what we have proved in this paper, we have

$$cl(\mathrm{supp}(\pi|_{ZN_k})) = cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi)))).$$

Therefore

$$cl(\mathrm{WF}(\pi|_{ZN_k})) = cl(j^*(cl(\mathrm{WF}(\pi))))$$

At this moment it is not clear how to relate $\mathrm{WF}(\pi|_{ZN_k})$ to $\mathrm{WF}(\pi)$. Nevertheless, we make the following conjecture.

Conjecture: Let G be a connected classical group of type I. Let π be an irreducible unitary representation of G. Let $j^* : \mathfrak{g}^* \to \mathfrak{z}n_k^*$ be the canonical projection. Then

$$\mathrm{WF}(\pi|_{ZN_k}) = j^*(\mathrm{WF}(\pi)).$$

This paper is essentially the first part of my Ph.D thesis. I wish to thank my advisor David Vogan for guidance.

1. Associated Variety under Restriction

A filtered (noncommutative) algebra $\mathcal{D}$ over $\mathbb{C}$ is an algebra endowed with a filtration $\{\mathcal{D}_i\}_{i\in\mathbb{Z}}$ such that

$$\mathcal{D}_i.\mathcal{D}_j \subseteq \mathcal{D}_{i+j} \qquad (i, j \in \mathbb{Z}).$$

Let $gr(\mathcal{D}) = \oplus \mathcal{D}_{i+1}/\mathcal{D}_i$ be the associated graded algebra. An element $x \in gr(\mathcal{D})$ is said to be *homogeneous* of degree i if there exists an $i \in \mathbb{Z}$ such that $x \in \mathcal{D}_i/\mathcal{D}_{i-1}$. Let $\sigma_i : \mathcal{D}_i \to \mathcal{D}_i/\mathcal{D}_{i-1}$ be the natural projection. We call it the *symbol map*. Then $gr(\mathcal{D}) = \oplus_i \sigma_i(\mathcal{D}_i)$.

Throughout this paper, our filtered algebra will be *assumed* to have the following property:

(1) $\mathcal{D}_0 = \mathbb{C}1$, where 1 is the identity element;
(2) $\mathcal{D}_n = \{0\}$ for every $n < 0$;
(3) $gr(\mathcal{D})$ is a commutative affine algebra ([Ei]).

Notice that $gr(\mathcal{D})$ being commutative is equivalent to

$$[\mathcal{D}_i, \mathcal{D}_j] \subseteq \mathcal{D}_{i+j-1}.$$

Definition 1.1. *Let* $\mathrm{spec}(\mathcal{D})$ *be the* **maximal spectrum** *of* $gr(\mathcal{D})$. *Suppose that* $\mathcal{I}$ *is a (left) ideal of* $\mathcal{D}$. *Then* $\mathcal{I}$ *inherits a filtration from* $\mathcal{D}$, *i.e.,*

$$\mathcal{I}_i = \mathcal{D}_i \cap \mathcal{I} \qquad (i \in \mathbb{N}).$$

Let $gr(\mathcal{I}) = \oplus \sigma_i(\mathcal{I}_i)$ be the graded algebra of $\mathcal{I}$. Then $gr(\mathcal{I})$ is an ideal of $gr(\mathcal{D})$. Let $\mathcal{V}(gr(\mathcal{I}))$ be the set of maximal ideals in $gr(\mathcal{D})$ containing $gr(\mathcal{I})$. Define $\mathcal{V}(\mathcal{I}) = \mathcal{V}(gr(\mathcal{I}))$. $\mathcal{V}(\mathcal{I})$ is called the associated variety of $\mathcal{I}$.

Now suppose that $\mathcal{C}$ is a subalgebra of $\mathcal{D}$ with identity. $\mathcal{C}$ inherits a filtration from $\mathcal{D}$. Thus we have an injection:

$$j : gr(\mathcal{C}) \to gr(\mathcal{D})$$

Automatically, $gr(\mathcal{C})$ becomes an affine, commutative algebra. The associated map on the spaces of spectrum is

$$j^* : \mathrm{spec}(\mathcal{D}) \to \mathrm{spec}(\mathcal{C}).$$

If $\mathcal{M} \in \mathrm{spec}(\mathcal{D})$, then $j^*(\mathcal{M}) = \mathcal{M} \cap gr(\mathcal{C})$ which is again a maximal ideal in $gr(\mathcal{C})$. Let $\mathcal{J}$ be a left ideal of $\mathcal{D}$. Let $\mathcal{I} = \mathcal{J} \cap \mathcal{C}$. We would like to study the relationship between $\mathcal{V}(\mathcal{J})$ and $\mathcal{V}(\mathcal{I})$. Strictly speaking, we should have written $\mathcal{V}_{\mathcal{D}}(\mathcal{J})$ and $\mathcal{V}_{\mathcal{C}}(\mathcal{I})$ instead of $\mathcal{V}(\mathcal{J})$ and $\mathcal{V}(\mathcal{I})$ to indicate the difference of the ambient space. However, within the context, it is clear that $\mathcal{I}$ is an ideal of C and $\mathcal{J}$ is an ideal of $\mathcal{D}$. And we will only be discussing the associated variety of an ideal. So it is clear that $\mathcal{V}(\mathcal{J})$ is a subvariety of $\mathrm{spec}(\mathcal{D})$ and $\mathcal{V}(\mathcal{I}))$ is a subvariety of $\mathrm{spec}(\mathcal{C})$.

Lemma 1.1. *Let $\mathcal{D}$ be a filtered algebra with the properties specified at the beginning of this section. Let $\mathcal{C}$ be a subalgebra of $\mathcal{D}$. Let $\mathcal{J}$ be an left ideal in $\mathcal{D}$ and $\mathcal{I} = \mathcal{C} \cap \mathcal{J}$. Then $\mathcal{I}$ is a left ideal of $\mathcal{C}$. In addition,*

$$j^*(\mathcal{V}(\mathcal{J})) \subseteq \mathcal{V}(\mathcal{I}).$$

Proof: Obviously, $\mathcal{I}$ is a left ideal of $\mathcal{C}$. By definition, $gr(\mathcal{I})$ is a direct sum of homogeneous elements. Suppose $f \in gr(\mathcal{I})$ is homogeneous of degree k. Then there exists $U \in \mathcal{I} \subseteq \mathcal{J}$, such that $\sigma_k(U) = f$. This implies that $j(f) \in gr(\mathcal{J})$. Therefore $j(gr(\mathcal{I})) \subseteq gr(\mathcal{J})$. So $j^*(\mathcal{V}(\mathcal{J})) \subseteq \mathcal{V}(\mathcal{I})$. Q.E.D.

Corollary 1.1. *Let $\mathcal{D}$ be a filtered algebra with the properties specified at the beginning of this section. Let M be a $\mathcal{D}$-module, N a linear subspace of M. Let $\mathcal{C}$ be a subalgebra of $\mathcal{D}$. Let $\mathrm{Ann}_{\mathcal{D}}(N)$ and $\mathrm{Ann}_{\mathcal{C}}(N)$ be the annihilators of N in $\mathcal{D}$ and $\mathcal{C}$ respectively. Then $\mathrm{Ann}_{\mathcal{D}}(N)$ and $\mathrm{Ann}_{\mathcal{C}}(N)$ are left ideals of $\mathcal{D}$ and of $\mathcal{C}$ respectively. In addition,*

$$j^*\mathcal{V}(\mathrm{Ann}_{\mathcal{D}}(N)) \subseteq \mathcal{V}(\mathrm{Ann}_{\mathcal{C}}(N)).$$

Now let $\mathcal{D} = U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ with complex coefficients. Since $U(\mathfrak{g})$ has a natural filtration

$$\mathbb{C}.1 \subseteq U_1(\mathfrak{g}) \subseteq U_2(\mathfrak{g}) \subseteq \ldots \subseteq U_i(\mathfrak{g}) \subseteq \ldots,$$

the associated graded algebra $gr(U(\mathfrak{g}))$ can be identified with the symmetric algebra $S(\mathfrak{g})$. Thus

$$\mathrm{spec}(U(\mathfrak{g})) = \mathfrak{g}^*_{\mathbb{C}}.$$

Here $\mathfrak{g}^*_{\mathbb{C}}$ is the complex dual of $\mathfrak{g}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then j^* is simply the projection of $\mathfrak{g}^*_{\mathbb{C}}$ onto $\mathfrak{h}^*_{\mathbb{C}}$ (through restriction). In this setting, we have

Corollary 1.2. *Let $\mathfrak{h}$ be a Lie subalgebra of a Lie algebra $\mathfrak{g}$. Let M be a $\mathfrak{g}$-module. Let N be a linear subspace of M. Then*

$$j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N))) \subseteq \mathcal{V}(\mathrm{Ann}_{U(\mathfrak{h})}(N)).$$

We are interested in the equalities of the following type:

$$cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N)))) = \mathcal{V}(\mathrm{Ann}_{U(\mathfrak{h})}(N)).$$

At this stage, we only have a very limited understanding about the behavior of j^* for associated varieties. Nevertheless, we have the following theorem.

