Reconstruction and Higher Dimensional Geometry

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Abstract

Tutte proved that, if two graphs, both with more than two vertices, have the same collection of vertex-deleted subgraphs, then the determinants of the two corresponding adjacency matrices are the same. In this paper, we give a geometric proof of Tutte's theorem using vectors and angles. We further study the lowest eigenspaces of these adjacency matrices.

1 Introduction

Given the graph $G = \{V, E\}$, let G_i be the graph obtained by deleting the $i-$th vertex v_i. Fix $n \geq 3$ from now on. Let G and H be two graphs of n vertices. The main conjecture in reconstruction theory, states that if G_i is isomorphic to H_i for every i, then G and H are isomorphic (up to a reordering of V). This conjecture is also known as the Ulam's conjecture.

The reconstruction conjecture can be formulated in purely algebraic terms. Consider two $n \times n$ real symmetric matrices A and B. Let A_i and B_i be the matrices obtaining by deleting the i-th row and i-th column of A and B, respectively.

Definition 1 *Let σ_i be a $n-1$ by $n-1$ permutation matrix. Let A and B be two $n \times n$ real symmetric matrices. We say that A and B are hypomorphic if there exists a set of $n-1 \times n-1$ permutation matrices*

$$\{\sigma_1, \sigma_2, \ldots, \sigma_n\},$$

such that $B_i = \sigma_i A_i \sigma_i^t$ for every i. Put $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. We write $B = \Sigma(A)$. Σ is called a hypomorphism.

The algebraic version of the reconstruction conjecture can be stated as follows.

Conjecture 1 *Let A and B be two $n \times n$ symmetric matrices. If there exists a hypomorphism Σ such that $B = \Sigma(A)$, then there exists a $n \times n$ permutation matrix τ such that $B = \tau A \tau^t$.*

We start by fixing some notations. If M is a symmetric real matrix, then the eigenvalues of M are real. We write

$$eigen(M) = (\lambda_1(M) \geq \lambda_2(M) \geq \ldots \geq \lambda_n(M)).$$

If α is an eigenvalue of M, we denote the corresponding eigenspace by $eigen_\alpha(M)$. Let $\mathbf{1}_n$ be the n-dimensional row vector $(1, 1, \ldots, 1)$. We may drop the subscript n if it is implicit. Put $J = \mathbf{1}^t\mathbf{1}$. If A and B are hypomorphic, so are $A + tJ$ and $B + tJ$.

Theorem 1 (Tutte) *Let B and A be two real $n \times n$ symmetric matrices. If B and A are hypomorphic then* $\det(B - \lambda I + tJ) = \det(A - \lambda I + tJ)$ *for all* $t, \lambda \in \mathbb{R}$.

In this paper, we will study the geometry related to Conjecture 1. Out main result can be stated as follows.

Theorem 2 (Main Theorem) *Let B and A be two real $n \times n$ symmetric matrices. Let Σ be a hypomorphism such that $B = \Sigma(A)$. Let t be a real number. Then there exists an open interval T such that for $t \in T$ we have*

1. $\lambda_n(A + tJ) = \lambda_n(B + tJ)$;
2. *eigen*$_{\lambda_n}(A + tJ)$ *and eigen*$_{\lambda_n}(B + tJ)$ *are both one dimensional;*
3. *eigen*$_{\lambda_n}(A + tJ) =$ *eigen*$_{\lambda_n}(B + tJ)$.

A similar statement holds for the highest eigenspaces.

Since the sets of majors of $A + tJ$ and of $B + tJ$ are the same, for every $t \in T$ and $\lambda \in \mathbb{R}$,

$$\det(A + tJ - \lambda I) - \det(B + tJ - \lambda I) = \det(A + tJ) - \det(B + tJ). \tag{1}$$

If $t \in T$, by taking $\lambda = \lambda_n(A + tJ)$, we obtain

$$\det(A + tJ) - \det(B + tJ) = \det(A + tJ - \lambda I) - \det(B + tJ - \lambda I) = 0.$$

Since the above statement is true for $t \in T$, $\det(A + tJ) = \det(B + tJ)$ for every t. By Equation. 1, we obtain $\det(B - \lambda I + tJ) = \det(A - \lambda I + tJ)$ for all $t, \lambda \in \mathbb{R}$. This is Tutte's theorem, which was proved using rank polynomials and Hamiltonian circuits. I should also mention that Kocay [1] found a simpler way to deduce the reconstructibility of characteristic polynomials.

