Eigenvectors and Reconstruction

Hongyu He
Department of Mathematics & Statistics
Louisiana State University
email: hongyu@math.lsu.edu

Abstract

In this paper, we study the simple eigenvectors of two hypomorphic matrices using linear algebra. We also give a new proof of a result of Godsil-McKay.

1 Introduction

We start by fixing some notations. Let A be a $n \times n$ real symmetric matrix. Let A_i be the matrice obtaining by deleting the i-th row and i-th column of A. We say that two symmetric matrices A and B are hypomorphic if B_i can be obtained by permuting the rows and columns of A_i simultaneously. Let Σ be the set of permutations. We write $B = \Sigma(A)$.

If M is a symmetric real matrix, then the eigenvalues of M are real. We write

$$eigen(M) = (\lambda_1(M) \geq \lambda_2(M) \geq \ldots \geq \lambda_n(M)).$$

If α is an eigenvalue of M, we denote the corresponding eigenspace by $eigen_\alpha(M)$. Let $\mathbf{1}$ be the n-dimensional vector $(1, 1, \ldots, 1)$. Put $\mathbf{J} = \mathbf{1}^t\mathbf{1}$.

Theorem 1 ([HE1]) *Let B and A be two real $n \times n$ symmetric matrices. Let Σ be a hypomorphism such that $B = \Sigma(A)$. Let t be a real number. Then there exists an open interval T such that for $t \in T$ we have*

1. $\lambda_n(A + t\mathbf{J}) = \lambda_n(B + t\mathbf{J})$;
2. $eigen_{\lambda_n}(A + t\mathbf{J})$ *and* $eigen_{\lambda_n}(B + t\mathbf{J})$ *are both one dimensional;*
3. $eigen_{\lambda_n}(A + t\mathbf{J}) = eigen_{\lambda_n}(B + t\mathbf{J})$.

As proved in [HE1], our result implies Tutte's theorem which says that $eigen(A + tJ) = eigen(B + tJ)$.

In this paper, we shall study the eigenvectors of A and B. We first prove that the squares of the entries of simple unit eigenvectors of A can be reconstructed as functions of $eiegn(A)$ and $eigen(A_i)$. This yields a proof of a Theorem of Godsil-McKay. We also study how the eigenvectors of A change after a purturbation of a rank 1 symmetric matrices. Combined with Theorem 1, we prove another result of Godsil-McKay which states that the simple eigenvectors that are perpenticular to $\mathbf{1}$ are reconstructible. We further show that the orthogonal projection of $\mathbf{1}$ onto higher dimensional eigenspaces is reconstructible.

Our investigation indicates that the following conjecture could be true.

Conjecture 1 *Let A be a real $n \times n$ symmetric matrix. Then there exists a subgroup $G(A) \subseteq O(n)$ such that a real symmetric matrix B satisfies the properties that $eigen(B) = eigen(A)$ and $eigen(B_i) = eigen(A_i)$ for each i if and only if $B = UAU^t$ for some $U \in G(A)$.*

This conjecture is clearly true if $rank(A) = 1$. For $rank(A) = 1$, the group $G(A)$ can be chosen as $\mathbb{Z}_2^n$, all in the form of diagonal matrices. In some other cases, $G(A)$ can be a subgroup of the permutation group S_n.

Conjecture 2 *The group $G(A)$ can be chosen to be a twisted product of a subgroup of S_n with $\mathbb{Z}_2^n$.*

Clearly, this conjecture implies the reconstruction conjecture.

2 Reconstruction of Square Functions

Theorem 2 *Let A be a $n \times n$ real symmetric matrix. Let $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ be the eigenvalues of A. Suppose λ_i is a simple eigenvalue of A. Let $\mathbf{p}_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t$ be a unit vector in $eigen_{\lambda_i}(A)$. Then for every m, $p_{m,i}^2$ can be expressed as a function of $eigen(A)$ and $eigen(A_m)$.*

