# Bounds on Smooth Matrix Coefficients on $L^2$-spaces

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**Abstract.** Let $G$ be a semisimple Lie group with a finite number of connected components and a finite center. Let $K$ be a maximal compact subgroup. In this paper, we give an upper bound for $K$-finite and $\mathfrak{k}$-smooth matrix coefficients of the regular representation $L^2(X)$ where $X$ is a differentiable $G$-space equipped with a $G$-invariant measure, under an assumption about $\mathrm{supp}(L^2(X)) \cap \hat{G}_K$. Furthermore, we show that this bound holds for unitary representations that are weakly contained in $L^2(X)$. Our result generalizes a result of Cowling-Haagerup-Howe [2]. As an example, we discuss the matrix coefficients of the $O(p,q)$ representation $L^2(\mathbb{R}^{p+q})$.

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## 1. Introduction

Let $G$ be a semisimple Lie group with a finite number of connected components and a finite center. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of $G$. One important problem in harmonic analysis is to decompose $(\pi, \mathcal{H}_\pi)$ into a direct integral of irreducible unitary representations with multiplicities. More precisely, there exists a Borel measure $d\sigma$ on the unitary dual $\hat{G}$ such that

$$\mathcal{H}_\pi = \int_{(\sigma, \mathcal{H}_\sigma) \in \hat{G}} \mathcal{H}_\sigma \hat{\otimes} M_\sigma d\sigma.$$

Here $\hat{G}$ is equipped with the Fell topology and $M_\sigma$ records the mutiplicity of $\sigma$ ( [4] [14]). Very often, to determine the direct integral decomposition, one has to first determine the support of $\pi$, namely, the closed subset of $\hat{G}$ consisting of all

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representations that are weakly contained in $(\pi, \mathcal{H}_\pi)$ ( [4], [14]). Then one can define a certain transform for each $\sigma \in \text{supp}(\pi)$ to decompose $\mathcal{H}_\pi$.

Let $X$ be a (differentiable) $G$-space that carries a $G$-invariant measure. Then $L^2(X)$ becomes a unitary representation of $G$. For many $X$, determining $\text{supp}(L^2(X))$ remains an open problem, especially the discrete part. In this paper, we want to point a way that may lead to some new development. For a set of vectors $S$, let $\langle S \rangle$ be the complex linear space spanned by $S$. Let $u$ be a vector in $\mathcal{H}_\pi$. If $u$ is cyclic, that is, $\langle \pi(G)u \rangle$ is dense in $\mathcal{H}_\pi$, then the matrix coefficient $(\pi(g)u, u)$ determines $\text{supp}(\pi)$ uniquely. The purpose of this paper is to give some basic estimate of the smooth matrix coefficients of $L^2(X)$. Smooth matrix coefficients here mean the matrix coefficients for smooth vectors. We show that all $K$-finite and $\mathfrak{k}$-smooth matrix coefficients are bounded above by some function related to Harish-Chandra's $\Xi$ function. Our estimate equally applies to representations that are weakly contained in $L^2(X)$, in particular those in $\text{supp}(L^2(X))$. We follow the approach taking by Cowling, Haagerup and Howe in treating the tempered representations ( [2]).

Before we state our result, we fix some notations. Fix an Iwasawa decompostion $KAN$. Let $\Sigma^+$ be the set of positive restricted roots from $N$. Let $\rho$ be the half sum of positive restricted roots. Let $\mathfrak{a}_{\mathbb{C}}^* = \text{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C})$ and $\mathfrak{a}^* = \text{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{R})$. Let $\mathfrak{a}^+$ be a closed Weyl chamber defined by $\Sigma^+$ and by $W(G, \mathfrak{a})$ (See Page 124 [9]). Let $\lambda, \lambda' \in \mathfrak{a}_{\mathbb{C}}^*$. We say that $\lambda$ is dominated by $\lambda'$ if

$$\Re(\lambda')(H) \geq \Re(\lambda)(H) \qquad (\forall H \in \mathfrak{a}^+).$$

We write $\lambda \preceq \lambda'$. $\preceq$ defines a partial ordering on $\mathfrak{a}_{\mathbb{C}}^*$.

Let $\hat{G}_K$ be the spherical unitary dual. Then $\hat{G}_K$ can be identified with a closed subset of

$$\mathfrak{a}_{\mathbb{C}}^* // W(G : \mathfrak{a}).$$

Fix a dominant Weyl chamber in $\mathfrak{a}^*$ corresponding to $\mathfrak{a}^+$. We say that $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is dominant if $\Re(\lambda)$ is in the dominant Weyl chamber. **Identify $\hat{G}_K$ with a closed subset of dominant $\mathfrak{a}_{\mathbb{C}}^*$.** Let $\Xi$ be Harish-Chandra's basic spherical function.

Fix a maximal torus $T$ in $K$ and a positive root system. Let $r_K$ be the rank of $K$ and $l_K$ be the number of positive roots of $K$. Let $\rho_K$ be the half sum of the positive roots. Let $V_\lambda$ be an irreducible unitary representation of $K$ with highest weight $\lambda$. Let $C(\mathfrak{k})$ be the Casimir element in $U(\mathfrak{k})$ if $\mathfrak{k}$ is semisimple. If $\mathfrak{k}$ contains a nontrivial center, define $C(\mathfrak{k})$ to be the element in the center of the universal enveloping algebra $U(\mathfrak{k})$ satisfying

$$C(\mathfrak{k})|_{V_\lambda} = [-(\rho_K + \lambda, \rho_K + \lambda) + (\rho_K, \rho_K)]I$$

for every $V_\lambda \in \hat{K}$. Let $d_\lambda$ be the dimension of $V_\lambda$.