Here is the content of this paper. We begin by presenting a positive semidefinite matrix $A + \lambda I$ by n vectors in $\mathbb{R}^n$. We then interpret the reconstruction conjecture as a generalization of a congruence theorem in Eulidean geometry. Next we study the presentations of $A + \lambda I$ under the perturbation by tJ. We define a norm of angles in higher dimensions and establish a comparison theorem. Our comparison theorem then forces hypomorphic matrices to have the same lowest eigenvalue and eigenvector.

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2 Notations

Unless stated otherwise,

1. all linear spaces in this paper will be finite dimensional real Euclidean spaces;
2. all linear subspaces will be equipped with the induced Euclidean metric;
3. all vectors will be column vectors;
4. vectors are sometimes regarded as points in $\mathbb{R}^n$.

Let $U = \{u_1, u_2, \dots u_m\}$ be an ordered set of m vectors in $\mathbb{R}^n$. U is also interpreted as a $n \times m$ matrix.

1. Let conv U be the convex hull spanned by U, namely,

$$\{\sum_{i=1}^{m} \alpha_i u_i \mid \alpha_i \geq 0, \sum_{i=1}^{m} \alpha_i = 1\}.$$

2. Let aff U be the affine space spanned by U, namely,

$$\{\sum_{i=1}^{m} \alpha_i u_i \mid \sum_{i=1}^{m} \alpha_i = 1\}.$$

3. Let span U be the linear span of U, namely,

$$\{\sum_{i=1}^{m} \alpha_i u_i \mid \alpha_i \in \mathbb{R}\}.$$

Then conv $U \subset$ aff $U \subset$ span U.

Let A be a matrix. We denote the (i,j)-th entry of A by a_{ij}. We denote the transpose of A by A^t. Let $\mathbb{R}^{+n}$ be the set of vectors with only positive coordinates.

3 Geometric Interpretation

Fix a standard Euclidean space $(\mathbb{R}^n, (,))$.

Definition 2 *Let A be a symmetric positive semidefinite real matrix. An ordered set of vectors $V = \{v_1, v_2, \dots v_n\}$ is said to be a presentation of A if and only if $(v_i, v_j) = a_{ij}$.*

Regarding v_i as column vectors and V as a $n \times n$ matrix, V is a presentation of A if and only if $V^t V = A$. Every positive semidefinite real matrix A has a presentation. In addition, the presentation V is unique up to a left multiplication by an orthogonal matrix.

Definition 3 *Let S and T be two sets of vectors in $\mathbb{R}^n$. S and T are said to be congruent if there exists an orthogonal linear transformation in $\mathbb{R}^n$ that maps S onto T.*

So $A = \sigma B \sigma^t$ for some permutation σ if and only if A and B are presented by two congruent subsets in $\mathbb{R}^n$.

Now consider two hypomorphic matrices $B = \Sigma(A)$. Observe that $B + \lambda I = \Sigma(A + \lambda I)$. Without loss of generality, assume A and B are both positive semidefinite. Let U and V be their presentations respectively. Since $B_i = \sigma_i A_i \sigma_i^t$, $U - \{u_i\}$ is congruent to $V - \{v_i\}$. Then the reconstruction conjecture can be stated as follows.

Conjecture 2 (Geometric reconstruction) *Let*

$$S = \{u_1, u_2, \dots, u_n\}$$

and

$$T = \{v_1, v_2, \dots v_n\}$$

be two finite sets of vectors in $\mathbb{R}^m$. Assume that $S - \{u_i\}$ is congruent to $T - \{v_i\}$ for every i. Then S and T are congruent.

Generically, $m = n$.