Proof: Let λ_i be a simple eigenvalue of A. Let $\mathbf{p}_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t$ be a unit vector in $eigen_{\lambda_i}(A)$. There exists an orthogonal matrix P such that $P = (\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n)$ and $A = PDP^t$ where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Then

$$A - \lambda_i I = PDP^t - \lambda_i I = P(D - \lambda_i I)P^t = \sum_{j \neq i} (\lambda_j - \lambda_i)\mathbf{p}_j \mathbf{p}_j^t.$$

which equals

$$\begin{pmatrix} p_{1,1} & \cdots & \widehat{p_{1,i}} & \cdots & p_{1,n} \\ p_{2,1} & \cdots & \widehat{p_{2,i}} & \cdots & p_{2,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{n,1} & \cdots & \widehat{p_{n,i}} & \cdots & p_{n,n} \end{pmatrix} \begin{pmatrix} \lambda_1 - \lambda_i & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \widehat{\lambda_i - \lambda_i} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \lambda_n - \lambda_i \end{pmatrix} \begin{pmatrix} p_{1,1} & p_{2,1} & \cdots & p_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{p_{1,i}} & \widehat{p_{2,i}} & \cdots & \widehat{p_{n,i}} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1,n} & p_{2,n} & \cdots & p_{n,n} \end{pmatrix}.$$

Deleting the m-th row and m-th column, we obtain

$$\begin{pmatrix} p_{1,1} & \cdots & \widehat{p_{1,i}} & \cdots & p_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{p_{m,1}} & \cdots & \widehat{p_{m,i}} & \cdots & \widehat{p_{m,n}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{n,1} & \cdots & \widehat{p_{n,i}} & \cdots & p_{n,n} \end{pmatrix} \begin{pmatrix} \lambda_1 - \lambda_i & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \widehat{\lambda_i - \lambda_i} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \lambda_n - \lambda_i \end{pmatrix} \begin{pmatrix} p_{1,1} & \cdots & \widehat{p_{m,1}} & \cdots & p_{n,1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{p_{1,i}} & \cdots & \widehat{p_{m,i}} & \cdots & \widehat{p_{n,i}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{1,n} & \cdots & \widehat{p_{m,n}} & \cdots & p_{n,n} \end{pmatrix}.$$

This is $A_m - \lambda_i I_{n-1}$. Notice that P is orthogonal. Taking the determinant, we have

$$\det(A_m - \lambda_i I_{n-1}) = p_{m,i}^2 \prod_{j \neq m} (\lambda_j - \lambda_i).$$

It follows that

$$p_{m,i}^2 = \frac{\prod_{j=1}^{n-1}(\lambda_j(A_m) - \lambda_i)}{\prod_{j\neq m}(\lambda_j - \lambda_i)}.$$

Q.E.D.

Corollary 1 *Let A and B be two $n \times n$ real symmetric matrices. Suppose that $eigen(A) = eigen(B)$ and $eigen(A_i) = eigen(B_i)$. Let λ_i be a simple eigenvalue of A and B. Let $\mathbf{p}_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t$ be a unit vector in $eigen_{\lambda_i}(A)$ and $\mathbf{q}_i = (q_{1,i}, q_{2,i}, \ldots, q_{n,i})^t$ be a unit vector in $eigen_{\lambda_i}(B)$. Then*

$$p_{j,i}^2 = q_{j,i}^2 \ \forall j \in [1, n].$$

Corollary 2 (Godsil-McKay, see Theorem 3.2, [GM]) *Let A and B be two $n \times n$ real symmetric matrices. Suppose that A and B are hypomorphic. Let λ_i be a simple eigenvalue of A and B. Let $\mathbf{p}_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t$ be a unit vector in $eigen_{\lambda_i}(A)$ and $\mathbf{q}_i = (q_{1,i}, q_{2,i}, \ldots, q_{n,i})^t$ be a unit vector in $eigen_{\lambda_i}(B)$. Then*

$$p_{j,i}^2 = q_{j,i}^2 \ \forall j \in [1, n].$$

3 Eigenvalues and Eigenvectors under the perturbation of a rank one symmetric matrix

Let A be a $n \times n$ real symmetric matrix. Let x be a n-dimensional row column vector. Let $M = xx^t$. Now consider $A + tM$. We have

$$A + tM = PDP^t + tM = P(D + tP^tMP)P^t = P(D + tP^txx^tP)P^t.$$

Let $P^tx = q$. So $q_i = (\mathbf{p}_i, x)$ for each $i \in [1, n]$. Then

$$A + t\mathbf{J} = P(D + tqq^t)P^t.$$

Put $D(t) = D + tqq^t$.