**Theorem 1.1 (Main Theorem).** *Let $G$ be a semisimple Lie group with a finite number of connected components and a finite center. Let $X$ be a $G$-space endowed with a $G$-invariant measure $dx$. Suppose that* $\mathrm{supp}(L^2(X)) \cap \hat{G}_K$*, as a subset of dominant $\mathfrak{a}^*_{\mathbb{C}}$, is dominated by a real $\lambda_0$. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation that is weakly contained in $L^2(X)$ (see* [4], [14]*).*

1. *Let $u$, $v$ be two $K$-finite vectors in $\mathcal{H}_\pi$. Let $S_1$ be the $K$-types appearing in $\langle \pi(K)u \rangle$. Let $S_2$ be the $K$-types appearing in $\langle \pi(K)v \rangle$. Then for any $H \in \mathfrak{a}^+$ and $k_1, k_2 \in K$, we have*

$$|(\pi(k_1 \exp H k_2)u, v)| \leq (\sum_{\sigma \in S_1} (d_\sigma)^2)^{\frac{1}{2}} (\sum_{\tau \in S_2} (d_\tau)^2)^{\frac{1}{2}} \|u\| \|v\| \exp \lambda_0(H) \Xi(\exp H). \tag{1.1}$$

2. *Let $C(\mathfrak{k})$ be the Casimir element in $U(\mathfrak{k})$. Let $u, v$ be two $\mathfrak{k}$ smooth vectors (See Definition* 5.1*). Then there exists a positive constant $C$, independent of $u, v$, such that for any $k_1, k_2 \in K$, $H \in \mathfrak{a}^+$,*

$$|(\pi(k_1 \exp H k_2)u, v)| \leq$$
$$C \exp \lambda_0(H) \Xi(\exp H) \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{l_K + r_K} u\| \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{l_K + r_K} v\|. \tag{1.2}$$

*In particular, these estimates hold for irreducible unitary representations in* $\mathrm{supp}(\pi)$*.*

Note that in many cases, the spherical support $\mathrm{supp}(L^2(X)) \cap \hat{G}_K$ is easier to determine than $\mathrm{supp}(L^2(X))$.

The proof of the main theorem contains three ingredients. The first ingredient comes from a uniform bound for the spherical functions in [9]. The second ingredient comes from a paper by Cowling-Haagerup-Howe that bounds the $K$-finite matrix coefficients of tempered representations by $\Xi(g)$. The third ingredient comes from a bound on the dimension of an irreducible unitary representations of $K$. In Theorem 6.1, we give a result only assuming that $X$ has a $K$-invariant measure.

There are bounds for smooth matrix coefficients for unitary representations in [1], [7]. The tempered case, that is $\lambda_0 = 0$, was treated in [2]. The bound for the smooth matrix coefficients of tempered representations was treated recently by Sun [13]. The ideas in this paper are quite standard, not new. Nevertheless, we believe that our estimates can shed lights on the structure of $\mathrm{supp}(L^2(X))$, as well as some other applications. Let us take the example of $L^2(\mathbb{R}^{p+q})$ as a unitary representation of $O(p, q)$. The spectral decomposition of $L^2(\mathbb{R}^{p+q})$ was established by Strichartz in general and others in some special cases. See [12] and the references therein. Applying our main theorem, we have

**Theorem 1.2.** *Suppose that $q \geq p$ and $pq > 1$. Let $G = O(p, q)$ and $K = O(p) \times O(q)$. Let $\mathcal{C}(\mathfrak{k})$ be the Casimir operator. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation that is weakly contained in $L^2(\mathbb{R}^{p+q})$. Let $u, v$ be two $\mathfrak{k}$-smooth vectors in $\mathcal{H}_\pi$. Let*

$\lambda_t = (\frac{p+q}{2}-2, \frac{p+q}{2}-3, \ldots, \frac{q-p}{2}, t)$. *Then for any* $H \in \mathfrak{a}^+$, $k_1, k_2 \in K$, *if* $q-p>2$, *we have*

$$|(\pi(k_1 \exp H k_2)u, v)| \leq$$
$$C \exp \lambda_{\frac{q-p}{2}-1}(H)\Xi(\exp H)\|(\mathcal{C}(\mathfrak{k})-2\|\rho_K\|^2-1)^{p^2+q^2}u\|\|(\mathcal{C}(\mathfrak{k})-2\|\rho_K\|^2-1)^{p^2+q^2}v\|; \tag{1.3}$$

*if* $q-p=0,1,2$, *we have*

$$|(\pi(g)u, v)| \leq C\phi_{\lambda_0}(g)\|(\mathcal{C}(\mathfrak{k})-2\|\rho_K\|^2-1)^{p^2+q^2}u\|\|(\mathcal{C}(\mathfrak{k})-2\|\rho_K\|^2-1)^{p^2+q^2}v\|. \tag{1.4}$$

*Here* $\phi_{\lambda_0}(g)$ *is the spherical function corresponding to* $\lambda_0 \in \mathfrak{a}^*_{\mathbb{C}}/W(G,\mathfrak{a})$.