Definition 4 *We say that $U = \{u_i\}_1^n$ is in good position if the point 0 is in the interior of the convex hull of U and the convex hull of U is of dimension $n-1$.*

Lemma 1 *Let A be a symmetric positive semidefinite matrix. The following are equivalent.*

1. *A has a presentation in good position.*

2. *Every presentation of A is in good position.*
3. $rank(A) = n-1$ *and* $eigen_0(A) = \mathbb{R}\alpha$ *for some* $\alpha \in (\mathbb{R}^+)^n$.

Proof: Since A is symmetric positive semidefinite, A has a presentation. Let U be a presentation of A.

If U is in good position, then every presentation obtained from an orthogonal linear transformation is also in good position. Since a presentation is unique up to an orthogonal linear transformation, (1) $\leftrightarrow$ (2).

Suppose U is in good position. Then $rank(U) = n-1$. So $rank(A) = n-1$. Since 0 is in the interior of the convex hull of U, there exists $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)^t$ such that

$$0 = \sum_1^n \alpha_i u_i; \qquad \sum_1^n \alpha_i = 1; \qquad \alpha_i > 0 \ \forall\ i.$$

Since $rank(U) = n-1$, α is unique. Now $U\alpha = 0$ implies

$$A\alpha = U^t U \alpha = U^t 0 = 0.$$

Since $rank(A) = rank(U) = n-1$, $eigen_0(A) = \mathbb{R}\alpha$. So (2) $\rightarrow$ (3).

Conversely, suppose $rank(A) = n-1$ and $eigen_0(A) = \mathbb{R}\alpha$ with $\alpha \in \mathbb{R^+}^n$. Then $\sum_i \alpha_i u_i = 0$ and the linear span span U is of dimension $n-1$. Thus, 0 is in conv U. It follows that aff U = span U. So dim(conv U) = dim(aff U) = dim(span U) $= n-1$. So (3) $\rightarrow$ (1). Q.E.D.

Lemma 2 *Let U be a presentation of A. Suppose that U is in good position. Let α_i be the volume of the convex hull of $\{0, u_1, u_2, \ldots, \hat{u_i}, \ldots, u_n\}$. Then*

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^t$$

is a lowest eigenvector.

The proof can be found in many places. For the sake of completeness, I will give a proof using the language of exterior product.

Proof: Choosing an orthonormal basis properly, we may assume that every $u_i \in \mathbb{R}^{n-1}$. U becomes a $(n-1) \times n$ matrix. Let $x_1, x_2 \ldots x_{n-1}$ be the row vectors of U. Consider the exterior product

$$x_1 \wedge x_2 \wedge \ldots \wedge x_{n-1}.$$

Let β_i be the i-th coordinate in terms of the standard basis

$$\{(-1)^{i-1} e_1 \wedge e_2 \wedge \ldots \wedge \hat{e_i} \wedge \ldots \wedge e_n \mid i \in [1, n]\}.$$

Put $\beta = (\beta_1, \beta_2, \ldots, \beta_n)^t$. Notice that $x_i \wedge (x_1 \wedge x_2 \wedge \ldots \wedge x_{n-1}) = 0$ for $1 \leq i \leq n-1$. Therefore, $(x_i, \beta) = 0$ for every i. So $U\beta = 0$. It follows that $\sum_{i=1}^n \beta_i u_i = 0$. Since 0 is in the convex hull of $\{u_i\}_1^n$, β_i must be either all negative or all positive. Clearly,

$$|\beta_i| = |u_1 \wedge \ldots \wedge \hat{u_i} \wedge \ldots \wedge u_n| = (n-1)! \alpha_i.$$

Therefore, we have $U\alpha = 0$. Then $A\alpha = U^t U \alpha = 0$. α is a lowest eigenvector. Q.E.D.

Theorem 3 *Suppose that $B = \Sigma(A)$. Suppose that A and B have presentations in good position. Then $eigen_0(A) = eigen_0(B) \cong \mathbb{R}$.*

Proof: Let U and V be presentations of A and B respectively. Then U and V are in good position. Notice that the volume of the convex hull of

$$\{0, u_1, u_2, \ldots, \hat{u_i}, \ldots u_n\}$$

equals the volume of the convex hull of

$$\{0, v_1, v_2, \ldots, \hat{v_i}, \ldots v_n\}$$

By Lemma 2 and Lemma 1, $eigen_0(A) = eigen_0(B) \cong \mathbb{R}$. So the lowest eigenspace of A is equal to the lowest eigenspace of B. Q.E.D.

