Eigenvectors and Reconstruction

Hongyu He
Department of Mathematics & Statistics
Louisiana State University
email: hongyu@math.lsu.edu

Abstract

In this paper, we study the simple eigenvectors of two hypomorphic matrices using linear algebra. We also give a new proof of a result of Godsil-McKay.

1 Introduction

We start by fixing some notations. Let $A$ be a $n \times n$ real symmetric matrix. Let $A_i$ be the matrix obtained by deleting the $i$-th row and $i$-th column of $A$. We say that two symmetric matrices $A$ and $B$ are hypomorphic if $B_i$ can be obtained by permuting the rows and columns of $A_i$ simultaneously. Let $\Sigma$ be the set of permutations. We write $B = \Sigma(A)$.

If $M$ is a symmetric real matrix, then the eigenvalues of $M$ are real. We write

$$eigen(M) = (\lambda_1(M) \geq \lambda_2(M) \geq \ldots \geq \lambda_n(M)).$$

If $\alpha$ is an eigenvalue of $M$, we denote the corresponding eigenspace by $eigen_\alpha(M)$. Let $1$ be the $n$-dimensional vector $(1, 1, \ldots, 1)$. Put $J = 1'1$.

**Theorem 1 (HE1)** Let $B$ and $A$ be two real $n \times n$ symmetric matrices. Let $\Sigma$ be a hypomorphism such that $B = \Sigma(A)$. Let $t$ be a real number. Then there exists an open interval $T$ such that for $t \in T$ we have

1. $\lambda_n(A + tJ) = \lambda_n(B + tJ)$;
2. $eigen_{\lambda_n}(A + tJ)$ and $eigen_{\lambda_n}(B + tJ)$ are both one dimensional;
3. $eigen_{\lambda_n}(A + tJ) = eigen_{\lambda_n}(B + tJ)$.

As proved in [HE1], our result implies Tutte’s theorem which says that $eigen(A + tJ) = eigen(B + tJ)$.

In this paper, we shall study the eigenvectors of $A$ and $B$. We first prove that the squares of the entries of simple unit eigenvectors of $A$ can be reconstructed as functions of $eigen(A)$ and $eigen(A_i)$. This yields a proof of a Theorem of Godsil-McKay. We also study how the eigenvectors of $A$ change after a perturbation of a rank 1 symmetric matrices. Combined with Theorem 1, we prove another result of Godsil-McKay which states that the simple eigenvectors that are perpendicular to $1$ are reconstructible. We further show that the orthogonal projection of $1$ onto higher dimensional eigenspaces is reconstructible.

Our investigation indicates that the following conjecture could be true.
Conjecture 1  Let $A$ be a real $n \times n$ symmetric matrix. Then there exists a subgroup $G(A) \subseteq O(n)$ such that a real symmetric matrix $B$ satisfies the properties that $\text{eigen}(B) = \text{eigen}(A)$ and $\text{eigen}(B_i) = \text{eigen}(A_i)$ for each $i$ if and only if $B = UAU^t$ for some $U \in G(A)$.

This conjecture is clearly true if $\text{rank}(A) = 1$. For $\text{rank}(A) = 1$, the group $G(A)$ can be chosen as $\mathbb{Z}_2^n$, all in the form of diagonal matrices. In some other cases, $G(A)$ can be a subgroup of the permutation group $S_n$.

Conjecture 2  The group $G(A)$ can be chosen to be a twisted product of a subgroup of $S_n$ with $\mathbb{Z}_2^n$.

Clearly, this conjecture implies the reconstruction conjecture.

2  Reconstruction of Square Functions

Theorem 2  Let $A$ be a $n \times n$ real symmetric matrix. Let $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ be the eigenvalues of $A$. Suppose $\lambda_i$ is a simple eigenvalue of $A$. Let $p_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t$ be a unit vector in $\text{eigen}_{\lambda_i}(A)$. Then for every $m$, $p_{m,i}$ can be expressed as a function of $\text{eigen}(A)$ and $\text{eigen}(A_m)$.