Let $\epsilon$ be a small positive number. Our theorem implies that $(\pi(k_1 \exp H k_2)u, v)$ decays faster that $C\exp(-1+\epsilon)(|H_1|+|H_2|+\ldots+|H_p|)$ if $q-p \geq 2$. If $q=p+1$ then $(\pi(k_1 \exp H k_2)u, v)$ decays faster than $C\exp(-1+\epsilon)(|H_1|+|H_2|+\ldots+\frac{1}{2}|H_p|)$. If $q=p$, then $(\pi(k_1 \exp H k_2)u, v)$ decays faster than $C\exp(-1+\epsilon)(|H_1|+|H_2|+\ldots+|H_{p-1}|)$. These results are slightly different from what one would expect.

A more intriguing problem is to find a bound for the $\mathfrak{k}$-smooth matrix coefficients from below. Clearly, the $\mathfrak{k}$-smooth matrix coefficients of $L^2(\mathbb{R}^{p+q})$ cannot decay arbitrarily fast unless $\min(p,q)=1$. Having an upper bound, if one can find a bound from below, one can potentially narrow down the possible $\tau$ in $\operatorname{supp}(L^2(\mathbb{R}^{p+q}))$, which is already known. For those $X$ that $\operatorname{supp}(L^2(X))$ is not known, we hope that this approach will yield some new results.

## 2. Bounds for $K$-invariant Matrix Coefficients

Let $G$ be a semisimple Lie group with a finite number of connected components and a finite center. $G$ may be disconnected. Let $K$ be a maximal compact subgroup. Fix an Iwasawa decomposition $KAN$. Let $\Sigma^+$ be the positive restricted roots corresponding to $N$. For any $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, let $\phi_\lambda(g)$ be the corresponding spherical function. $\phi_\lambda(g)$ is both left and right $K$-invariant. We have the following (see Ch. 7.8 [9])

1. For $\lambda$ real, $\phi_\lambda(g) > 0$ for all $g$;
2. For $\phi_\lambda(g) = \phi_{w\lambda}(g)$ for any $w \in W(G:\mathfrak{a})$;
3. $|\phi_\lambda(g)| \leq \phi_{\Re\lambda}(g)$.

Let $\mathfrak{a}^+$ be a closed positive Weyl chamber satisfying the property that

$$\alpha(H) \geq 0, \qquad (\forall\ H \in \mathfrak{a}^+, \alpha \in \Sigma^+).$$

$\mathfrak{a}^+$ determines a dominant Weyl chamber in $\mathfrak{a}^*$ by identifying $\mathfrak{a}$ with $\mathfrak{a}^*$. $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ is said to be **dominant** if $\Re\lambda$ is in the dominant Weyl chamber. If $\lambda$ is dominant and real, we have

$$\phi_\lambda(\exp H) \leq \exp(\lambda(H))\phi_0(\exp H)$$

for any $H \in \mathfrak{a}^+$ (see Ch. 7.8 [9]). Here $\phi_0(g)$ is Harish-Chandra's $\Xi$ function. Essentially, the formulae above give bounds for $K$-invariant functions for each irreducible representation.

Let $\hat{G}$ be the unitary dual of $G$. Let $(\pi, \mathcal{H})$ be a unitary representation of $G$. Let $\mathrm{supp}(\pi)$ or sometimes $\mathrm{supp}(\mathcal{H})$ be the support of $\pi$, namely the closed subset of $\hat{G}$ consisting of those that are weakly contained in $\pi$ (See Ch 18.1 [4] or Ch 14.10 [14]). If $\mathrm{supp}(\pi)$ is a subset of $\mathrm{supp}(\pi')$, we say that $\pi$ is weakly contained in $\pi'$.

An irreducible admissible representation is said to be spherical if it has a $K$-fixed vector. Infinitesimal equivalence classes of spherical admissible representations are in one-to-one correspondence with

$$\mathfrak{a}_{\mathbb{C}}^*/W(G : \mathfrak{a}).$$

See Ch .IV [6] for example. The unitary spherical dual is often denoted by $\hat{G}_K$. We parametrize $\hat{G}_K$ by a closed subset of **dominant** $\lambda$. We write the corresponding spherical unitary representation as $(\pi_\lambda, \mathcal{H}_\lambda)$.

**Theorem 2.1.** *Let $G$ be a semisimple Lie group with a finite number of connected components and a finite center. Let $(\pi, \mathcal{H})$ be a unitary representation of $G$. Suppose that* $\mathrm{supp}(\pi) \cap \hat{G}_K$ *is dominated by a real $\lambda_0$. Then*

$$|(\pi(k_1 \exp H k_2)u, v)| \leq \exp \lambda_0(H) \Xi(\exp H) \|u\| \|v\|$$

*for any $k_1, k_2 \in K$, $H \in \mathfrak{a}^+$ and $K$-fixed vectors $u, v \in \mathcal{H}$.*

The proof will be based on local results about $\phi_\lambda(g)$ we mentioned earlier and the direct integral theory (see for example Ch 14. [14]).

*Proof.* Decompose the unitary representation $(\pi, \mathcal{H})$ into a direct integral

$$\int_{\hat{G}} \mathcal{H}_s \hat{\otimes} M_s d\mu_s$$

where $M_s$ records the multiplicity. Write

$$u = \int_{\hat{G}_K \cap \mathrm{supp}(\pi)} u_s d\mu_s, \qquad v = \int_{\hat{G}_K \cap \mathrm{supp}(\pi)} v_s d\mu_s.$$

Then we have

$$\|u\|^2 = \int_{\hat{G}_K \cap \mathrm{supp}(\pi)} \|u_s\|^2 d\mu_s, \qquad \|v\|^2 = \int_{\hat{G}_K \cap \mathrm{supp}(\pi)} \|v_s\|^2 d\mu_s$$

$$(\pi(g)u, v) = \int_{\hat{G}_K \cap \mathrm{supp}(\pi)} (\pi(g)u_s, v_s) d\mu_s.$$