Theorem 1.1. *Suppose a is a semisimple element in an arbitrary Lie algebra $\mathfrak{g}$ such that $ad(a)$ has only real eigenvalues. Let r be the maximal eigenvalue. Suppose $r > 0$. Let $\mathfrak{h} = \mathfrak{g}_r$. Then $\mathfrak{h}$ is Abelian. Let M be a $\mathfrak{g}$-module, and N a subspace of M such that $a.N \subseteq N$. Then*

$$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{h})}(N)) = cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N))))$$

where $cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N))))$ is the Zariski closure of $j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N)))$.

Proof: First of all, under the eigendecomposition with respect to $ad(a)$, we have

$$[\mathfrak{g}_r, \mathfrak{g}_r] = \mathfrak{g}_{2r} = \{0\}.$$

Therefore $\mathfrak{h} = \mathfrak{g}_r$ is Abelian. Now it suffices to show that

$$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{h})}(N)) \subseteq cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N)))).$$

Suppose that $f \in S^i(\mathfrak{h})$ vanishes on $cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N))))$. In other words, $j(f) = f$ vanishes on $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N))$. Here f is regarded as a linear function on $\mathfrak{g}_{\mathbb{C}}^*$. Thus by Hilbert's Nulstellensatz, there exists $n \in \mathbb{N}$, such that $f^n \in gr(\mathrm{Ann}_{U(\mathfrak{g})}(N))$. Therefore,

$$\exists\ P \in U_{ni}(\mathfrak{g}) \cap \mathrm{Ann}_{U(\mathfrak{g})}(N), \sigma_{ni}(P) = f^n.$$

Since $ad(a)$ is semisimple, $U(\mathfrak{g})$ is completely reducible as an $ad(a)-$module. Also notice that N is an a-module. Thus $\mathrm{Ann}_{U(\mathfrak{g})}(N)$ is also an $ad(a)-$module. Now $U_{ni}(\mathfrak{g}) \cap \mathrm{Ann}_{U(\mathfrak{g})}(N)$ possesses an eigen (weight) decomposition with respect to $ad(a)$:

$$U_{ni}(\mathfrak{g}) \cap \mathrm{Ann}_{U(\mathfrak{g})}(N) = \oplus_{k \in \mathbb{R}} (U_{ni}(\mathfrak{g}) \cap \mathrm{Ann}_{U(\mathfrak{g})}(N))_k.$$

This implies that every eigencomponent of P with respect to $ad(a)$ is again in $\mathrm{Ann}_{U(\mathfrak{g})}(N)$.

Since $\mathfrak{h}$ is Abelian, $S^{ni}(\mathfrak{h})$ can be regarded as a subspace of $U_{ni}(\mathfrak{h})$, which in turn is a subspace of $U_{ni}(\mathfrak{g})$. In addition, $S^{ni}(\mathfrak{h})$ is the highest eigenspace of $ad(a)|_{U_{ni}(\mathfrak{g})}$. Let P_0 be the eigenprojection of $P \in U_{ni}(\mathfrak{g})$ onto $S^{ni}(\mathfrak{h})$. Clearly, $P_0 \in \mathrm{Ann}_{U(\mathfrak{g})}(N)$. Since the action of $ad(a)$ intertwins the symbol map

$$\sigma_{ni} : U_{ni}(\mathfrak{g}) \to S^{ni}(\mathfrak{g}),$$

by comparing the eigendecompositions for P and $\sigma_{ni}(P) = f^n$, we obtain $\sigma_{ni}(P_0) = f^n$. Now $P_0 \in \mathrm{Ann}_{U(\mathfrak{h})}(N)$, and

$$\sigma_{ni}(P_0) = f^n \in gr(\mathrm{Ann}_{U(\mathfrak{h})}(N)).$$

This implies that f vanishes at $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{h})}(N))$. So

$$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{h})}(N)) \subseteq cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N)))).$$

By Cor. 1.2,

$$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{h})}(N)) = cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N)))).$$

Q.E.D.

When $r = 0$, $\mathfrak{h}$ will no longer be Abelian. We can define P_0 to be the eigenprojection of $U_{ni}(\mathfrak{g})$ onto the highest eigenspace $U_{ni}(\mathfrak{h})$ with respect to $ad(a)$. It is still true that $\sigma_{ni}(P_0) = f^n$ and $P_0 \in \mathrm{Ann}_{U(\mathfrak{h})}(N)$. We obtain the following.

Theorem 1.2. *Suppose a is a semisimple element in an arbitrary Lie algebra $\mathfrak{g}$ such that $ad(a)$ has only real eigenvalues. Suppose that 0 is the highest eigenvalue of $ad(a)$. Let $\mathfrak{h}$ be the 0-eigenspace of $ad(a)$. Let M be a $\mathfrak{g}$-module, and N a subspace of M such that $a.N \subseteq N$. Then*

$$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{h})}(N)) = cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N))))$$

where $cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N))))$ is the Zariski closure of $j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(N)))$.

Before we continue on, we want to examine the definition of the annihilator of a unitary representation for an arbitrary Lie group G.

Theorem 1.3. *Let (π, H) be a unitary representation of a Lie group G. Let M be any dense subset of the space of smooth vectors H^∞. Then*

$$\mathrm{Ann}_{U(\mathfrak{g})}(H^\infty) = \mathrm{Ann}_{U(\mathfrak{g})}(M).$$

Proof: If $D \in U(\mathfrak{g})$ and $\pi(D)H^\infty = 0$, then $\pi(D)M = 0$. Thus

$$\mathrm{Ann}_{U(\mathfrak{g})}(M) \supseteq \mathrm{Ann}_{U(\mathfrak{g})}(H^\infty).$$

If $D \in \mathrm{Ann}_{U(\mathfrak{g})}(M)$, then

$$\forall\, u \in M, v \in H^\infty, \qquad (\pi(D)u, v) = 0.$$

Since $\mathfrak{g}$ act as skew-adjoint operators, i.e. ,

$$\forall\, X \in \mathfrak{g}, \pi(X)^* = \pi(-X),$$

we have

$$(\pi(D)u, v) = (u, \pi(D^*)v) = 0 \qquad (u \in M, v \in H^\infty).$$

Here $D \to D^*$ is the natural real involution defined by

$$\alpha X_1 X_2 \dots X_n \to (-1)^n \overline{\alpha} X_n X_{n-1} \dots X_2 X_1 \qquad (X_i \in \mathfrak{g}).$$

Since M is dense in H^∞, M is dense in H. Hence $\pi(D^*)v = 0$ for every $v \in H^\infty$. We have

$$(\pi(D)u, v) = (u, \pi(D^*)v) = 0 \qquad (u \in H^\infty, v \in H^\infty).$$

Thus for every $u \in H^\infty$, $\pi(D)u = 0$. Therefore

$$D \in \mathrm{Ann}_{U(\mathfrak{g})}(H^\infty).$$

This implies that

$$\mathrm{Ann}_{U(\mathfrak{g})}(M) \subseteq \mathrm{Ann}_{U(\mathfrak{g})}(H^\infty).$$

Q.E.D.