Proof: Let $\lambda_i$ be a simple eigenvalue of $A$. Let $p_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t$ be a unit vector in $\text{eigen}_{\lambda_i}(A)$. There exists an orthogonal matrix $P$ such that $P = (p_1, p_2, \cdots, p_n)$ and $A = PD^tP^t$ where

$$D = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}.$$  

Then

$$A - \lambda_iI = PD^t - \lambda_iI = P(D - \lambda_iI)P^t = \sum_{j \neq i} (\lambda_j - \lambda_i)p_j p_j^t,$$

which equals

$$\begin{pmatrix}
p_{1,1} & \cdots & \hat{p}_{1,i} & \cdots & p_{1,n} \\
p_{2,1} & \cdots & \hat{p}_{2,i} & \cdots & p_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
p_{n,1} & \cdots & \hat{p}_{n,i} & \cdots & p_{n,n}
\end{pmatrix} \begin{pmatrix}
\lambda_1 - \lambda_i & \cdots & 0 & \cdots & 0 \\
0 & \ddots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \cdots & \cdots \\
0 & \cdots & 0 & \lambda_n - \lambda_i
\end{pmatrix} \begin{pmatrix}
p_{1,1} & p_{2,1} & \cdots & p_{n,1} \\
p_{1,1} & \cdots & \hat{p}_{1,i} & \cdots & p_{1,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
p_{n,n} & \cdots & \hat{p}_{n,i} & \cdots & p_{n,n}
\end{pmatrix}.$$  

Deleting the $m$-th row and $m$-th column, we obtain

$$\begin{pmatrix}
p_{1,1} & \cdots & \hat{p}_{1,i} & \cdots & p_{1,n} \\
p_{m,1} & \cdots & \hat{p}_{m,i} & \cdots & p_{m,n} \\
p_{n,1} & \cdots & \hat{p}_{n,i} & \cdots & p_{n,n}
\end{pmatrix} \begin{pmatrix}
\lambda_1 - \lambda_i & \cdots & 0 & \cdots & 0 \\
0 & \ddots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \cdots & \cdots \\
0 & \cdots & 0 & \lambda_n - \lambda_i
\end{pmatrix} \begin{pmatrix}
p_{1,1} & \cdots & \hat{p}_{m,1} & \cdots & p_{n,1} \\
p_{1,1} & \cdots & \hat{p}_{m,1} & \cdots & p_{1,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
p_{n,n} & \cdots & \hat{p}_{n,i} & \cdots & p_{n,n}
\end{pmatrix}.$$  

This is $A_m - \lambda_iI_{n-1}$. Notice that $P$ is orthogonal. Taking the determinant, we have

$$\det(A_m - \lambda_iI_{n-1}) = p_{m,i}^2 \prod_{j \neq m} (\lambda_j - \lambda_i).$$

2
It follows that

\[ P_{m,i}^2 = \frac{\prod_{j=1}^{n-1}(\lambda_j(A_m) - \lambda_i)}{\prod_{j \neq m}(\lambda_j - \lambda_i)}. \]

Q.E.D.

**Corollary 1** Let A and B be two \( n \times n \) real symmetric matrices. Suppose that \( \text{eigen}(A) = \text{eigen}(B) \) and \( \text{eigen}(A_i) = \text{eigen}(B_i) \). Let \( \lambda_i \) be a simple eigenvalue of A and B. Let \( p_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t \) be a unit vector in \( \text{eigen}_{\lambda_i}(A) \) and \( q_i = (q_{1,i}, q_{2,i}, \ldots, q_{n,i})^t \) be a unit vector in \( \text{eigen}_{\lambda_i}(B) \). Then

\[ p_i^2 = q_i^2, \forall i \in [1, n]. \]