Notice that here $s$ are all dominant in $\mathfrak{a}_{\mathbb{C}}^*$ and $u_s, v_s$ are $K$-invariant. Now by our assumption, for every $H \in \mathfrak{a}^+$,

$$\begin{aligned}
&|(\pi(k_1 \exp H k_2)u, v)| \\
&\leq \int_{\hat{G}_K \cap \mathrm{supp}(\pi)} |(\pi(\exp H)u_s, v_s)| d\mu_s \\
&= \int_{\hat{G}_K \cap \mathrm{supp}(\pi)} |\phi_s(\exp H)||(u_s, v_s)| d\mu_s \\
&\leq \int_{\hat{G}_K \cap \mathrm{supp}(\pi)} \exp s(H)\Xi(\exp H)\|u_s\|\|v_s\| d\mu_s \qquad (2.1) \\
&\leq \int_{\hat{G}_K \cap \mathrm{supp}(\pi)} \exp \lambda_0(H)\Xi(\exp H)\|u_s\|\|v_s\| d\mu_s \\
&\leq \exp \lambda_0(H)\Xi(\exp H)\left(\int_{\hat{G}_K \cap \mathrm{supp}(\pi)} \|u_s\|^2 d\mu_s \int_{\hat{G}_K \cap \mathrm{supp}(\pi)} \|v_s\|^2 d\mu_s\right)^{\frac{1}{2}} \\
&= \exp \lambda_0(H)\Xi(\exp H)\|u\|\|v\|
\end{aligned}$$

□

In the case that $\pi$ is supported on the tempered dual of $G$, $\lambda_0 = 0$. So we have

$$|(\pi(g)u, v)| \leq \Xi(g)\|u\|\|v\|.$$

This is proved in [2].

For $u, v$ in other $K$-types of $\mathcal{H}$, it is not easy to bound $(\pi(g)u, v)$ by $\|u\|$ and $\|v\|$. Even if $\pi$ is spherical, it is still not clear whether the type of bound in Theorem 2.1 is true. However, if $\pi$ is supported on $\mathrm{supp}(L^2(X))$ with $X$ a $G$-space equipped with a $G$-invariant measure, we can find such a bound.

## 3. Bounds for $K$-finite Matrix Coefficients of $L^2(X)$

Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation that is weakly contained in $L^2(G)$. Cowling, Haagerup and Howe obtain a sharp bound on the $K$-finite matrix coefficients of $\pi$.

**Theorem 3.1 (Cowling-Haagerup-Howe** [2]**).** *Let $G$ be a semisimple Lie group with a finite number of connected components and a finite center. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation that is weakly contained in $L^2(G)$. Let $\xi$ and $\eta$ be two $K$-finite vectors. Decompose the $K$ invariant subspaces $\langle \pi(K)\xi \rangle$ and $\langle \pi(K)\eta \rangle$:*

$$\langle \pi(K)\xi \rangle = \oplus_{\tau \in \hat{K}} \oplus^{m(\tau)} \mathcal{H}_\tau, \qquad \langle \pi(K)\eta \rangle = \oplus_{\tau \in \hat{K}} \oplus^{n(\tau)} \mathcal{H}_\tau.$$

*Let $\dim(\mathcal{H}_\tau) = d_\tau$. Then $m(\tau) \leq d_\tau, n(\tau) \leq d_\tau$ and*

$$|(\pi(g)\xi, \eta)| \leq (\dim\langle \pi(K)\xi \rangle)^{\frac{1}{2}} (\dim\langle \pi(K)\eta \rangle)^{\frac{1}{2}} \Xi(g).$$

In particular, if $\langle \pi(K)u\rangle \cong \oplus^{m(\tau)}\mathcal{H}_\tau$ and $\langle \pi(K)v\rangle \cong \oplus^{n(\sigma)}\mathcal{H}_\sigma$, then

$$|(\pi(g)u, v)| \leq d_\tau d_\sigma \|u\|\|v\|\Xi(g).$$

Let $X$ be a differentiable $G$-space equipped with a $G$-invariant measure $dx$. Let $G$ act on $L^2(X)$ by

$$L(g)f(x) = f(g^{-1}x) \qquad (g \in G, x \in X).$$

We call $(L, L^2(X))$ a regular representation. One of the most important problems in harmonic analysis is to find the $\mathrm{supp}(L^2(X))$. In many cases, the set $\mathrm{supp}(L^2(X)) \cap \hat{G}_K$ is relatively easy to find, since $\hat{G}_K$ is better understood than $\hat{G}$. In Theorem 2.1, we find a bound for the $K$-invariant matrix coefficients, assuming that $\mathrm{supp}(L^2(X)) \cap \hat{G}_K$ is dominated by a $\lambda_0 \in \mathfrak{a}^*$. Borrowing an idea from [2], we can show that similar bounds apply to all $K$-finite matrix coefficients of $L^2(X)$. Now this does not tell you much if $X$ has finite volume because the trivial representation will appear in $L^2(X)$. But if $X$ has infinite volume, bounds on $K$-finite matrix coefficients can shed lights on the structure of $\mathrm{supp}(L^2(X))$. At the end of this paper, we will use the hyperboloid as an example to illustrate our point.