Lemma 1 $\det(D + tqq^t - \lambda I) = \det(A - \lambda I)(1 + \sum_i \frac{tq_i^2}{\lambda_i - \lambda})$.

Proof: $\det(D - \lambda I + tqq^t)$ can be written as a sum of products of $\lambda_i - \lambda$ and q_iq_j. For each S a subset of $[1, n]$, combine the terms containing only $\prod_{i\in S}(\lambda_i - \lambda)$. Since the rank of qq^t is one, only for $|S| = n, n-1$, the coefficients may be nonzero. We obtain

$$det(D + tqq^t - \lambda I) = \prod_{i=1}^{n}(\lambda_i - \lambda) + \sum_{i=1}^{n} tq_i^2 \prod_{j\neq i}(\lambda_i - \lambda).$$

The Lemma follows. □

Put $P_t(\lambda) = 1 + \sum_i \frac{tq_i^2}{\lambda_i - \lambda}$.

Lemma 2 *Fix $t < 0$. Suppose that for each i, λ_i is a simple eigenvalue and $q_i \neq 0$. Then $P_t(\lambda)$ has exactly n roots $(\mu_1, \mu_2, \cdots, \mu_n)$ satisfying a interlacing relation:*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \mu_{n-1} > \lambda_n > \mu_n.$$

Proof: Clearly, $\frac{dP_t(\lambda)}{dt} = \sum_i \frac{tq_i^2}{(\lambda_i - \lambda)^2} < 0$. So $P_t(\lambda)$ is always decreasing. On the interval $(-\infty, \lambda_n)$, $P_t(-\infty) = 1$ and $P_t(\lambda_n^-) = -\infty$. So $P_t(\lambda)$ has a unique root $\mu_n \in (-\infty, \lambda_n)$. Similar statement holds for each $(\lambda_{i-1}, \lambda_i)$. Q.E.D.

Theorem 3 *Fix $t < 0$. Let l be the number of distinct eigenvalues satisfying $(x, eigen_\lambda(A)) \neq 0$. Without loss of generalities, suppose that $A = PDP^t$ such that there exists a*

$$S = \{i_1 > i_2 > \cdots > i_l\}$$

satisfying $(x, \mathbf{p}_{i_j}) \neq 0$ and $(x, \mathbf{p}_i) = 0$ for every $i \notin S$. Then there exists $(\mu_1, \ldots, \mu_l)$ such that

$$\lambda_{i_1} > \mu_1 > \lambda_{i_2} > \mu_2 > \cdots > \lambda_{i_l} > \mu_l$$

and

$$eigen(A + tM) = \{\lambda_i(A) \mid i \notin S\} \cup \{\mu_1, \mu_2 \ldots, \mu_l\}.$$

Furthermore, $eigen_{\mu_j}(A + tM)$ contains

$$\sum_{i \in S} \mathbf{p}_i \frac{q_i}{\lambda_i - \mu_j}.$$

Here the index set $\{i_1, i_2, \cdots, i_l\}$ may not be unique. I shall also point out a similar statement holds for $t > 0$ with

$$\mu_1 > \lambda_{i_1} > \mu_2 > \lambda_{i_2} > \cdots > \mu_l > \lambda_{i_l}.$$

Proof: Since $(x, eigen_{\lambda_{i_j}}(A)) \neq 0$, $q_{i_j} \neq 0$. For $i \notin S$, $q_i = 0$. Notice

$$P_t(\lambda) = 1 + \sum_{j=1}^{l} \frac{t q_{i_j}^2}{\lambda_{i_j} - \lambda}.$$

Applying Lemma 2 to S, we obtain the roots of $P_t(\lambda)$ $\{\mu_1, \mu_2, \ldots, \mu_l\}$ satisfying

$$\lambda_{i_1} > \mu_1 > \lambda_{i_2} > \mu_2 > \cdots > \lambda_{i_l} > \mu_l.$$