**Corollary 2** (Godsil-McKay, see Theorem 3.2, [GM]) Let A and B be two \( n \times n \) real symmetric matrices. Suppose that A and B are hypomorphic. Let \( \lambda_i \) be a simple eigenvalue of A and B. Let \( p_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t \) be a unit vector in \( \text{eigen}_{\lambda_i}(A) \) and \( q_i = (q_{1,i}, q_{2,i}, \ldots, q_{n,i})^t \) be a unit vector in \( \text{eigen}_{\lambda_i}(B) \). Then

\[ p_i^2 = q_i^2, \forall i \in [1, n]. \]

### 3 Eigenvalues and Eigenvectors under the perturbation of a rank one symmetric matrix

Let A be a \( n \times n \) real symmetric matrix. Let \( x \) be a \( n \)-dimensional row column vector. Let \( M = xx^t \). Now consider \( A + tM \). We have

\[ A + tM = PD(P^t) + tM = P(D + tP^tM)P^t = P(D + tP^txx^tP)P^t. \]

Let \( P^tx = q \). So \( q_i = (p_i, x) \) for each \( i \in [1, n] \). Then

\[ A + tJ = P(D + tqq^t)P^t. \]

Put \( D(t) = D + tqq^t \).

**Lemma 1** \( \det(D + tqq^t - \lambda I) = \det(A - \lambda I)(1 + \sum_i \frac{tq_i^2}{\lambda_i - \lambda}) \).

Proof: \( \det(D - \lambda I + tqq^t) \) can be written as a sum of products of \( \lambda_i - \lambda \) and \( q_iq_j \). For each \( S \) a subset of \([1, n]\), combine the terms containing only \( \prod_{i \in S}(\lambda_i - \lambda) \). Since the rank of \( qq^t \) is one, only for \( |S| = n, n - 1 \), the coefficients may be nonzero. We obtain

\[ \det(D + tqq^t - \lambda I) = \prod_{i=1}^n(\lambda_i - \lambda) + \sum_{i=1}^n tq_i^2 \prod_{j \neq i}(\lambda_i - \lambda). \]

The Lemma follows. \( \square \)

Put \( P_t(\lambda) = 1 + \sum_i \frac{tq_i^2}{\lambda_i - \lambda} \).

**Lemma 2** Fix \( t < 0 \). Suppose that for each \( i, \lambda_i \) is a simple eigenvalue and \( q_i \neq 0 \). Then \( P_t(\lambda) \) has exactly \( n \) roots (\( \mu_1, \mu_2, \ldots, \mu_n \)) satisfying an interlacing relation:

\[ \lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \mu_{n-1} > \lambda_n > \mu_n. \]

Proof: Clearly, \( \frac{dP_t(\lambda)}{dt} = \sum_i \frac{tq_i^2}{(\lambda_i - \lambda)^2} < 0. \) So \( P_t(\lambda) \) is always decreasing. On the interval \( (-\infty, \lambda_n) \), \( P_t(-\infty) = 1 \) and \( P_t(\lambda_n^+) = -\infty \). So \( P_t(\lambda) \) has a unique root \( \mu_n \in (-\infty, \lambda_n) \). Similar statement holds for each \( (\lambda_{i-1}, \lambda_i) \). Q.E.D.
**Theorem 3** Fix \( t < 0 \). Let \( l \) be the number of distinct eigenvalues satisfying \((x, \text{eigen}_x(A)) \neq 0\). Without loss of generalities, suppose that \( A = PDP^t \) such that there exists a

\[
S = \{i_1 > i_2 > \cdots > i_l\}
\]

satisfying \((x, p_{i_j}) \neq 0\) and \((x, p_i) = 0\) for every \( i \notin S \). Then there exists \((\mu_1, \ldots, \mu_l)\) such that

\[
\lambda_{i_1} > \mu_1 > \lambda_{i_2} > \mu_2 > \cdots > \lambda_{i_l} > \mu_l
\]

and

\[
\text{eigen}(A + tM) = \{\lambda_i(A) \mid i \notin S\} \cup \{\mu_1, \mu_2, \ldots, \mu_l\}.
\]