**Theorem 3.2.** *Let $G$ be a semisimple Lie group with a finite number of connected components and a finite center. Let $X$ be a $G$-space equipped with a $G$-invariant measure $dx$. Suppose that* $\mathrm{supp}(L^2(X)) \cap \hat{G}_K$ *is dominated by $\lambda_0 \in \mathfrak{a}^*$. Let $\phi$, $\psi$ be continuous $K$-finite functions. Then for any $H \in \mathfrak{a}^+$,*

$$|(L(k_1 \exp H k_2)\phi, \psi)| \leq \dim(\langle L(K)\phi\rangle)^{\frac{1}{2}} \dim(\langle L(K)\psi\rangle)^{\frac{1}{2}} \|\phi\|_2 \|\psi\|_2 \exp \lambda_0(H) \Xi(\exp H).$$

Before we give the proof, let us recall the following lemma (See [2], for example).

**Lemma 3.3.** *Let $\phi$ be a continuous function on a $K$-homogeneous space $X$. Suppose that $\langle L(K)\phi\rangle$ is finite dimensional. Then*

$$\|\phi\|_\infty \leq \dim(\langle L(K)\phi\rangle)^{\frac{1}{2}} \|\phi\|_2.$$

*Here $L^2$-norm $\| * \|_2$ is taken over the $K$-invariant probability measure on $X$. In addition, if $\langle L(K)\phi\rangle$ consists of $K$-types from the set $S \subset \hat{K}$,*

$$\dim(\langle L(K)\phi\rangle) \leq \sum_{\sigma_i \in S} d_{\sigma_i}^2.$$

Proof of Theorem 3.2: Use the notation from [2]. Let

$$\tilde{\phi}(x) = \sup_{k\in K} |\phi(kx)|, \qquad \tilde{\psi}(x) = \sup_{k \in K} |\psi(kx)| \qquad (x \in X).$$

Consider any $K$-orbit $Kx_0$ equipped with the $K$-invariant probability measure. We have

$$\int_{Kx_0} |\tilde{\phi}(kx_0)|^2 d[k] = (\sup_{k\in K} |\phi(kx_0)|)^2 \leq \dim(\langle L(K)\phi\rangle) \int_{Kx_0} |\phi(kx_0)|^2 d[k].$$

It follows that

$$\|\tilde{\phi}\|_2^2 \leq \dim(\langle L(K)\phi\rangle)\|\phi\|_2^2.$$

Hence $\|\tilde{\phi}\|_2 \leq \dim(\langle L(K)\phi\rangle)^{\frac{1}{2}}\|\phi\|_2$. For any $H \in \mathfrak{a}^+$, we have

$$\begin{aligned}
&|(L(k_1 \exp H k_2)\phi, \psi)| \\
\leq &|(L(\exp H)\tilde{\phi}, \tilde{\psi})| \\
\leq &\exp \lambda_0(H)\Xi(\exp H)\|\tilde{\phi}\|_2\|\tilde{\psi}\|_2 \\
\leq &\dim(\langle L(K)\phi\rangle)^{\frac{1}{2}} \dim(\langle L(K)\psi\rangle)^{\frac{1}{2}}\|\phi\|_2\|\psi\|_2 \exp \lambda_0(H)\Xi(\exp H)
\end{aligned} \tag{3.1}$$

□

Now one can drop the requirement that $\phi, \psi$ are continuous.

**Corollary 3.4.** *Let $G$ be a semisimple Lie group with a finite number of connected components and a finite center. Let $X$ be a $G$-space equipped with a $G$-invariant measure $dx$. Suppose that* $\operatorname{supp}(L^2(X)) \cap \hat{G}_K$ *is dominated by $\lambda_0 \in \mathfrak{a}^*$. Let $\phi$, $\psi$ be two $L^2$ $K$-finite functions on $X$. Let $S_1$ be the $K$-types appearing in $\langle L(K)\phi\rangle$. Let $S_2$ be the $K$-types appearing in $\langle L(K)\psi\rangle$. Then for any $H \in \mathfrak{a}^+$, $k_1, k_2 \in K$, we have*

$$|(L(k_1 \exp H k_2)\phi, \psi)| \leq \exp \lambda_0(H)\Xi(\exp H)\|\phi\|_2\|\psi\|_2(\sum_{\sigma\in S_1}(d_\sigma)^2)^{\frac{1}{2}}(\sum_{\tau\in S_2}(d_\tau)^2)^{\frac{1}{2}}.$$

*Proof.* Choose two sequences of continuous functions

$$\phi_i \to \phi \qquad \psi_i \to \psi$$

in $L^2$-norm. Without loss of generality, suppose that

$$\operatorname{supp}(\langle L(K)\phi_i\rangle) = S_1, \qquad \operatorname{supp}(\langle L(K)\psi_i\rangle) = S_2.$$

Otherwise, we can always project $\phi_i$ and $\psi_i$ to respective $K$-types. By Theorem 3.2 and Lemma 3.3, for $H \in \mathfrak{a}^+$, we have

$$\begin{aligned}
&|(L(k_1 \exp H k_2)\phi_i, \psi_i)| \\
\leq &\dim(\langle L(K)\phi_i\rangle)^{\frac{1}{2}} \dim(\langle L(K)\psi_i\rangle)^{\frac{1}{2}}\|\phi_i\|_2\|\psi_i\|_2 \exp \lambda_0(H)\Xi(\exp H) \\
\leq &(\sum_{\sigma\in S_1}(d_\sigma)^2)^{\frac{1}{2}}(\sum_{\tau\in S_2}(d_\tau)^2)^{\frac{1}{2}}\|\phi_i\|_2\|\psi_i\|_2 \exp \lambda_0(H)\Xi(\exp H).
\end{aligned} \tag{3.2}$$