It follows that the roots of $\det(A + tM - \lambda I) = \det(D(t) - \lambda I) = P_t(\lambda) \prod_{i=1}^n (\lambda_i - \lambda)$ can be obtained from $eigen(A)$ be changing $\{\lambda_{i_1} > \lambda_{i_2} > \cdots > \lambda_{i_l}\}$ to $\{\mu_1, \mu_2 \ldots, \mu_l\}$. Therefore,

$$eigen(A + tM) = \{\lambda_i(A) \mid i \notin S\} \cup \{\mu_1, \mu_2 \ldots, \mu_l\}.$$

For the sake of convenience, suppose that $\mu_i \notin eigen(A)$. Then

$$\sum_{i \in S} \frac{q_i}{\lambda_i - \mu_j} \mathbf{p}_i = \sum_{i=1}^{n} \frac{q_i}{\lambda_i - \mu_j} \mathbf{p}_i.$$

Here for $\lambda_i \notin S$, $q_i = 0$. Notice that

$$(A + tM) \sum_{i=1}^{n} \frac{q_i}{\lambda_i - \mu_j} \mathbf{p}_i = P(D + tqq^t) P^t \sum_{i=1}^{n} \frac{q_i}{\lambda_i - \mu_j} \mathbf{p}_i = P(D + tqq^t) \begin{pmatrix} \frac{q_1}{\lambda_1 - \mu_j} \\ \vdots \\ \frac{q_n}{\lambda_n - \mu_j} \end{pmatrix},$$

which equals

$$P\left(\begin{pmatrix} \frac{\lambda_1 q_1}{\lambda_1 - \mu_j} \\ \vdots \\ \frac{\lambda_n q_n}{\lambda_n - \mu_j} \end{pmatrix} + t \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \sum_{i=1}^{n} \frac{q_i^2}{\lambda_i - \mu_j} \right) = P\left(\begin{pmatrix} \frac{\lambda_1 q_1}{\lambda_1 - \mu_j} \\ \vdots \\ \frac{\lambda_n q_n}{\lambda_n - \mu_j} \end{pmatrix} - \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \right) = P \begin{pmatrix} \frac{\mu_j q_1}{\lambda_1 - \mu_j} \\ \vdots \\ \frac{\mu_j q_n}{\lambda_n - \mu_j} \end{pmatrix}.$$

We have obtained that

$$(A+tM)\sum_{i=1}^{n}\frac{q_i}{\lambda_i-\mu_j}\mathbf{p}_i=\mu_j P\begin{pmatrix}\frac{q_1}{\lambda_1-\mu_j}\\ \vdots\\ \frac{q_n}{\lambda_n-\mu_j}\end{pmatrix}=\mu_j\sum_{i=1}^{n}\frac{q_i}{\lambda_i-\mu_j}\mathbf{p}_i.$$

If $\mu_j \in eigen(A)$, we still have $(A+tM)\sum_{\lambda_i\in S}\frac{q_i}{\lambda_i-\mu_j}\mathbf{p}_i=\sum_{i\in S}\frac{q_i}{\lambda_i-\mu_j}\mathbf{p}_i$. Therefore,

$$\sum_{i\in S}\frac{q_i}{\lambda_i-\mu_j}\mathbf{p}_i \in eigen_{\mu_j}(A+tM).$$

Q.E.D.

4 Reconstruction of Simple Eigenvectors not perpenticular to 1

Now let $M=\mathbf{J}=\mathbf{1}\mathbf{1}^t$. Theorem 3 applies to $A+t\mathbf{J}$ and $B+t\mathbf{J}$.