Definition 1.2. *Let $(\pi, \mathcal{H})$ be a unitary representation of G. Let M be any dense subset of $\mathcal{H}^\infty$. Define $\mathrm{Ann}_{U(\mathfrak{g})}(\pi) = \mathrm{Ann}_{U(\mathfrak{g})}(M)$. Let N be a connected closed subgroup of G. Define $\mathrm{Ann}_{U(\mathfrak{n})}(\pi) = \mathrm{Ann}_{U(\mathfrak{n})}(M)$. We call $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{n})}(\pi)) \subseteq \mathfrak{n}_{\mathbb{C}}^*$ the N-associated variety of π.*

Let $N_G(N)$ be the normalizer of N in G. One can easily see that the N-associated variety is $N_G(N)$-stable.

2. Associated Variety and Support of a Unitary Representation: Abelian Case

In this section, we review the basic theory of unitary representations of Abelian groups and Abelian Lie groups. When G is an Abelian Lie group, the Lie algebra $\mathfrak{g}$ acts as mutually commuting (unbounded) skew-self adjoint operators. Both the Lie group action and Lie algebra action can be represented by spectral integrals. This allows us to relate the associated variety and the support of a unitary representation π of G.

Theorem 2.1. *Suppose that (π, H) is a unitary representation of a connected Abelian Lie group G. If we identify $\hat{G}$ with a subset of $i\mathfrak{g}^*$, then*

$$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi)) = cl(\mathrm{supp}_G(\pi)).$$

Let G be a locally compact Abelian group. Let $\hat{G}$ be the set of unitary characters of G endowed with the Pontryagin topology. Then $\hat{G}$ is a locally compact Abelian group under pointwise multiplication.

Theorem 2.2 (Stone)**.** *If H is a Hilbert space and μ a regular projection-valued Borel measure on $\hat{G}$, then the equation*

$$T_g = \int_{\hat{G}} \xi(g) d\mu(\xi) \qquad (g \in G) \tag{2}$$

defines a unitary representation T of G on H. Conversely, every unitary representation of G determines a unique regular projection-valued Borel measure μ on H such that Equation 2 holds.

We define the support of a unitary representation H of G to be the (closed) support of the projection-valued measure μ. In other words, $\mathrm{supp}_G(\pi)$ is the complement of the biggest open subset U of $\hat{G}$ such that $\mu(U) = 0$. Equivalently, $\mathrm{supp}_G(\pi)$ is the smallest closed subset K of $\hat{G}$ such that $\mu(K) = id$. Of course if we remove the closedness of $\mathrm{supp}_G(\pi)$, $\mathrm{supp}_G(\pi)$ is only unique up to a set of measure zero.

For arbitrary Borel measurable set $K \subseteq \hat{G}$, let

$$\mu_v(K) = \mu(K).v$$

$$\mu_{u,v}(K) = (\mu(K)u, v)$$

Then μ_v defines a vector-valued regular Borel measure and $\mu_{u,v}$ defines a complex regular Borel measure.

Suppose G is a connected Abelian Lie group and $\mathfrak{g}$ is the (real) Lie algebra of G. Let $\mathfrak{g}^*$ be the real dual of $\mathfrak{g}$. Each $\xi \in \hat{G}$ corresponds to a smooth function $\xi(g)$ on G. We can define

$$\xi(x) = \frac{d}{dt}|_{t=0} \xi(\exp(tx)) \qquad (x \in \mathfrak{g}).$$

This defines a map from $\hat{G}$ to $\mathfrak{g}^*_{\mathbb{C}}$. Since $\xi(\exp(tx))\overline{\xi(\exp(tx))} = 1$, we have $\xi(x) + \overline{\xi(x)} = 0$. So $\xi(x) \in i\mathbb{R}$. We denote the pure imaginary dual by $i\mathfrak{g}^*$. Then we have defined a map from $\hat{G}$ to $i\mathfrak{g}^*$. Now, we want to study the Lie algebra action π of $\mathfrak{g}$. This involves integral of unbounded functions. We recall the following definition of spectral integral.

Definition 2.1. *Let (μ, X) be a projection-valued spectral measure on a Hilbert space. Let $f : X \to \mathbb{C}$ be a μ-measurable function. Then we may find a sequence $\{A_n\}$ of pairwise disjoint measurable sets such that*

- *$\cup_1^\infty A_n = X$;*
- *f is $\mu-$essentially bounded on each A_n*

Let $P_n = \mu(A_n)$, $H_n = range(P_n)$, $T_n = \int_{A_n} f d\mu$. Then there exists a unique normal operator $T = \Sigma T_n$ on $\hat{\oplus} H_n$. T is often written as $\int f d\mu$, called the spectral integral of f.

In the framework of spectral integral, the action of the Abelian Lie group G is presented in Stone's theorem as integral of bounded functions. We will first find a presentation of the Lie algebra action in terms of the spectral integral. Let us recall the following theorems in [Fell-Doran].

Theorem 2.3. *If $f : \hat{G} \to \mathbb{C}$ is a $\mu-$measurable function. Let*

$$T_f = \int_{\hat{G}} f d\mu$$

Then $v \in Dom(T)$ if and only if

$$\int |f(\xi)|^2 d\mu_{v,v}(\xi) < \infty.$$

In this case,

$$\|T_f v\|^2 = \int |f(\xi)|^2 d\mu_{v,v}(\xi),$$

$$(T_f v, u) = \int f(\xi) d\mu_{v,u}(\xi) \qquad (u, v \in H).$$

Theorem 2.4. *Let f_1, f_2 be μ-measurable functions on $\hat{G}$. Then in terms of the graphs of linear operators,*

$$(\int f_1 d\mu)(\int f_2 d\mu) \subset \int f_1 f_2 d\mu,$$

$$(\int f_1 d\mu)^* = \int \overline{f_1} d\mu.$$

We can derive the following

Proposition 2.1. *Let (π, H) be a unitary representation of a connected Abelian Lie group G. Let μ be the projection-valued regular Borel measure from Stone's theorem. We denote the Lie algebra $\mathfrak{g}$ actions by π. Then*

$$\int_{\hat{G}} \xi(X) d\mu(\xi) \subset \pi(X) \qquad (X \in \mathfrak{g}).$$

Here $\xi \in \hat{G} \cong i\mathfrak{g}^*$.

Proof: Let $T_X = \int_{\hat{G}} \xi(X) d\mu(\xi)$. Suppose $u \in Dom(T_X)$. It suffices to show that $\forall\ v \in Dom(\pi(X))$,

$$(T_X u, v) = -(u, \pi(X)v).$$

In other words,

$$-(u, \pi(X)v) = \int_{\hat{G}} \xi(X) d\mu_{u,v}(\xi).$$

Notice that

$$\begin{aligned} -(u, \pi(X)v) &= -\,(u, \frac{d}{dt}|_{t=0} \pi(\exp tX) v) \\ &= \frac{d}{dt}|_{t=0} (\pi(\exp(tX)u, v)) \\ &= \frac{d}{dt}|_{t=0} \int_{\hat{G}} \xi(\exp(tX)) d\mu_{u,v}(\xi) \end{aligned} \tag{3}$$

We would like to interchange the integration and differentiation, obtaining

$$\begin{aligned} -(u, \pi(X)v) &= \int \frac{d}{dt}|_{t=0}\xi(\exp(tX))d\mu_{u,v}(\xi) \\ &= \int \xi(X)d\mu_{u,v}(\xi). \end{aligned} \tag{4}$$

To show that the integration is interchangeable with the differentiation, first we observe that

$$|\frac{d}{dt}\xi(\exp(tX))| = |\frac{d}{dt}\exp(t\xi(X))| \leq |\xi(X)| \qquad (\xi \in \hat{G}).$$

For a complex measure λ on $\hat{G}$, we define $|\lambda|(U)$ to be the supremum of $\{\Sigma_{j=1}^{m}|\lambda(E_j)|\}$, where $\{E_j\}_1^m$ is any measurable partition of U. Since

$$|(\mu(U)u, v)|^2 = |(\mu(U)u, \mu(U)v)|^2 \leq \|\mu(U)u\|^2\|\mu(U)v\|^2,$$

we have

$$|\mu_{u,v}|(U)^2 \leq |\mu_{u,u}|(U)|\mu_{v,v}|(U) = \mu_{u,u}(U)\mu_{v,v}(U).$$