Taking pointwise limits, we obtain

$$|(L(k_1 \exp H k_2)\phi, \psi)| \leq \exp \lambda_0(H)\Xi(\exp H)\|\phi\|_2\|\psi\|_2(\sum_{\sigma\in S_1}(d_\sigma)^2)^{\frac{1}{2}}(\sum_{\tau\in S_2}(d_\tau)^2)^{\frac{1}{2}}.$$

□

## 4. Bounds for $K$-finite Matrix Coefficients

**Theorem 4.1.** *Let $G$ be a semisimple Lie group with a finite number of connected components and a finite center. Let $X$ be a $G$-space equipped with a $G$-invariant measure $dx$. Suppose that* $\operatorname{supp}(L^2(X)) \cap \hat{G}_K$ *is dominated by $\lambda_0 \in \mathfrak{a}^*$. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation that is weakly contained in $L^2(X)$. Let $u$, $v$ be two*

*$K$-finite vectors in $\mathcal{H}_\pi$. Let $S_1$ be the $K$-types appearing in $\langle \pi(K)u \rangle$. Let $S_2$ be the $K$-types appearing in $\langle \pi(K)v \rangle$. Then for any $H \in \mathfrak{a}^+$,*

$$|(\pi(k_1 \exp H k_2)u, v)| \leq (\sum_{\sigma \in S_1} (d_\sigma)^2)^{\frac{1}{2}} (\sum_{\tau \in S_2} (d_\tau)^2)^{\frac{1}{2}} \|u\| \|v\| \exp \lambda_0(H) \Xi(\exp H).$$

*Proof.* The ideas in this proof are essentially from [2]. For a unitary representation $\mathcal{H}$ of $K$, let $\mathcal{H}(S_j)$ be the direct sum of its $\sigma$-isotypic subspaces with $\sigma \in S_j$.

Since $(\pi, \mathcal{H}_\pi)$ is weakly contained in $L^2(X)$, $(\pi(g)u, v)$ can be approximated by finite sums

$$\sum_i (L(g)\phi_i, \psi_i) \qquad (\phi_i, \psi_i \in L^2(X))$$

uniformly on compacta, subject to the condition that

$$\sum_i \|\phi_i\|_2 \|\psi_i\|_2 \leq \|u\| \|v\|.$$

Let $P_{S_j}$ be the projector of $L^2(X)$ to $L^2(X)(S_j) (j = 1, 2)$. we can project $\phi_i$ and $\psi_i$ to $L^2(X)(S_1)$ and $L^2(X)(S_2)$ resepctively. Without loss of generality, assume that $\phi_i \in L^2(X)(S_1)$ and $\psi_i \in L^2(X)(S_2)$. By Cor. 3.4, we have the bound

$$|(L(k_1 \exp H k_2)\phi_i, \psi_i)| \leq \exp \lambda_0(H) \Xi(\exp H) \|\phi_i\|_2 \|\psi_i\|_2 (\sum_{\sigma \in S_1} (d_\sigma)^2)^{\frac{1}{2}} (\sum_{\tau \in S_2} (d_\tau)^2)^{\frac{1}{2}}.$$

Hence

$$\begin{aligned} & |\sum (L(k_1 \exp H k_2)\phi_i, \psi_i)| \\ \leq & \sum |(L(k_1 \exp H k_2)\phi_i, \psi_i)| \\ \leq & \exp \lambda_0(H) \Xi(\exp H) \| (\sum_{\sigma \in S_1} (d_\sigma)^2)^{\frac{1}{2}} (\sum_{\tau \in S_2} (d_\tau)^2)^{\frac{1}{2}} \sum \|\phi_i\|_2 \|\psi_i\|_2. \qquad (4.1) \\ \leq & \exp \lambda_0(H) \Xi(\exp H) \| (\sum_{\sigma \in S_1} (d_\sigma)^2)^{\frac{1}{2}} (\sum_{\tau \in S_2} (d_\tau)^2)^{\frac{1}{2}} \|u\| \|v\| \end{aligned}$$

It follows that

$$|(\pi(k_1 \exp H k_2)u, v)| \leq (\sum_{\sigma \in S_1} (d_\sigma)^2)^{\frac{1}{2}} (\sum_{\tau \in S_2} (d_\tau)^2)^{\frac{1}{2}} \|u\| \|v\| \exp \lambda_0(H) \Xi(\exp H).$$

□

I shall point out that our estimate clearly holds if $\pi$ is in the support of $L^2(X)$. So our estimate can be used to exclude those $\pi$ that are not in the support of $L^2(X)$.

## 5. Bounds for Smooth Matrix Coefficients

Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation weakly contained in $L^2(X)$. Now we can move forward to give a bound for $\mathfrak{k}$-smooth matrix coefficients of $\pi$. Very recently, B. Sun found a bound for the tempered representations for a bigger class of group $G$ ( [13]). Our idea is essentially the same.

**Definition 5.1.** Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of a Lie group $H$. We say that a vector $v$ is $\mathfrak{h}$ smooth if $\pi(D)v$ is well-defined in $\mathcal{H}_\pi$ for any $D \in U(\mathfrak{h})$.