Theorem 4 (Godsil-McKay, [GM]) *Let B and A be two real $n\times n$ symmetric matrices. Let Σ be a hypomorphism such that $B=\Sigma(A)$. Then there exists a subset $S\subseteq[1,n]$ such that $A=PDP^t$ and $B=UDU^t$ as in Theorem 3. For $i\in S$, we have $\mathbf{p}_i=\mathbf{u}_i$ or $\mathbf{p}_i=-\mathbf{u}_i$. In particular, if λ_i is a simple eigenvalue of A and $(eigen_{\lambda_i}(A),\mathbf{1})\neq 0$, then $eigen_{\lambda_i}(A)=eigen_{\lambda_i}(B)$.*

Proof: • By Tutte's theorem, $eigen(A)=eigen(B)$. Let $A=PDP^t$ and $B=UDU^t$. Since $\det(A+t\mathbf{J}-\lambda I)=\det(B+t\mathbf{J}-\lambda I)$. By Lemma 1,

$$\det(A-\lambda I)(1+\sum_i\frac{t(\mathbf{1},\mathbf{p}_i)^2}{\lambda_i-\lambda})=\det(B-\lambda I)(1+\sum_i\frac{t(\mathbf{1},\mathbf{u}_i)^2}{\lambda_i-\lambda}).$$

It follows that for every λ_i, $\sum_{\lambda_j=\lambda_i}(\mathbf{1},\mathbf{p}_j)^2=\sum_{\lambda_j=\lambda_i}(\mathbf{1},\mathbf{u}_j)^2$. Consequently, the l for A is the same as the l for B. Let S be as in Theorem 3 for both A and B. Without loss of generality, suppose that $A=PDP^t$ and $B=UDU^t$ as in Theorem 3. In particular, for every $i\in[1,n]$, we have

$$(\mathbf{p}_i,\mathbf{1})^2=(\mathbf{u}_i,\mathbf{1})^2. \quad (1)$$

• Let T be as in Theorem 1 for A and B. Without loss of generality, suppose $T=(t_1,t_2)\subseteq\mathbb{R}^-$. Let $\mu_l(t)$ be the μ_l in Theorem 3 for A and B. Notice that the lowest eigenvectors of $A+t\mathbf{J}$ and $B+t\mathbf{J}$ are in ${\mathbb{R}^+}^n$ and they are not perpenticular to $\mathbf{1}$. By Theorem 3, $\mu_l(t)=\lambda_n(A+t\mathbf{J})=\lambda_n(B+t\mathbf{J})$. By Theorem 1,

$$eigen_{\mu_1(t)}(A+t\mathbf{J})=eigen_{\mu_l(t)}(B+t\mathbf{J})\cong\mathbb{R}.$$

So

$$\sum_{i\in S}\mathbf{p}_i\frac{(\mathbf{p}_i,\mathbf{1})}{\lambda_i-\mu_l(t)}//\sum_{i\in S}\mathbf{u}_i\frac{(\mathbf{u}_i,\mathbf{1})}{\lambda_i-\mu_l(t)}.$$

Since $\{\mathbf{p}_i\}$ and $\{\mathbf{u}_i\}$ are orthogonal, by Equation 1,

$$\|\sum_{i\in S}\mathbf{p}_i\frac{(\mathbf{p}_i,\mathbf{1})}{\lambda_i-\mu_l(t)}\|^2=\|\sum_{i\in S}\mathbf{u}_i\frac{(\mathbf{u}_i,\mathbf{1})}{\lambda_i-\mu_l(t)}\|^2.$$

It follows that for every $t \in T$,

$$\sum_{i\in S} \mathbf{p}_i \frac{(\mathbf{p}_i, \mathbf{1})}{\lambda_i - \mu_l(t)} = \pm \sum_{i\in S} \mathbf{u}_i \frac{(\mathbf{u}_i, \mathbf{1})}{\lambda_i - \mu_l(t)}.$$

• Recall that $-\frac{1}{t} = \sum_i \frac{q_i^2}{\lambda_i - \mu_l(t)}$. Notice that the function $\rho \to \sum_i \frac{q_i^2}{\lambda_i - \rho}$ is a continuous and one-to-one mapping from $(-\infty, \lambda_n)$ onto $(0, \infty)$. There exists a nonempty interval $T_0 \subseteq (-\infty, \lambda_n)$ such that if $\rho \in T_0$, then $\sum_i \frac{q_i^2}{\lambda_i - \rho} \in (-\frac{1}{t_1}, -\frac{1}{t_2})$. So every $\rho \in T_0$ is a $\mu_l(t)$ for some $t \in (t_1, t_2)$. It follow that for every $\rho \in T_0$,

$$\sum_{i\in S} \mathbf{p}_i \frac{(\mathbf{p}_i, \mathbf{1})}{\lambda_i - \rho} = \pm \sum_{i\in S} \mathbf{u}_i \frac{(\mathbf{u}_i, \mathbf{1})}{\lambda_i - \rho}.$$