Therefore

$$\begin{aligned} (\int |\xi(X)|d|\mu_{u,v}|(\xi))^2 \leq &(\int |\xi(X)|^2 d\mu_{u,u}(\xi))(\int d\mu_{v,v}(\xi)) \\ =&(\int |\xi(X)|^2 d\mu_{u,u}(\xi))\|v\|^2. \end{aligned} \tag{5}$$

From Theorem 2.3, $u \in Dom(T_X)$ implies that

$$\int |\xi(X)|^2 d\mu_{u,u}(\xi) < \infty$$

Hence $\xi(X)$ as a function on $\hat{G}$ is absolutely integrable with respect to $\mu_{u,v}$. But $\frac{d}{dt}\xi(\exp(tX))$ is dominated by $|\xi(X)|$. Thus integration and differentiation in Equation 4 are interchangeable. We obtain

$$(T_X u, v) = -(u, \pi(X)v) \qquad (\forall\, v \in Dom(\pi(X)).$$

So $T_X u$ is a bounded linear functional on $Dom(\pi(X))$ and $T_X u = -\pi(X^*)u$. Since X is skew self-adjoint, $T_X u = \pi(X)u$ and $u \in Dom(\pi(X))$. $\square$

Now for $X_1, X_2, \ldots, X_n \in \mathfrak{g}$, we define

$$T_{X_1X_2\ldots X_n} = \int_{\hat{G}} \xi(X_1)\xi(X_2)\ldots\xi(X_n)d\mu(\xi).$$

We can extend this definition by linearality to all $D \in U(\mathfrak{g})$. One can easily obtain the following corollary concerning the universal enveloping algebra $U(\mathfrak{g})$.

Corollary 2.1. *Let (π, H) be a unitary representation of a connected Abelian Lie group G, and μ its projection-valued regular Borel measure. Suppose $X_1, X_2, \ldots, X_n \in \mathfrak{g}$. Then*

$$T_{X_1}T_{X_2}\ldots T_{X_n} \subset \pi(X_1X_2\ldots X_n),$$

$$T_{X_1X_2\ldots X_n} \supset T_{X_1}T_{X_2}\ldots T_{X_n}.$$

Since $U(\mathfrak{g})$ is commutative, we may identify it with $S(\mathfrak{g})$. Thus for every $\xi \in \mathfrak{g}^*$, $D \in U(\mathfrak{g})$, $\xi(D)$ is well-defined. We will also denote $\xi(D)$ by $D(\xi)$, just to indicate the fact that D can be regarded as a function on $\mathfrak{g}^*$.

Corollary 2.2. *If $u \in Dom(T_D)$ for every $D \in U(\mathfrak{g})$, then u is smooth. Furthermore,*

$$\pi(D)u = T_D u.$$

Proof: Suppose $u \in Dom(T_D)$ for every $D \in U(\mathfrak{g})$. By Cor. 2.1, $u \in Dom(\pi(D))$. So u is smooth and $\pi(D)u = T_D u$. Q.E.D.

By Theorem 1.3, we may define $\mathrm{Ann}_{U(\mathfrak{g})}(\pi)$ to be the annihilator of any smooth dense subset M of H. In particular, in our context, for G an Abelian Lie group, we choose

$$M = \{\int_{\hat{G}} f(\xi) d\mu_u(\xi) \mid f \in B_c(\hat{G}), u \in H\}.$$

where $B_c(\hat{G})$ is the space of bounded measurable functions with compact support. M here has some property similar to the Gårding space.

Theorem 2.5. *Let (π, H) be a unitary representation of a connected Abelian Lie group G, μ the projection-valued regular Borel measure on $\hat{G}$. Then M is dense in H, and $M \subseteq H^\infty$. Suppose $D \in U(\mathfrak{g}) = S(\mathfrak{g})$ such that*

$$D(\xi) = 0 \qquad (\forall\, \xi \in \mathrm{supp}_G(\pi)).$$

Then $D \in \mathrm{Ann}_{U(\mathfrak{g})}(\pi)$.

Proof: We will show that $M \subseteq Dom(T_D)$ for every $D \in U(\mathfrak{g})$. $\forall\ f \in B_c(\hat{G}), u \in H, D \in S(\mathfrak{g})$, let $v = (\int f(\xi) d\mu(\xi))u$. Then for every $U \subset \hat{G}$ measurable, we have

$$\mu_{v,v}(U) = (\int_U d\mu(\xi) v, v) = \int_U |f(\xi)|^2 d\mu_{u,u}(\xi)$$

This implies that

$$d\mu_{v,v}(\xi) = |f(\xi)|^2 d\mu_{u,u}(\xi).$$

Notice that

$$\int |D(\xi)|^2 d\mu_{v,v}(\xi) = \int |D(\xi) f(\xi)|^2 d\mu_{u,u}(\xi) \tag{6}$$

converges since f is compactly supported. Thus by Theorem 2.3,

$$(\int f(\xi) d\mu(\xi))u \in Dom(T_D) \qquad (\forall\ D \in U(\mathfrak{g})).$$

Therefore $\int f(\xi) d\mu_u(\xi) \in H^\infty$. We have $M \subseteq H^\infty$. Approximate the constant function $1_{\hat{G}}$ by bounded functions $\{f_i\}_1^\infty$ with compact support. Since the measure μ is regular, $u \in H$ can be approximated by $\int f_i(\xi) d\mu_u(\xi)$. Therefore M is dense in H. Now suppose

$$D(\xi) = 0 \qquad (\forall\ \xi \in \mathrm{supp}_G(\pi)).$$

Then we have $\forall\, f \in B_c(\hat{G})$,

$$\pi(D)(\int f(\xi) d\mu(\xi))u = T_D(\int f(\xi) d\mu(\xi))u = (\int D(\xi) f(\xi) d\mu(\xi))u.$$

Notice that the integral above is over $\mathrm{supp}_G(\pi)$. It must vanish. Hence

$$D \in \mathrm{Ann}_{U(\mathfrak{g})}(M) = \mathrm{Ann}_{U(\mathfrak{g})}(\pi).$$

Q.E.D.

Theorem 2.6. *Under the same assumptions as in Theorem 2.5, if $D \in \mathrm{Ann}_{U(\mathfrak{g})}(\pi)$, then*

$$D(\mathrm{supp}_G(\pi)) = 0.$$

Proof: Let $D \in \mathrm{Ann}_{U(\mathfrak{g})}(\pi)$.

(1) First, we want to show that

$$\mu(zero(D) \cap \mathrm{supp}_G(\pi)) = id.$$

Suppose not. Then there exist a complex number $a \neq 0$, a compact $K \subset \mathrm{supp}_G(\pi)$, $\mu(K) \neq 0$, such that

$$|D(\xi) - a| < \frac{1}{2}|a| \qquad (\forall\, \xi \in K).$$

It follows that

$$\begin{aligned} \| \int_K D(\xi) d\mu(\xi) - a\mu(K) \| =& \| \int_K (D(\xi) - a) d\mu(\xi) \| \\ \leq& \| \int_K |D(\xi) - a| d\mu(\xi) \| \\ \leq& \| \int_K \frac{1}{2}|a| d\mu(\xi) \| \\ \leq& \frac{1}{2}|a| \|\mu(K)\| \end{aligned} \tag{7}$$

Thus $\int_K D(\xi)d\mu(\xi) \neq 0$. On the other hand, for every $v \in H$, by Theorem 2.5

$$(\int_K d\mu(\xi))v \in M \subseteq \cap_{D \in U(\mathfrak{g})} Dom(T_D).$$

We have

$$0 = \pi(D)(\int_K d\mu(\xi))v = T_D(\int_K d\mu(\xi))v = (\int_K D(\xi) d\mu(\xi))v.$$

This is a contradiction.

(2) Therefore, we have $\mu(zero(D) \cap \mathrm{supp}_G(\pi)) = id$. Notice that for a connected Abelian Lie group G, the Gelfand topology is just the induced Euclidean topology. Thus $zero(D) = \{\xi \in \hat{G} \mid D(\xi) = 0\}$ is closed in the Eulidean topology (not necessarily in the Zariski topology). Therefore $zero(D) \cap \mathrm{supp}_G(\pi)$ is closed. According to the minimality of $\mathrm{supp}_G(\pi)$, we have

$$zero(D) \cap \mathrm{supp}_G(\pi) = \mathrm{supp}_G(\pi).$$

Thus $zero(D) \supseteq \mathrm{supp}_G(\pi)$. Hence

$$D(\mathrm{supp}_G(\pi)) = 0$$

Q.E.D.