4 Perturbation by J

Recall that $J = \mathbf{1}_n^t \mathbf{1}_n$. We know that $B = \Sigma(A)$ if and only if $B + tJ = \Sigma(A + tJ)$. Let us see how presentations of $A + tJ$ depend on t. Let A be a positive definite matrix. Let $U = \{u_i\}_1^n$ be a presentation of A.

Let aff U be the affine space spanned by U. Then $\{u_i\}$ are affinely independent. Let u_0 be the orthogonal projection of the origin onto aff U. Then $(u_0, u_i - u_0) = 0$ for every i. We obtain

$$U^t u_0 = \|u_0\|^2 \mathbf{1}.$$

It follows that $u_0 = \|u_0\|^2 (U^t)^{-1} \mathbf{1}$. Consequently,

$$\|u_0\|^2 = (u_0, u_0) = \|u_0\|^4 \mathbf{1}^t U^{-1} (U^t)^{-1} \mathbf{1} = \|u_0\|^4 \mathbf{1}^t A^{-1} \mathbf{1}.$$

Clearly, $\|u_0\|^2 = \frac{1}{\mathbf{1}^t A^{-1} \mathbf{1}}$. We obtain the following lemma.

Lemma 3 *Let A be a positive definite matrix. Let $U = \{u_i\}_1^n$ be a presentation of A. Let u_0 be the orthogonal projection of the origin onto* aff U. *Then $\|u_0\|^2 = \frac{1}{\mathbf{1}^t A^{-1} \mathbf{1}}$ and*

$$u_0 = \frac{1}{\mathbf{1}^t A^{-1} \mathbf{1}} (U^t)^{-1} \mathbf{1}.$$

Consider $\{u_i - s u_0\}_1^n$. Notice that

$$(u_i - su_0, u_j - su_0) = (u_i - u_0 + (1-s)u_0, u_j - u_0 + (1-s)u_0) = (u_i - u_0, u_j - u_0) + (1-s)^2 (u_0, u_0).$$

Taking $s = 0$, we have

$$(u_i, u_j) = (u_i - u_0, u_j - u_0) + (u_0, u_0).$$

Therefore

$$(u_i - su_0, u_j - su_0) = (u_i, u_j) - (u_0, u_0) + (1-s)^2 (u_0, u_0) = (u_i, u_j) + (s^2 - 2s)\|u_0\|^2.$$

We see clearly that $A + (s^2 - 2s)\|u_0\|^2 J$ is presented by $\{u_i - su_0\}_1^n$. Observe that

$$\operatorname{span}(u_1 - su_0, u_2 - su_0, \ldots, u_n - su_0)$$

is of dimension n for all $s \neq 1$. So $A + (s^2 - 2s)\|u_0\|^2 J$ is positive definite for all $s \neq 1$. If $s = 1$, we see that $A - \|u_0\|^2 J$ is presented by $\{u_i - u_0\}_1^n$ whose linear span is of dimension $n - 1$. We obtain the following lemma.

Lemma 4 *Let A be a symmetric positive definite matrix. Let U be a presentation of A. Let u_0 be the orthogonal projection of the origin onto* aff U. *Then $\{u_i - su_0\}_1^n$ is a presentation of $A + (s^2 - 2s)\|u_0\|^2 J$. Let $t = (s^2 - 2s)\|u_0\|^2$. Then $A + tJ$ is positive definite for all $t > -\|u_0\|^2$ and positive semidefinite for $t = -\|u_0\|^2$.*

Notice that

$$u_0 = \frac{1}{\mathbf{1}^t A^{-1}\mathbf{1}}(U^t)^{-1}\mathbf{1} = \frac{1}{\mathbf{1}^t A^{-1}\mathbf{1}}U(U^{-1}(U^t)^{-1})\mathbf{1} = \frac{1}{\mathbf{1}^t A^{-1}\mathbf{1}}UA^{-1}\mathbf{1}.$$