Notice that both vectors are nonzero and depend continuously on ρ. Either,

$$\sum_{i\in S} \mathbf{p}_i \frac{(\mathbf{p}_i, \mathbf{1})}{\lambda_i - \rho} = \sum_{i\in S} \mathbf{u}_i \frac{(\mathbf{u}_i, \mathbf{1})}{\lambda_i - \rho} \qquad \forall\ (\rho \in T_0);$$

or,

$$\sum_{i\in S} \mathbf{p}_i \frac{(\mathbf{p}_i, \mathbf{1})}{\lambda_i - \rho} = -\sum_{i\in S} \mathbf{u}_i \frac{(\mathbf{u}_i, \mathbf{1})}{\lambda_i - \rho} \qquad \forall\ (\rho \in T_0);$$

•. Notice that the functions $\{\rho \to \frac{1}{\lambda_{i_j} - \rho}\}|_{i_j \in S}$ are linearly independent. For every $i \in S$, we have

$$\mathbf{p}_i(\mathbf{p}_i, \mathbf{1}) = \pm \mathbf{u}_i(\mathbf{u}_i, \mathbf{1}).$$

Because $\mathbf{p}_i$ and $\mathbf{u}_i$ are both unit vectors, $\mathbf{p}_i = \pm \mathbf{u}_i$. In particular, for every simple λ_i with $(\mathbf{p}_i, \mathbf{1}) \neq 0$ we have $eigen_{\lambda_i}(A) = eigen_{\lambda_i}(B)$. Q.E.D.

Corollary 3 *Let B and A be two real $n \times n$ symmetric matrices. Let Σ be a hypomorphism such that $B = \Sigma(A)$. Let λ_i be an eigenvalue of A such that $(eigen_{\lambda_i}(A), \mathbf{1}) \neq 0$. Then the orthogonal projection of $\mathbf{1}$ onto $eigen_{\lambda_i}(A)$ equals the orthogonal projection of $\mathbf{1}$ onto $eigen_{\lambda_i}(B)$.*

Proof: Notice that the projections are $\mathbf{p}_i(\mathbf{p}_i, \mathbf{1})$ and $\mathbf{u}_i(\mathbf{u}_i, \mathbf{1})$. Whether $\mathbf{p}_i = \mathbf{u}_i$ or $\mathbf{p}_i = -\mathbf{u}_i$,

$$\mathbf{p}_i(\mathbf{p}_i, \mathbf{1}) = \mathbf{u}_i(\mathbf{u}_i, \mathbf{1}).$$

Q.E.D.

Conjecture 3 *Let A and B be two hypomorphic matrices. Let λ_i be a simple eigenvalue of A. Then there exists a permutation matrix τ such that $\tau eigen_{\lambda_i}(A) = eigen_{\lambda_i}(B)$.*

This conjecture is apparently true if $eigen_{\lambda_i}(A)$ is not perpenticular to $\mathbf{1}$.

References

[Tutte] W. T. Tutte, "All the King's Horses (A Guide to Reconstruction)", *Graph Theory and Related Topics*, Academic Press, 1979, (15-33).

[GM] C. D. Godsil and B. D. McKay, "Spectral Conditions for the Reconstructiblity of a graph", *J. Combin. Theory Ser. B* 30. 1981, No. 3, (285-289).

[HE1] H. He, "Reconstruction and Higher Dimensional Geometry", to appear in *Journal of Combinatorial Theory, Series B.*

[Ko] W. Kocay, "Some New Methods in Reconstruction Theory", *Combinatorial mathematics, IX (Brisbane, 1981)*, LNM 952, (89-114).