What we have shown is that for $D \in U(\mathfrak{g})$,

$$D(\mathrm{supp}_G(\pi)) = 0 \Longleftrightarrow D \in \mathrm{Ann}_{U(\mathfrak{g})}(\pi).$$

But

$$D \in \mathrm{Ann}_{U(\mathfrak{g})}(\pi) \Longleftrightarrow D(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))) = 0$$

Thus we have Theorem 2.1.

3. Structure Theory of the Parabolic subgroups of Classical Groups of Type I

In this section, we summerize some *known* results about the structure of parabolic subgroups of a classical group of type I. We also sketch some proofs when they are needed. Notations are mainly adopted from [Li].

Definition 3.1. *A type I classical group $G(V)$ consists of the following data.*

- *A division algebra D of a field $\mathbb{F}$ with involution $\sharp$, and $a^\sharp b^\sharp = (ba)^\sharp$;*
- *A (right) vector space V over D, with a nondegenerate (D-valued) sesquilinear form $(,)_\epsilon$, $\epsilon = \pm 1$, i.e.,*
$$(u, v) = \epsilon(v, u)^\sharp \qquad (u, v \in V)$$
$$(u\lambda, v) = (u, v)\lambda \qquad (u, v \in V, \lambda \in D);$$
- *G is the isometry group of $(,)$, i.e.,*
$$g.(u\lambda) = (g.u)\lambda \qquad (\lambda \in D, u \in V, g \in G)$$
$$(gu, gv) = (u, v) \qquad (u, v \in V).$$

Here we allow $\sharp$ to be trivial. We call the identity component of G connected classical group of type I. For $\mathbb{F} = \mathbb{C}$, $\sharp$ trivial, we obtain all the complex simple groups of type I, namely, $Sp_{2n}(\mathbb{C})$, and $O(n, \mathbb{C})$. If $D = \mathbb{H}$, $\mathbb{F} = \mathbb{R}$, $\sharp$ the usual involution, we obtain $Sp(p, q)$ and $O^*(2n)$ depending on the sesquilinear form. For $\mathbb{F} = \mathbb{R}$, $D = \mathbb{C}$ and $\sharp$ the usual conjugation, we obtain $U(p, q)$ depending on the signature of the Hermitian form. For $\mathbb{F} = \mathbb{R}$, $D = \mathbb{R}$ with trivial involution, we obtain $Sp_{2n}(\mathbb{R})$ and $O_{p,q}(\mathbb{R})$. If $(V, (\, , \,))$ is implicitly understood, we write G or $G(n)$ if $V \cong D^n$. Let V_0 be a linear subspace of V, we write $V_0^\perp$ for the subspace of V that is orthogonal to V_0 under $(\, , \,)$. If $(\, , \,)$ is nondegenerate on V_0, we let $G(V_0)$ denote the subgroup of G consisting of elements which act by identity on $V_0^\perp$.

Definition 3.2. *A flag $\mathcal{F}$ of $V = D^n$ is a sequence of strictly increasing (D-)linear subspaces of V*
$$0 = V_0 \subsetneqq V_1 \subsetneqq V_2 \subsetneqq \ldots \subsetneqq V_k \subsetneqq V$$
such that
$$V_i^\perp = V_{k+1-i}.$$
Suppose $\dim(V_i) = d_i$. *$\mathcal{F}$ is said to be a flag of type*
$$\mathcal{I} = (0 < d_1 < d_2 < \ldots < d_k < n) \qquad (d_i \in \mathbb{N}).$$

We denote the space of flags of type $\mathcal{I}$ by $\mathcal{B}_{\mathcal{I}}$. We fix once for all a maximal set of linearly independent vectors
$$\{e_1, e_2, \ldots, e_r, e_1^*, e_2^*, \ldots, e_r^*\} \qquad (e_i, e_i^* \in V)$$
such that
$$(e_i, e_j) = 0 = (e_i^*, e_j^*), \qquad (e_i, e_j^*) = \delta_{ij}$$
where r is the real rank of G. For each integer $1 \leq k \leq r$, we let X_k be the linear span of $\{e_1, \ldots, e_k\}$, and X_k^* be the linear span of $\{e_1^*, \ldots, e_k^*\}$. We set $W_k = X_k \oplus X_k^*$. We define a map $\tau \in G$ as follows
$$\tau(e_i) = e_i^*, \qquad \tau(e_i^*) = \epsilon e_i \qquad (i \in [1, r]),$$
$$\tau|_{W_r^\perp} = id.$$
Let $\mathcal{I}_0 = \{0 < 1 < 2 < \ldots < r \leq n - r < n - r + 1 < \ldots < n - 1 < n\}$. We fix a flag
$$\mathcal{F}_0 = \{0 \subsetneqq X_1 \subsetneqq \ldots \subsetneqq X_r \subseteq X_r^\perp \subsetneqq \ldots \subsetneqq X_1^\perp \subsetneqq V\}.$$

For an arbitrary $\lambda = (\lambda_1, \dots \lambda_r) \in (\mathbb{R}^+)^r$, we define a linear isomorphism $A(\lambda) \in GL_D(V)$ as follows,

$$A(\lambda)e_i = \lambda_i e_i; \qquad A(\lambda)e_i^* = \lambda_i^{-1} e_i^* \qquad (i \in [1, r])$$
$$A(\lambda)u = u \qquad (u \in W_r^{\perp}).$$

It is easy to check that $A(\lambda) \in G(V)$. Let A be the group consisting of all $A(\lambda)$. Then A is a maximal split Abelian subgroup of $G(V)$.

For $h = (h_1, \dots, h_r) \in \mathbb{R}^r$, we define $a(h) \in \mathrm{End}_D(V)$ such that

$$a(h)e_i = h_i e_i, \qquad a(h)e_i^* = -h_i e_i^* \qquad (i \in [1, r])$$
$$a(h)u = u \qquad (u \in W_r^{\perp}).$$

It is easy to see that the Lie algebra $\mathfrak{a}$ of A consists of all $a(h)$. Let $\Delta(\mathfrak{g}, \mathfrak{a})$ be the restricted root system. For $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$, let $\mathfrak{g}_\alpha$ be the root space. Then we have

$$\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha} \qquad (\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})).$$

Lemma 3.1. *The isotropic group $P_0 = G_{\mathcal{F}_0}$ is a minimal parabolic subgroup of G. Its Levi factor*

$$\begin{aligned} MA =& P_0 \cap \tau(P_0) \\ =& \{g \in G(V) \mid g.X_i = X_i,\ g.X_i^* = X_i^*,\ g.W_r^{\perp} = W_r^{\perp}\} \\ =& \{g \in G(V) \mid g.(e_i D) = e_i D,\ g.(e_i^* D) = e_i^* D,\ g.W_r^{\perp} = W_r^{\perp}\}. \end{aligned} \tag{8}$$

Similarly, we can define a flag $\mathcal{F}_{\mathcal{I}}$ of type

$$\mathcal{I} = \{0 < i_1 < i_2 \dots < i_l < n\}$$

by

$$V_j = X_{i_j} \qquad (j \leq \frac{l+1}{2})$$
$$V_j = X_{i_{l+1-j}}^{\perp} \qquad (j \geq \frac{l+1}{2}).$$

Lemma 3.2. *$P_{\mathcal{I}} = G_{\mathcal{F}_{\mathcal{I}}}$ are all the parabolic subgroups containing P_0. If $G \neq O_{1,1}, O(2, \mathbb{C})$ (in these two cases, no proper parabolic subgroup exists), the maximal parabolic subgroups correspond to $\mathcal{I} = \{0 < k \leq n - k < n\}$.*

Proof: Obviously $P_{\mathcal{I}} \supseteq P_0$. Now we observe that for $G \neq O(1,1), O(2, \mathbb{C})$, $P_{\mathcal{I}}$ and $P_{\mathcal{I}'}$ are different if $\mathcal{I} \neq \mathcal{I}'$. The cardinality of all the $\mathcal{I}$ is 2^r. But the cardinality of parabolic groups containing P_0 is also 2^r. Thus $P_{\mathcal{I}}$ exhaust all the parabolic subgroups containing P_0.