Fix a maximal torus $\mathfrak{t}$ and positive roots $\Sigma^+$ for the Lie algebra $\mathfrak{k}$. Let $r_K$ be the dimension of $\mathfrak{t}$, $l_K$ be the cardinality of $\Sigma^+$, and $\rho_K$ be the half sum of positive roots. Let $\mathcal{C}(\mathfrak{k})$ be the Casimir operator in $U(\mathfrak{k})$. Paramatrize $\hat{K}$ by the highest weight $\lambda$. Then

$$\mathcal{C}(\mathfrak{k})|_{V_\lambda} = -\|\rho_K + \lambda\|^2 + \|\rho_K\|^2.$$

Clearly, for each positive root $\alpha$,

$$(\lambda + \rho_K, \alpha) \leq (\lambda + \rho_K, 2\rho_K) \leq (\lambda + \rho_K, \lambda + \rho_K) + (\rho_K, \rho_K).$$

If $K$ is not Abelian, by Weyl's character formula,

$$\dim(V_\lambda) \leq (\|\lambda + \rho_K\|^2 + \|\rho_K\|^2)^{l_K}.$$

For all compact $K$,

$$\dim(V_\lambda) \leq (\|\lambda + \rho_K\|^2 + \|\rho_K\|^2 + 1)^{l_K}.$$

So for $u \in V_\lambda$, we have

$$\dim(V_\lambda)\|u\| \leq \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{l_K} u\|.$$

**Theorem 5.2.** *Let $G$ be a semisimple Lie group with a finite number of connected components and a finite center. Let $K$ be a maximal compact subgroup of $G$. Let $r_K$ be the rank of $K$ and $l_K$ be the number of positive roots for $\mathfrak{k}$. Let $\mathcal{C}(\mathfrak{k})$ be the Casimir operator. Let $X$ be a $G$-space equipped with a $G$-invariant measure. Suppose that $\operatorname{supp}(L^2(X)) \cap \hat{G}_K$ is dominated by $\lambda_0 \in \mathfrak{a}^*$. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation that is weakly contained in $L^2(X)$. Let $u, v$ be two $\mathfrak{k}$-smooth vectors in $\mathcal{H}_\pi$. Then there exists a positive constant $C$, independent of $u, v$, such that for any $k_1, k_2 \in K$, $H \in \mathfrak{a}^+$*

$$|(\pi(k_1 \exp H k_2)u, v)| \leq$$
$$C \exp \lambda_0(H) \Xi(\exp H) \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{l_K + r_K} u\| \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{l_K + r_K} v\|. \tag{5.1}$$

*Here in the place of $r_K$ one can use any integer greater than $\frac{r_K}{2}$.*

*Proof.* Suppose that $u, v$ are $\mathfrak{k}$-smooth. Decompse $u, v$ according to the $K$-types:

$$u = \sum_{\lambda \in \hat{K}} u_\lambda, \qquad v = \sum_{\lambda \in \hat{K}} v_\lambda.$$

Let $g = k_1 \exp H k_2$ and $H \in \mathfrak{a}^+$. Put $\rho_0 = \rho_K$. Then

$$\begin{aligned}
&|(\pi(g)u, v)| \\
\leq & \sum_{\lambda,\mu} |(\pi(g)u_\lambda, v_\mu)| \\
\leq & \sum d_\mu d_\lambda \|u_\lambda\| \|v_\mu\| \Xi(g) \exp \lambda_0(H) \\
= & \Xi(g) \exp \lambda_0(H) [\sum d_\lambda \|u_\lambda\|][\sum d_\mu \|v_\lambda\|] \\
\leq & \Xi(g) \exp \lambda_0(H) [\sum (\|\lambda + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{l_K} \|u_\lambda\|][\sum (\|\mu + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{l_K} \|v_\mu\|] \\
= & \Xi(g) \exp \lambda_0(H) [\sum (\|\lambda + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{l_K + r_K} \|u_\lambda\| (\|\lambda + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{-r_K}] \\
& [\sum (\|\mu + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{l_K + r_K} \|v_\mu\| (\|\mu + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{-r_K}] \\
\leq & \Xi(g) \exp \lambda_0(H) [\sum (\|\lambda + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{2l_K + 2r_K} \|u_\lambda\|^2]^{\frac{1}{2}} [\sum (\|\lambda + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{-2r_K}]^{\frac{1}{2}} \\
& [\sum (\|\mu + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{2l_K + 2r_K} \|v_\mu\|^2]^{\frac{1}{2}} [\sum (\|\mu + \rho_0\|^2 + \|\rho_0\|^2 + 1)^{-2r_K}]^{\frac{1}{2}} \\
= & C\Xi(g) \exp \lambda_0(H) [\sum \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_0\|^2 - 1)^{l_K + r_K} u_\lambda\|^2]^{\frac{1}{2}} [\sum \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_0\|^2 - 1)^{l_K + r_K} v_\mu\|^2]^{\frac{1}{2}} \\
= & C\Xi(g) \exp \lambda_0(H) \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_0\|^2 - 1)^{l_K + r_K} u\| \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_0\|^2 - 1)^{l_K + r_K} v\|
\end{aligned} \tag{5.2}$$

Here $C = \sum (\|\mu + \rho_0\|^2 + \|\rho_0\|^2)^{-2r_K}$ which converges absolutely. □

Of course, the estimate we obtain here can be improved substantially. For the purpose of this paper, it is sufficient. I shall also point out that for $\mathfrak{k}$-smooth vectors in $L^2(X)$, our bound can be established directly by bounding sup norm by the $L^2$ norm of some derivative. But this bound can not be passed from $L^2(X)$ to $(\pi, \mathcal{H}_\pi)$. Therefore, for $(\pi, \mathcal{H}_\pi)$, we must bound the $K$-finite matrix coefficients first and then pass this bound to all $\mathfrak{k}$ smooth vectors.