Furthermore, \(\text{eigen}_{\mu_i}(A + tM)\) contains

\[
\sum_{i \in S} p_i \frac{q_i}{\lambda_i - \mu_j}.
\]

Here the index set \(\{i_1, i_2, \cdots, i_l\}\) may not be unique. I shall also point out a similar statement holds for \( t > 0 \) with

\[
\mu_1 > \lambda_{i_1} > \lambda_{i_2} > \cdots > \lambda_{i_l} > \mu_l.
\]

**Proof:** Since \((x, \text{eigen}_{\lambda_{i_j}}(A)) \neq 0, q_{i_j} \neq 0\). For \( i \notin S, q_i = 0 \). Notice

\[
P_t(\lambda) = 1 + \sum_{j=1}^{l} \frac{tq_j^2}{\lambda_{i_j} - \lambda}.
\]

Applying Lemma 2 to \( S \), we obtain the roots of \( P_t(\lambda) \{\mu_1, \mu_2, \ldots, \mu_l\} \) satisfying

\[
\lambda_{i_1} > \mu_1 > \lambda_{i_2} > \mu_2 > \cdots > \lambda_{i_l} > \mu_l.
\]

It follows that the roots of \( \det(A + tM - \lambda I) = \det(D(t) - \lambda I) = P_t(\lambda) \prod_{k=1}^{n} (\lambda_k - \lambda) \) can be obtained from \( \text{eigen}(A) \) by changing \( \{\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_l}\} \) to \( \{\mu_1, \mu_2, \ldots, \mu_l\} \). Therefore,

\[
\text{eigen}(A + tM) = \{\lambda_i(A) \mid i \notin S\} \cup \{\mu_1, \mu_2, \ldots, \mu_l\}.
\]

For the sake of convenience, suppose that \( \mu_i \notin \text{eigen}(A) \). Then

\[
\sum_{i \in S} q_i \frac{p_i}{\lambda_i - \mu_j} = \sum_{i=1}^{n} q_i \frac{p_i}{\lambda_i - \mu_j}
\]

Here for \( \lambda_i \notin S, q_i = 0 \). Notice that

\[
(A + tM) \sum_{i=1}^{n} \frac{q_i}{\lambda_i - \mu_j} p_i = P(D + tqq^t)P^t \sum_{i=1}^{n} \frac{q_i}{\lambda_i - \mu_j} p_i = P(D + tqq^t) \begin{pmatrix}
\frac{q_1}{\lambda_1 - \mu_j} \\
\vdots \\
\frac{q_n}{\lambda_n - \mu_j}
\end{pmatrix},
\]

which equals

\[
P \begin{pmatrix}
\frac{\lambda_1 q_1}{\lambda_1 - \mu_j} \\
\vdots \\
\frac{\lambda_n q_n}{\lambda_n - \mu_j}
\end{pmatrix} + t \begin{pmatrix}
q_1 \\
\vdots \\
\sum_{i=1}^{n} \frac{q_i^2}{\lambda_i - \mu_j}
\end{pmatrix} = P \begin{pmatrix}
\frac{\lambda_1 q_1}{\lambda_1 - \mu_j} \\
\vdots \\
\frac{\lambda_n q_n}{\lambda_n - \mu_j}
\end{pmatrix} - \begin{pmatrix}
q_1 \\
\vdots \\
q_n
\end{pmatrix} = P \begin{pmatrix}
\frac{\mu_1 q_1}{\lambda_1 - \mu_j} \\
\vdots \\
\frac{\mu_n q_n}{\lambda_n - \mu_j}
\end{pmatrix}.
\]
We have obtained that
\[(A + tM) \sum_{i=1}^{n} \frac{q_i}{\lambda_i - \mu_j} p_i = \mu_j P \begin{pmatrix} \frac{q_1}{\lambda_1 - \mu_j} \\ \vdots \\ \frac{q_n}{\lambda_n - \mu_j} \end{pmatrix} = \mu_j \sum_{i=1}^{n} \frac{q_i}{\lambda_i - \mu_j} p_i.\]