Observe that $P_{\mathcal{I}} \supseteq P_{\mathcal{I}'}$ if and only if $\mathcal{I}'$ is a refinement of $\mathcal{I}$. Therefore the maximal parabolic subgroups correspond to $\mathcal{I} = \{0 < k \leq n - k < n\}$. Q.E.D.

We denote the maximal parabolic subgroup $P_{\{0 < k \leq n-k < n\}}$ by P_k.

Lemma 3.3. *The Levi factor $M_{\mathcal{I}} A_{\mathcal{I}}$ can be given by*

$$P_{\mathcal{I}} \cap \tau(P_{\mathcal{I}}) = \{g \in G(V) \mid g.X_{i_j} = X_{i_j}; g.X_{i_j}^* = X_{i_j}^*, j \in [1, \frac{l+1}{2}]\}.$$

For P_k maximal parabolic, let $M_k A_k N_k$ be the Langlands decomposition. Then A_k is 1-dimensional and

$$A_k = \{a(t), t \in \mathbb{R}^+ \mid a(t)|_{X_k} = t; a(t)|_{X_k^*} = t^{-1}; a(t)|_{W_k^{\perp}} = 1\},$$
$$M_k A_k = \{g \in G(V) \mid g.X_k = X_k; g.X_k^* = X_k^*\} \cong GL_D(k) \times G(W_k^{\perp}).$$

Now we fix an $h_k \in \mathfrak{a}_k$, such that h_k is identity on X_k, and -1 on X_k^*, and zero on $W_k^{\perp}$. Then V can be decomposed into eigenspaces of h_k:

$$V_{-1} = X_k^*, \qquad V_1 = X_k, \qquad V_0 = W_k^{\perp}.$$

Thus $\mathfrak{g}$ can be decomposed into eigenspaces of h_k as follows.

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2. \tag{9}$$

where

$$\begin{gathered} \mathfrak{g}_0 = \{x \in \mathfrak{g} \mid x.X_k \subseteq X_k; x.X_k^* \subseteq X_k^*; x.W_k^{\perp} \subseteq W_k^{\perp}\} \\ \mathfrak{g}_1 = \{x \in \mathfrak{g} \mid x.X_k = 0; x.W_k^{\perp} \subseteq X_k; x.X_k^* \subseteq W_k^{\perp}\} \\ \mathfrak{g}_2 = \{x \in \mathfrak{g} \mid x.X_k = 0; x.W_k^{\perp} = 0; x.X_k^* \subseteq X_k\} \\ \mathfrak{g}_{-i} = \tau(\mathfrak{g}_i) \qquad (i = 1, 2). \end{gathered}$$

Moreover,

$$\mathfrak{g}_0 = \mathfrak{m}_k \oplus \mathfrak{a}_k \qquad \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{n}_k.$$

Since our argument is valid for every $k \leq r$, $\mathfrak{g}_i$ will denote the i-eigenspace of $ad(h_k)$ for a fixed (implicit) k. Notice that

$$x \in \mathfrak{g}_2 \Longleftrightarrow x|_{X_k \oplus W_k^{\perp}} = 0;\ (x.u, v) + (u, x.v) = 0\ (\forall\ u, v \in X_k^*).$$

If we define a sesquilinear form on X_k^* to be

$$B_x(u, v) = (x.u, v) \qquad (u, v \in X_k^*),$$

then

$$B_x(u, v) = -\epsilon B_x(v, u)^{\sharp}.$$

Therefore $\mathfrak{g}_2$ can be identified with a space of sesquilinear forms $(,)_{-\epsilon}$ on X_k^*. Similarly, $\mathfrak{g}_2^*$ can be identified with a space of sesquilinear forms $(,)_{-\epsilon}$ on X_k.

Lemma 3.4. *$\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module. Suppose $\mathfrak{g}_2 \neq \{0\}$. Then $\mathfrak{g}_2$ is the center of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.*

By Theorem 1.1, we have the following theorem.

Theorem 3.1. *Let $\mathfrak{g}$ be a real classical Lie algebra of type I. Let M be a $\mathfrak{g}$-module. Let j^* be the canonical projection from $\mathfrak{g}_{\mathbb{C}}^*$ onto $\mathfrak{g}_{2\mathbb{C}}^*$. Then*

$$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g}_2)}(M)) = cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(M)))).$$

By Theorem 1.2, we have the following theorem.

Theorem 3.2. *Let $P_k = M_k A_k N_k$ be as in Lemma 3.3. Let $\mathfrak{l}_k = \mathfrak{m}_k \oplus \mathfrak{a}_k$. Let V be a $\mathfrak{p}_k$-module. Let p^* be the canonical projection from $\mathfrak{p}_{k\mathbb{C}}^*$ onto $\mathfrak{l}_{k\mathbb{C}}^*$. Then*

$$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{l}_k)}(V)) = cl(p^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{p}_k)}(V)))).$$

We end this section with the following lemma.

Lemma 3.5. *P_k acts on $\mathfrak{g}_2^*$ with finitely many orbits. The orbits are uniquely determined by the rank and the signature of the corresponding sesquilinear form.*

Here $\mathfrak{g}_2^*$ is the dual space of $\mathfrak{g}_2$. It is not to be confused with $\mathfrak{g}_{-2}$. Following [Howe1], define the rank of any subset S of $\mathfrak{g}_2^*$ regarded as sesquilinear form to be the maximal rank of the elements of S.

4. Howe's N-spectrum and N-associated variety

Let G be a Lie group with a finite number of connected components, H be a closed subgroup. Let $\hat{G}$ be the unitary dual of G. Suppose that G and H are type I groups ([Di]). Take a unitary representation $(\pi, \mathcal{H})$ of G and consider its restriction to H. According to the direct integral theory, $\pi|_H$ uniquely determines a projection-valued Borel measure $\mu_H(\pi)$ on $\hat{H}$. R. Howe called such a measure the $H-$spectrum of π [Howe1]. Under the Fell topology, the (closed) support of $\mu_H(\pi)$ is called the geometric H-spectrum [Howe1]. Let $N_G(H)$ be the normalizer of H in G. Since $(\pi, \mathcal{H})$ is a unitary representation of $N_G(H)$, $\mathrm{supp}(\mu_H(\pi))$ is $N_G(H)$-stable.

To study H-spectrum, we have to have a well-understood unitary dual $\hat{H}$. For H nilpotent or solvable of type I, $\hat{H}$ is well-understood to some extent. For H connected Abelian, $\hat{H}$ can be identified with a subset of $i\mathfrak{h}^*$. In this section, we will deal with Abelian H and we identify $\hat{H}$ with a subset of $i\mathfrak{h}^*$.

Let G be a type I classical group as in the last section. Let N_k be the nilradical of P_k and ZN_k be the center of N_k. Then

$$\mathfrak{n}_k = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \qquad \mathfrak{z}\mathfrak{n}_k = \mathfrak{g}_2$$

where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are defined as eigenspaces of $ad(h_k)$. The main problem in this section is to study the relationship between Howe's ZN_k-spectrum and the associated variety $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))$.

Recall that $\mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{n}_k$ and $\mathfrak{g}_2 = \mathfrak{z}\mathfrak{n}_k$. By Theorem 2.1 and Theorem 3.1, we have

Theorem 4.1. *Let $(\pi, \mathcal{H})$ be a unitary representation of a type I classical group G. Then the ZN_k-associated variety of π is the Zariski closure of the geometric ZN_k-spectrum of π. Furthermore,*

$$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{z}n_k)}(\pi)) = cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$$

where j^* is the projection from $\mathfrak{g}^*_{\mathbb{C}}$ to $\mathfrak{z}\mathfrak{n}^*_{k\mathbb{C}}$. So

$$cl(\mathrm{supp}_{ZN_k}(\pi)) = cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi)))).$$

Since $\mathfrak{g}$ is a reductive linear Lie algebra, $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$ by an invariant bilinear form. For any subset S of $\mathfrak{g}^*$, we define $\mathrm{rank}(S)$ to be the $\max\{\mathrm{rank}_D(X) \mid X \in S\}$. Now for a type I classical group $G(V)$, for every $x \in \mathfrak{g}$, we define a sesquilinear form B_x such that

$$B_x(u, v) = (x.u, v) \qquad (u, v \in V)$$

Then

$$B_x(u, v) = -\epsilon B_x(v, u)^\sharp$$

Thus $\mathfrak{g}$ can be identified with a space of sesquilinear forms. Clearly, the rank of the sesquilinear form B_x is exactly the rank of the x.