Theorem 4 *Let A be a symmetric positive definite matrix. Let U be a presentation of A. Let u_0 be the orthogonal projection of the origin onto* aff U. *Then $u_0 = \frac{1}{\mathbf{1}^t A^{-1}\mathbf{1}}UA^{-1}\mathbf{1}$ and the following are equivalent.*

1. *$A - \|u_0\|^2 J$ has a presentation in good position;*
2. *u_0 is in the interior of* conv U*;*
3. *$A^{-1}\mathbf{1} \in \mathbb{R}^{+n}$.*

Corollary 1 *Let A be a real symmetric matrix. There exists λ_0 such that for every $\lambda \geq \lambda_0$ there exists a real number t such that $A + \lambda I + tJ$ has a presentation in good position.*

Proof: Instead, consider $I + sA$ with $s = \frac{1}{\lambda}$. $I + sA$ is related to $A + \lambda I$ by a constant multiplication:

$$\lambda(I + sA) = \lambda I + A.$$

Let $s_0 = \frac{1}{\|A\|+1}$ where $\|A\|$ denote the operator norm. Suppose that $0 \leq s \leq s_0$. Then $I + sA$ is positive definite. For $s = 0$, $(I + sA)^{-1}\mathbf{1} \in \mathbb{R}^{+n}$. Since

$$s \to (I + sA)^{-1}\mathbf{1}$$

is continuous on $(0, s_0)$, there exists a $s_1 \in (0, s_0)$ such that $(I + sA)^{-1}\mathbf{1} \in \mathbb{R}^{+n}$ for every $s \in (0, s_1]$. So for every $\lambda \in [\frac{1}{s_1}, \infty)$, $(A + \lambda I)^{-1}\mathbf{1} = \lambda^{-1}(sA + I)^{-1}\mathbf{1} \in \mathbb{R}^{+n}$. Let $\lambda_0 = \frac{1}{s_1}$. So for every $\lambda \geq \lambda_0$, $(A + \lambda I)^{-1}\mathbf{1} \in \mathbb{R}^{+n}$. By Theorem 4, for every $\lambda \geq \lambda_0$ there exists a t such that $A + \lambda I + tJ$ has a presentation in good position. Q.E.D.

5 Higher Dimensional Angle and Comparison Theorem

Definition 5 *Let $U = \{u_1, u_2, \ldots u_n\}$ be a subset in $\mathbb{R}^n$. $\mathbb{R}^n$ may be contained in some other Euclidean space. Let u be a point in $\mathbb{R}^n$. The angle $\angle(u, U)$ is defined to be the region*

$$\{\sum_1^n \alpha_i(u_i - u) \mid \alpha_i \geq 0\}.$$

Two angles are congruent if there exists an isometry that maps one angle to the other. Let $\mathcal{B}$ be the unit ball in $\mathbb{R}^n$. The norm of $\angle(u, U)$ is defined to be the volume of $\angle(u, U) \cap \mathcal{B}$, denote it by $|\angle(u, U)|$.

Let me make a few remarks.

1. Firstly, if two angles are congruent, their norms are the same. But, unlike the 2 dimensional case, if the norms of two angles are the same, these two angles may not be congruent.
2. Secondly, if $\{u_i - u\}_1^n$ are linearly dependent, then $|\angle(u, U)| = 0$. If u happens to be in aff U, then $|\angle(u, U)| = 0$.

3. According to our definition, $|\angle(u, U)|$ is always less than half of the volume of $\mathcal{B}$.
4. More generally, one can allow $\{\alpha_i\}_1^n$ to be in a collection of other sign patterns which correspond to quadrants in two dimensional case. Then the norm of an angle can be greater than half of the volume of $\mathcal{B}$.