If \(\mu_j \in \text{eigen}(A)\), we still have \((A + tM) \sum_{\lambda_i \in S} \frac{q_i}{\lambda_i - \mu_j} p_i = \sum_{\lambda_i \in S} \frac{q_i}{\lambda_i - \mu_j} p_i.\) Therefore,
\[
\sum_{\lambda_i \in S} \frac{q_i}{\lambda_i - \mu_j} p_i \in \text{eigen}_{\mu_j} (A + tM).
\]
Q.E.D.

4 Reconstruction of Simple Eigenvectors not perpendicular to 1

Now let \(M = J = 11^t\). Theorem 3 applies to \(A + tJ\) and \(B + tJ\).

**Theorem 4 (Godsil-McKay, [GM])** Let \(B\) and \(A\) be two real \(n \times n\) symmetric matrices. Let \(\Sigma\) be a hypomorphism such that \(B = \Sigma(A)\). Then there exists a subset \(S \subseteq [1, n]\) such that \(A = PDP^t\) and \(B = UD U^t\) as in Theorem 3. For \(i \in S\), we have \(p_i = u_i\) or \(p_i = -u_i.\) In particular, if \(\lambda_i\) is a simple eigenvalue of \(A\) and \((\text{eigen}_{\lambda_i}(A), 1) \neq 0\), then \(\text{eigen}_{\lambda_i}(A) = \text{eigen}_{\lambda_i}(B)\).

Proof: • By Tutte’s theorem, \(\text{eigen}(A) = \text{eigen}(B)\). Let \(A = PDP^t\) and \(B = UD U^t\). Since \(\text{det}(A + tJ - \lambda I) = \text{det}(B + tJ - \lambda I)\). By Lemma 1,
\[
\text{det}(A - \lambda I)(1 + \sum_{i} \frac{t(1,p_i)^2}{\lambda_i - \lambda}) = \text{det}(B - \lambda I)(1 + \sum_{i} \frac{t(1,u_i)^2}{\lambda_i - \lambda}).
\]
It follows that for every \(\lambda_i\), \(\sum_{\lambda_i = \lambda}(1,p_i)^2 = \sum_{\lambda_i = \lambda}(1,u_i)^2\). Consequently, the \(l\) for \(A\) is the same as the \(l\) for \(B\). Let \(S\) be as in Theorem 3 for both \(A\) and \(B\). Without loss of generality, suppose that \(A = PDP^t\) and \(B = UD U^t\) as in Theorem 3. In particular, for every \(i \in [1, n]\), we have
\[
(p_i, 1)^2 = (u_i, 1)^2. \tag{1}
\]

• Let \(T\) be as in Theorem 1 for \(A\) and \(B\). Without loss of generality, suppose \(T = (t_1, t_2) \subseteq \mathbb{R}^-.\) Let \(\mu_i(t)\) be the \(\mu_i\) in Theorem 3 for \(A\) and \(B\). Notice that the lowest eigenvectors of \(A + tJ\) and \(B + tJ\) are in \(\mathbb{R}^+\) and they are not perpendicular to \(1\). By Theorem 3, \(\mu_i(t) = \lambda_n(A + tJ) = \lambda_n(B + tJ)\). By Theorem 1,
\[
eigen_{\mu_i(t)} (A + tJ) = \text{eigen}_{\mu_i(t)} (B + tJ) \cong \mathbb{R}.
\]
So
\[
\sum_{i \in S} \frac{p_i}{\lambda_i - \mu_i(t)} (p_i, 1) = \sum_{i \in S} \frac{u_i}{\lambda_i - \mu_i(t)} (u_i, 1).
\]
Since \(\{p_i\}\) and \(\{u_i\}\) are orthogonal, by Equation 1,
\[
\| \sum_{i \in S} \frac{p_i}{\lambda_i - \mu_i(t)} (p_i, 1) \|^2 = \| \sum_{i \in S} \frac{u_i}{\lambda_i - \mu_i(t)} (u_i, 1) \|^2.
\]
It follows that for every \( t \in T \),
\[
\sum_{i \in S} p_i \frac{(p_i, 1)}{\lambda_i - \mu_i(t)} = \pm \sum_{i \in S} u_i \frac{(u_i, 1)}{\lambda_i - \mu_i(t)}.
\]