Recall that the parabolic subgroup P_k acts on $\mathfrak{z}\mathfrak{n}^*_k$ with finitely many orbits and that $\mathfrak{z}\mathfrak{n}^*_k$ can be identified with a subspace of sesquilinear forms. Howe and Li defined the ZN_k-rank to be the maximal rank of $\mathrm{supp}(\mu_{ZN_k}(\pi))$ regarded as sesqulinear forms. Notice that for each $x \in \mathrm{Hom}_D(X_k, X^*_k)$ the rank of the linear transform x is the same as the rank of the bilinear form B_x. In the rest of this paper, we will compute the Howe's ZN_k-rank using associated variety.

If we regard $\mathfrak{g}$ as a subset of $\mathrm{Hom}_D(V, V)$, then j^* can be regarded as the (eigen)-projection with respect to $ad(h_k)$, from $\mathfrak{g}$ onto $\mathfrak{g}_{-2} \cong \tau(\mathfrak{zn}_k)$ (see Equ. 9). We have the following list regarding $\mathfrak{g}_{-2}$ and its complexification:

(1) $G = U(p,q)$, $\mathfrak{zn}_k^*$ is the space of $k \times k$ skew-Hermitian matrices, its complexification is the space of $k \times k$ complex matrices;
(2) $G = O(p,q)$, $\mathfrak{zn}_k^*$ is the space of $k \times k$ real skew-symmetric matrices, its complexification is the space of $k \times k$ complex skew-symmetric matrices;
(3) $G = Sp_{2n}(\mathbb{R})$, $\mathfrak{zn}_k^*$ is the space of $k \times k$ real symmetric matrices, its complexification is the space of $k \times k$ complex symmetric matrices;
(4) $G = O^*(2n)$, $\mathfrak{zn}_k^*$ is the space of sesquilinear forms on $\mathbb{H}^k$, such that
$$(u,v) = (v,u)^\sharp \qquad (u, v \in \mathbb{H}^k)$$
Let $(u,v) = A(u,v) + jB(u,v)$ with A and B complex-valued. Then
$$A(v,u) + jB(v,u) = (A(u,v) + jB(u,v))^\sharp = \overline{A(u,v)} - jB(u,v)$$
Therefore
$$A(u,v) = \overline{A}(v,u) \qquad B(u,v) = -B(v,u).$$
Now $B(u,v)$ is a (right) $\mathbb{C}$-bilinear form. If we fix a basis $\{(e_i, je_i)\}_1^k$ for $\mathbb{H}^k$, $\mathfrak{zn}_k^*$ can be identified with
$$\{\begin{pmatrix} U & V \\ -\bar{V} & \bar{U} \end{pmatrix} \mid U^t = -U, \bar{V} = V^t\}.$$
Thus the complexification of $\mathfrak{zn}_k^*$ can be identified with the space of $2k \times 2k$ complex skew-symmetric matrices.
(5) $G = Sp(p,q)$, $\mathfrak{zn}_k^*$ can be identified with a space of $2k \times 2k$ symmetric matrices, its complexification is the space of $2k \times 2k$ complex symmetric matrices.
(6) $G = O(n, \mathbb{C})$, $\mathfrak{zn}_k^*$ is the space of $k \times k$ complex skew-symmetric matrices. It can be identified with
$$\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mid A^t = -A, B^t = -B, A, B \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^k)\}.$$
Therefore ${\mathfrak{zn}_k}_{\mathbb{C}}^*$ can be identified with
$$\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mid A^t = -A, B^t = -B, A, B \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^k)\}.$$
(7) $G = Sp(n, \mathbb{C})$, $\mathfrak{zn}_k^*$ can be identified with
$$\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mid A^t = A, B^t = B, A, B \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^k)\}$$
and ${\mathfrak{zn}_k}_{\mathbb{C}}^*$ can be identified with
$$\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mid A^t = A, B^t = B, A, B \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^k)\}.$$

For any $S \subseteq {\mathfrak{zn}_k}_{\mathbb{C}}^*$, we write $\mathrm{rank}_{\mathbb{C}}(S)$ for the maximal rank of the elements in S in this setting. We call it the $\mathbb{C}$-rank of S. Thus, we have
$$\mathrm{rank}(\mathrm{supp}(\mu_{ZN_k}(\pi))) = \mathrm{rank}_{\mathbb{C}}(\mathrm{supp}(\mu_{ZN_k}(\pi))) \qquad (G = U(p,q), O(p,q), Sp_{2n}(\mathbb{R}));$$
$$2\mathrm{rank}(\mathrm{supp}(\mu_{ZN_k}(\pi))) = \mathrm{rank}_{\mathbb{C}}(\mathrm{supp}(\mu_{ZN_k}(\pi))) \qquad (G = Sp(n,\mathbb{C}), O(n,\mathbb{C}), Sp(p,q), O^*(2n)).$$

In this setting, taking the Zariski closure of a subset of sesquilinear form would not change $\mathbb{C}$-rank of such a subset. Since $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{zn}_k)}(\pi))$ is the Zariski closure of $\mathrm{supp}(\mu_{ZN_k}(\pi))$,
$$\mathrm{rank}_{\mathbb{C}}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{zn}_k)}(\pi))) = \mathrm{rank}_{\mathbb{C}}(\mathrm{supp}(\mu_{ZN_k}(\pi))).$$

By Theorem 4.1, we have

$$\mathrm{rank}_{\mathbb{C}}(\mathrm{supp}(\mu_{ZN_k}(\pi))) = \mathrm{rank}_{\mathbb{C}}(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi)))).$$

To compute Howe's ZK_k-rank, we will have to compute $\mathrm{rank}_{\mathbb{C}}(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$.

Let us first recall the following theorem.

Theorem 4.2 (Borho-Brylinski-Joseph). *Suppose $\mathfrak{g}$ is a reductive Lie algebra, M a simple $\mathfrak{g}$-module. Then $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(M))$ is the closure of a single coadjoint orbit.*

So for a connected reductive group G and an irreducible unitary representation π, $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))$ is the closure if a single coadjoint orbit. Now concerning a linear reductive Lie group G with finitely many components, we can employ Mackey machine to show that for any irreducible unitary representation (π, H) of G, π splits into finitely many irreducible representations when restricted to the identity component G_0, namely,

$$\pi = \pi_1 \oplus \pi_2 \oplus \ldots \oplus \pi_s$$

Furthermore, G/G_0 permutes these irreducible factors. A more careful examination shows that the Harish-Chandra modules of π_i's are related by the algebra isomorphisms of $U(\mathfrak{g})$ defined by the adjoint action of G/G_0. Thus $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi_i))$ are related by automorphisms of $\mathfrak{g}$ defined by G/G_0. In fact, $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))$ is exactly the union of G/G_0-orbit of any chosen $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi_i))$. More precisely, we have

$$\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi)) = \bigcup_{xG_0 \in G/G_0} Ad(x)(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi_1)))$$

Thus, for the rest of this paper, even though some of the classical Lie group G is not connected, we may prove our results for the identity component G_0 first. Then all the results can be generalized to G.