Lemma 5 *If $\angle(u, U) \subseteq \angle(u, V)$, then $|\angle(u, U)| \leq |\angle(u, V)|$. If $|\angle(u, U)| > 0$ and $\angle(u, U)$ is a proper subset of $\angle(u, V)$ then $|\angle(u, U)| < |\angle(u, V)|$.*

Theorem 5 (Comparison Theorem) *Let $\angle(u, U)$ be an angle and $|\angle(u, U)| \neq 0$. Suppose that v is contained in the interior of the convex hull of $\{u\} \cup U$. Then $|\angle(u, U)| < |\angle(v, U)|$.*

Proof: Without loss of generality, assume $u = 0$. Suppose $|\angle(0, U)| > 0$. Let $U = \{u_1, u_2, \ldots u_n\}$. Then U is linearly independent. Since v is in the interior of $\text{conv}(0, U)$, v can be written as

$$\sum_{i=1}^{n} \alpha_i u_i$$

with $\alpha_i \in \mathbb{R}^+$ and $\sum_i^n \alpha_i < 1$.

Let $U' = \{u_i - v\}_1^n$. It suffices to prove that $\angle(0, U)$ is a proper subset of $\angle(0, U')$. Let x be a point in $\angle(0, U)$ with $x \neq 0$. Then $x = \sum_i x_i u_i$ for some $x_i \geq 0$ with $\sum_i x_i > 0$. Define for each i

$$y_i = x_i + \alpha_i \frac{\sum x_i}{1 - \sum \alpha_i}$$

The reader can easily verify that $\sum y_i(u_i - v) = x$. Observe that $y_i > x_i \geq 0$. So $\angle(0, U)$ is a proper subset of $\angle(0, U')$. It follows that

$$|\angle(0, U)| < |\angle(0, U')| = |\angle(v, U)|.$$

Q.E.D.

Theorem 6 *Let $U = \{u_1, u_2, \ldots, u_n\} \subset \mathbb{R}^m$ for some $m \geq n$. Suppose that $|\angle(u, U)| > 0$. Suppose that the orthogonal projection of u onto* aff U *is in the interior of* conv U. *Let v be a vector such that $(u - v, u - u_i) = 0$ for every i. If $u \neq v$ then $|\angle(u, U)| > |\angle(v, U)|$.*

Proof: Without loss of generality, assume that $u = 0$. Then U is linearly independent and $v \perp u_i$ for every i. Let u_0 be the orthogonal projection of u onto aff U. By our assumption, u_0 is in the interior of conv U and $\|u_0\| \neq 0$. Let

$$v' = (1 - \sqrt{\frac{\|v\|^2}{\|u_0\|^2} + 1})u_0.$$

Then

$$\|v' - u_0\|^2 = (\frac{\|v\|^2}{\|u_0\|^2} + 1)\|u_0\|^2 = \|v\|^2 + \|u_0\|^2.$$

Notice that $v \perp u_i$ and $u_0 \perp u_i - u_0$. We obtain

$$\begin{aligned}(u_i - v', u_j - v') =& (u_i - u_0 + \sqrt{\frac{\|v\|^2}{\|u_0\|^2} + 1} u_0, u_j - u_0 + \sqrt{\frac{\|v\|^2}{\|u_0\|^2} + 1} u_0) \\ =& (u_i - u_0, u_j - u_0) + (\frac{\|v\|^2}{\|u_0\|^2} + 1)\|u_0\|^2 \\ =& (u_i - u_0, u_j - u_0) + (u_0, u_0) + (v, v) \\ =& (u_i, u_j) + (v, v) \\ =& (u_i - v, u_j - v).\end{aligned} \tag{2}$$

Hence $\angle(v, U) \cong \angle(v', U)$. Notice that $1 - \sqrt{\frac{\|v\|^2}{\|u_0\|^2} + 1} < 0$. So the origin sits between v' and u_0 which is in the interior of conv U. Therefore, 0 is in the interior of $\angle(v', U)$. By the Comparison Theorem, $|\angle(v', U)| > |\angle(0, U)|$. Consequently, $|\angle(v, U)| > |\angle(0, U)|$. Q.E.D.