- Recall that \(-\frac{1}{t} = \sum_i \frac{q_i^2}{\lambda_i - \rho} \). Notice that the function \( \rho \rightarrow \sum_i \frac{q_i^2}{\lambda_i - \rho} \) is a continuous and one-to-one mapping from \((-\infty, \lambda_n)\) onto \((0, \infty)\). There exists a nonempty interval \( T_0 \subseteq (-\infty, \lambda_n) \) such that if \( \rho \in T_0 \), then \( \sum_i \frac{q_i^2}{\lambda_i - \rho} \in (-\frac{1}{t_1}, -\frac{1}{t_2}) \). So every \( \rho \in T_0 \) is a \( \mu_i(t) \) for some \( t \in (t_1, t_2) \). It follows that for every \( \rho \in T_0 \),
\[
\sum_{i \in S} p_i \frac{(p_i, 1)}{\lambda_i - \rho} = \pm \sum_{i \in S} u_i \frac{(u_i, 1)}{\lambda_i - \rho}.
\]

Notice that both vectors are nonzero and depend continuously on \( \rho \). Either,
\[
\sum_{i \in S} p_i \frac{(p_i, 1)}{\lambda_i - \rho} = \sum_{i \in S} u_i \frac{(u_i, 1)}{\lambda_i - \rho} \quad \forall (\rho \in T_0);
\]
or,
\[
\sum_{i \in S} p_i \frac{(p_i, 1)}{\lambda_i - \rho} = -\sum_{i \in S} u_i \frac{(u_i, 1)}{\lambda_i - \rho} \quad \forall (\rho \in T_0);
\]

- Notice that the functions \( \{ \rho \rightarrow \frac{1}{\lambda_i - \rho} \}_{i \in S} \) are linearly independent. For every \( i \in S \), we have
\[
p_i(p_i, 1) = \pm u_i(u_i, 1).
\]

Because \( p_i \) and \( u_i \) are both unit vectors, \( p_i = \pm u_i \). In particular, for every simple \( \lambda_i \) with \( (p_i, 1) \neq 0 \) we have \( \text{eigen}_{\lambda_i}(A) = \text{eigen}_{\lambda_i}(B) \). Q.E.D.

**Corollary 3** Let \( B \) and \( A \) be two real \( n \times n \) symmetric matrices. Let \( \Sigma \) be a hypomorphism such that \( B = \Sigma(A) \). Let \( \lambda_i \) be an eigenvalue of \( A \) such that \( (\text{eigen}_{\lambda_i}(A), 1) \neq 0 \). Then the orthogonal projection of \( 1 \) onto \( \text{eigen}_{\lambda_i}(A) \) equals the orthogonal projection of \( 1 \) onto \( \text{eigen}_{\lambda_i}(B) \).

**Proof:** Notice that the projections are \( p_i(p_i, 1) \) and \( u_i(u_i, 1) \). Whether \( p_i = u_i \) or \( p_i = -u_i \),
\[
p_i(p_i, 1) = u_i(u_i, 1).
\]

Q.E.D.

**Conjecture 3** Let \( A \) and \( B \) be two hypomorphic matrices. Let \( \lambda_i \) be a simple eigenvalue of \( A \). Then there exists a permutation matrix \( \tau \) such that \( \tau \text{eigen}_{\lambda_i}(A) = \text{eigen}_{\lambda_i}(B) \).

This conjecture is apparently true if \( \text{eigen}_{\lambda_i}(A) \) is not perpendicular to \( 1 \).

**References**