Now identify $\mathfrak{g}_{\mathbb{C}}^*$ with $\mathfrak{g}_{\mathbb{C}}$ via an invariant bilinear form. According to [CM], each nilpotent orbit in a (complex) simple Lie algebra $\mathfrak{g}(m) \subseteq \mathrm{End}_{\mathbb{C}}(\mathbb{C}^m)$ is parameterized by certain partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l > 0)$ of m. We denote the adjoint orbit corresponding to λ by $\mathcal{O}_\lambda$. Then

$$\mathrm{rank}_{\mathbb{C}}(\mathcal{O}_\lambda) = m - l.$$

Lemma 4.1. *Let $S \subseteq \mathfrak{g}(m)$. Then*

$$\mathrm{rank}_{\mathbb{C}}(j^*(S)) \leq \min(r_k, \mathrm{rank}_{\mathbb{C}}(S))$$

where $r_k = \mathrm{rank}_{\mathbb{C}}(\mathfrak{z}\mathfrak{n}_k^*)$. In particular,

$$\mathrm{rank}_{\mathbb{C}}(j^*(\mathcal{O}_\lambda)) \leq \min(r_k, \mathrm{rank}_{\mathbb{C}}(\mathcal{O}_\lambda)).$$

Now we restrict our attention to the non-complex groups, $O(p,q)$, $U(p,q)$, $Sp_{2n}(\mathbb{R})$, $O^*(2n)$, $Sp(p,q)$. We will deal with complex groups at the end. We treat Type A, C and Type B, D Lie algebras differently. We will follow the convention in [CM] regarding the order of nilpotent orbits.

Theorem 4.3 (Type A,C $\mathfrak{g}_{\mathbb{C}}$). *Let $\mathcal{O}_\lambda$ be a complex nilpotent orbit in a type A or C simple Lie algebra $\mathfrak{g}(m)$ parametrized by λ. Then* $\mathrm{rank}_{\mathbb{C}}(j^*(\mathcal{O}_\lambda)) = \min(k, \mathrm{rank}_{\mathbb{C}}(\mathcal{O}_\lambda))$.

Proof: If $\mathrm{rank}_{\mathbb{C}}(\mathcal{O}_\lambda) \geq k$, then $\lambda \geq (1^{m-2k}, 2^k)$. Thus

$$cl(\mathcal{O}_\lambda) \supseteq cl(\mathcal{O}_{(1^{m-2k}, 2^k)}).$$

Recall that $\mathfrak{g}_{-2} \subseteq \mathcal{O}_{(1^{m-2k}, 2^k)}$. Therefore

$$cl(j^*(\mathcal{O}_\lambda)) \supseteq j^*(cl((\mathcal{O}_\lambda))) \supseteq j^*(cl(\mathcal{O}_{(1^{m-2k}, 2^k)})) \supseteq j^*(\mathfrak{g}_{-2}) \supseteq \mathfrak{g}_{-2}.$$

Hence $\mathrm{rank}_{\mathbb{C}}(j^*(\mathcal{O}_\lambda)) = k$. If $\mathrm{rank}_{\mathbb{C}}(\mathcal{O}_\lambda) = s < k$, then

$$cl(\mathcal{O}_\lambda) \supseteq cl(\mathcal{O}_{(1^{m-2s},2^s)}).$$

Thus

$$cl(j^*(\mathcal{O}_\lambda)) \supseteq j^*(cl((\mathcal{O}_\lambda))) \supseteq j^*(cl(\mathcal{O}_{(1^{m-2s},2^s)})).$$

But $\mathrm{rank}_{\mathbb{C}}(cl(\mathcal{O}_{(1^{m-2s},2^s)}) \cap \mathfrak{g}_{-2}) = s$, because the elements in $\mathfrak{g}_{-2}$ of rank s are all contained in $\mathcal{O}_{(1^{m-2s},2^s)}$. Therefore

$$\mathrm{rank}_{\mathbb{C}}(j^*(\mathcal{O}_\lambda)) \geq \mathrm{rank}_{\mathbb{C}}(j^*(cl(\mathcal{O}_{(1^{m-2s},2^s)})) \cap \mathfrak{g}_{-2}) = \mathrm{rank}_{\mathbb{C}}(cl(\mathcal{O}_{(1^{m-2s},2^s)})) \cap \mathfrak{g}_{-2}) = s.$$

Combined with Lemma 4.1, we have

$$\mathrm{rank}_{\mathbb{C}}(j^*(\mathcal{O}_\lambda)) = \min(k, \mathrm{rank}_{\mathbb{C}}(\mathcal{O}_\lambda))$$

Q.E.D.

Theorem 4.4 (Type B,D $\mathfrak{g}_{\mathbb{C}}$)**.** *Let $\mathcal{O}_\lambda$ be a complex nilpotent orbit in a type B or D simple Lie algebra parametrized by λ. Then $\mathrm{rank}_{\mathbb{C}}(j^*(\mathcal{O}_\lambda))$ is always even and it is equal to $\min(r_k, \mathrm{rank}_{\mathbb{C}}(\mathcal{O}_\lambda))$. Here $r_k = \mathrm{rank}_{\mathbb{C}}(\mathfrak{z}\mathfrak{n}_k^*)$.*

Proof: For $O(p,q)$, the $\mathbb{C}$-rank of a real skew-symmetric form is always even. For $O^*(2n)$, the $\mathbb{C}$-rank of an $\mathbb{H}$-sesquilinear form is also even. Thus $\mathrm{rank}_{\mathbb{C}}(j^*(\mathcal{O}_\lambda))$ is always even. Recall that the partitions corresponding to Type B, D nilpotent orbits satisfy that even parts occur with even multiplicity. In other words, if we delete the first column in the Young diagram, then odd parts occur with even multiplicity. Therefore, $\mathrm{rank}_{\mathbb{C}}(\mathcal{O}_\lambda)$ has to be even as well. The rest of the proof is the same as the proof for type A, C groups. Q.E.D.

Now we want to deal with complex groups $O(n,\mathbb{C})$ and $Sp(n,\mathbb{C})$. In these cases, $\mathfrak{g}_{\mathbb{C}}$ is not simple. However, once we regard $\mathfrak{g}$ as a real matrix Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ is still a matrix algebra. Thus the $\mathbb{C}$-rank of $\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))$ is still valid. Recall that

$$cl(\mathrm{WF}(\pi)) = \mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi)).$$

Here $WF(\pi) \subseteq \mathfrak{g}$.

Theorem 4.5. *Let π be an irreducible representation of $O(n,\mathbb{C})$ or $Sp(n,\mathbb{C})$. Then*

$$\mathrm{rank}_{\mathbb{C}}(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi)))) = \min(r_k, \mathrm{rank}_{\mathbb{C}}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$$

where $r_k = \mathrm{rank}_{\mathbb{C}}(\mathfrak{z}\mathfrak{n}_k^*)$.

Proof: Notice that

$$cl(j^*(\mathrm{WF}(\pi))) = cl(j^*(cl(\mathrm{WF}(\pi)))) = cl(j^*(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi)))).$$

Since $\mathfrak{g}$ is already a complex linear space, for any $S \subseteq \mathfrak{g}^* \subseteq \mathfrak{g}_{\mathbb{C}}^*$,

$$\mathrm{rank}_{\mathbb{C}}(S) = 2\mathrm{rank}(S).$$

It suffices to show that

$$\mathrm{rank}(j^*(\mathrm{WF}(\pi))) = \min(\mathrm{rank}(\mathfrak{z}\mathfrak{n}_k), \mathrm{rank}(\mathrm{WF}(\pi))).$$

Since $\mathrm{WF}(\pi)$ is a finite union of nilpotent orbits in $\mathfrak{g}^*$, the statement above is just a corollary of Theorem 4.4 and 4.5. Q.E.D.

Finally, we come to a conclusion that for $Sp_{2n}, U(p,q)$, Howe's ZN_k-rank of (π, H) equals

$$\min(k, \mathrm{rank}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))));$$

for $O(p,q)$, Howe's ZN_k-rank of (π, H) equals

$$\min(k, \mathrm{rank}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$$

for k even, and

$$\min(k-1, \mathrm{rank}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))))$$

for k odd; for $Sp(p,q), O^*(2n)$, Howe's ZN_k-rank of (π, H) equals

$$\min(k, \frac{1}{2}\mathrm{rank}_{\mathbb{C}}(\mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})}(\pi))));$$

for $Sp(n,\mathbb{C})$, Howe's ZN_k-rank of (π, H) equals $\min(k, \mathrm{rank}(\mathrm{WF}(\pi)))$; for $O(n,\mathbb{C})$, Howe's ZN_k-rank of (π, H) equals $\min(k, \mathrm{rank}(\mathrm{WF}(\pi)))$ when k is even, and $\min(k-1, \mathrm{rank}(\mathrm{WF}(\pi)))$ when k is odd. Thus Theorem 0.4 is proved.

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