Theorem 7 *Suppose that $B = \Sigma(A)$. Let λ_0 be as in Cor. 1 for both B and A. Fix $\lambda \geq \lambda_0$. Let t_1 and t_2 be two real numbers such that $A + \lambda I + t_1 J$ and $B + \lambda I + t_2 J$ have presentations in good position. Then $t_1 = t_2$.*

Proof: We prove by contradiction. Without loss of generality, suppose that $t_1 > t_2$. Let U be a presentation of $A + \lambda I + t_1 J$. Then U is in good position. So 0 is in the interior of conv U. Let V be a representation of $B + \lambda I + t_2 J$. Then V is in good position. So 0 is in the interior of conv V and dim(span V) $= n - 1$. Let $v_0 \perp$ span V and $\|v_0\|^2 = t_1 - t_2$. Let $V' = \{v_i + v_0\}_1^n$. Clearly, V' is a presentation of $B + \lambda I + t_1 J$.

By Thm. 6, for every i,

$$|\angle(0, V\backslash\{v_i\})| > |\angle(-v_0, V\backslash\{v_i\})| = |\angle(0, V'\backslash\{v_i + v_0\})|.$$

Since $B + \lambda I + t_1 J = \Sigma(A + \lambda I + t_1 J)$, $V'\backslash\{v_i + v_0\}$ is congruent to $U\backslash\{u_i\}$ for every i. Therefore $|\angle(0, V'\backslash\{v_i + v_0\})| = |\angle(0, U\backslash\{u_i\})|$. Since 0 is in the interiors of the convex hulls of U and of V, we have

$$Vol(\mathcal{B}) = \sum_{i=1}^n |\angle(0, V\backslash\{v_i\})| > \sum_{i=1}^n |\angle(0, V'\backslash\{v_i + v_0\})| = \sum_{i=1}^n |\angle(0, U\backslash\{u_i\})| = Vol(\mathcal{B}).$$

This is a contradiction. Therefore, $t_1 = t_2$. Q.E.D.

6 Proof of the Main Theorem

Suppose $B = \Sigma(A)$. Suppose λ_0 satisfies Cor. 1 for both A and B. So for every $\lambda \geq \lambda_0$ there exist real numbers t_1 and t_2 such that $A + \lambda I + t_1 J$ has a presentation in good position and $B + \lambda I + t_2 J$ has a presentation in good position. By Theorem 7, $t_1 = t_2$. Because of the dependence on λ, put $t(\lambda) = t_1 = t_2$. By Theorem 3,

$$eigen_0(A + \lambda I + t(\lambda)J) = eigen_0(B + \lambda I + t(\lambda)J) \cong \mathbb{R}.$$

Since 0 is the lowest eigenvalue of $A + \lambda I + t(\lambda)J$ and $B + \lambda I + t(\lambda)J$, λ is the lowest eigenvalue of $A + t(\lambda)J$ and $B + t(\lambda)J$. In addition,

$$eigen_{-\lambda}(A + t(\lambda)J) = eigen_{-\lambda}(B + t(\lambda)J) \cong \mathbb{R}.$$

Now it suffices to show that $t([\lambda_0, \infty))$ covers a nonempty open interval.

By Lemme 4 and Lemma 3,

$$t(\lambda) = -\|u_0\|^2 = -\frac{1}{\mathbf{1}^t (A + \lambda I)^{-1} \mathbf{1}}.$$

So $t(\lambda)$ is a rational function. Clearly, $t([\lambda_0, \infty))$ contains a nonempty open interval T. For $t \in T$, we have $\lambda_n(A + tJ) = \lambda_n(B + tJ)$ and $eigen_{\lambda_n}(A + tJ) = eigen_{\lambda_n}(B + tJ) \cong \mathbb{R}$. This finishes the proof of Theorem 1. Q.E.D.

Tutte's proof involves certain polynomials associated with a graph. It is algebraic in nature. The main instrument in our proof is the comparison theorem. Presumably, there is a connection between the geometry in this paper and the polynomials defined in Tutte's paper. In particular, given n unit vectors $u_1, u_2, \dots u_n$, can we compute the function $|\angle(0, U)|$ explicitly in terms of $U^t U = A$? This question turns out to be hard to answer. The norm $|\angle(0, U)|$ as a function of A may be closely related to the functions studied in Tutte's paper [2].

References

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[2] [Tutte] W. T. Tutte, "All the King's Horses (A Guide to Reconstruction) ", *Graph Theory and Related Topics*, Academic Press, 1979, (15-33).