## 6. $X$ with $K$-invariant Measure

Sometimes, $G$-invariant measure does not exist for a $G$-space $X$. For example, when $X$ is a flag variety, there is no $G$-invariant measure. Nevertheless, $K$-invariant measure always exists. Now suppose that $X$ is equipped with only a $K$-invariant measure. Then $L^2(X)$ may no longer be a unitary representation of $G$. We can still define $K$-finite and $\mathfrak{k}$-smooth matrix coefficients. Suppose that there is a positive function $B(g)$ such that

$$|(L(g)\phi, \psi)| \leq B(g) \|\phi|_2 \|\psi\|_2,$$

for any $K$-invariant function $\phi$ and $\psi$ in $L^2(X)$. Then by similar arguments as in the proofs of Theorem 3.2 and Cor. 3.4 and Theorem 5.2, we obtain

**Theorem 6.1.** *Let $G$ be a semisimple Lie group with a finite number of connected components and a finite center. Let $X$ be a $G$-space endowed with a $K$-invariant measure $dx$. Suppose that there is positive function $B(g)$ such that*

$$|(L(g)\phi,\psi)| \leq B(g)\|\phi|_2\|\psi\|_2,$$

*for any $K$-invariant function $\phi$ and $\psi$ in $L^2(X)$.*

1. *Let $u$, $v$ be two $K$-finite functions in $L^2(X)$. Let $S_1$ be the $K$-types appearing in $\langle L(K)u\rangle$. Let $S_2$ be the $K$-types appearing in $\langle L(K)v\rangle$. Then we have*
$$|(L(g)u,v)| \leq (\sum_{\sigma\in S_1}(d_\sigma)^2)^{\frac{1}{2}}(\sum_{\tau\in S_2}(d_\tau)^2)^{\frac{1}{2}}\|u\|_2\|v\|_2 B(g).$$
2. *Let $C(\mathfrak{k})$ be the Casimir element in $U(\mathfrak{k})$. Let $u, v$ be two $\mathfrak{k}$ smooth functions in $L^2(X)$. Then there exists a $C>0$ such that for any $g\in G$*

$$|(\pi(g)u,v)| \leq CB(g)\|(\mathcal{C}(\mathfrak{k})-2\|\rho_K\|^2-1)^{l_K+r_K}u\|_2\|(\mathcal{C}(\mathfrak{k})-2\|\rho_K\|^2-1)^{l_K+r_K}v\|_2.$$

## 7. Bounds for Smooth Matrix Coefficients of $L^2(\mathbb{R}^{p+q})$

Now we shall give an example here. Let $O(p,q)$ be the orthogonal group preserving the standard symmetric form

$$(x,y)=\sum_{i=1}^{p}x_iy_i-\sum_{j=p+1}^{p+q}x_jy_j \qquad (x,y\in\mathbb{R}^{p+q}).$$

Consider $L^2(\mathbb{R}^{p+q})$, a regular representation of $O(p,q)$. R. Strichartz computed the spectrum of the pseudo Laplacian $\Box$ on $L^2(\mathbb{R}^{p+q})$ in full generality. Special cases were treated earlier. See [12] and the references therein. If $pq>1$, besides the continuous spectrum, there are also discrete spectrum. Essentially, this determines the support of $L^2(\mathbb{R}^{p+q})$. The continuous spectrum comes from degenerate principal series and the discrete spectrum comes from some quotients of degenerate principal series. $L^2(\mathbb{R}^{p+q})$ was later studied by Rallis-Schiffman ( [11]) and Howe ( [3]) under the framework of dual reductive pair $(O(p,q),SL_2(\mathbb{R}))$. Howe proved that

$$L^2(\mathbb{R}^{p+q})\cong\int_{s\in\widehat{\widetilde{SL}_2(\mathbb{R})}}\mathcal{H}_{\theta(s)}\otimes\mathcal{H}_s ds$$

Here $\widetilde{SL}_2(\mathbb{R})$ is the double cover of $SL_2(\mathbb{R})$, $ds$ is a Borel measure on the unitary dual of $\widetilde{SL}_2(\mathbb{R})$, and $\mathcal{H}_{\theta(s)}$ is an irreducible unitary representation of $O(p,q)$. The structure of the representation $\mathcal{H}_{\theta(s)}$ was studied by Molcanov ( [10]) and later by Howe and Tan in greater details ( [8]).

Let $G = O(p,q)$ and $K = O(p) \times O(q)$. Suppose that $pq > 1$. Then the real rank $r = \min(p,q)$. The half sum of positive restricted root

$$\rho = (\frac{p+q}{2} - 1, \frac{p+q}{2} - 2, \ldots, |\frac{p-q}{2}|).$$

For the purpose of giving a bound for $\mathfrak{k}$-smooth matrix coefficients, we will need to know $\operatorname{supp}(L^2(\mathbb{R}^{p+q})) \cap \hat{G}_K$. We will assume that $p \leq q$. This assumption won't effect our estimation. $\hat{G}_K$ is parametrized by certain dominant $\lambda$, i.e.,

$$\Re(\lambda_1) \geq \Re(\lambda_2) \geq \ldots \geq \Re(\lambda_p) \geq 0$$

up to a permutation and sign change. $\operatorname{supp}(L^2(\mathbb{R}^{p+q})) \cap \hat{G}_K$ can be described as follows.

1. the continuous spectrum consists of $\lambda_{it} = (\frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \ldots, \frac{q-p}{2}, it)$ with $t \geq 0$;
2. If $\frac{q-p}{2} > 1$, then the discrete spectrum consists of

$$\lambda_{\frac{q-p}{2}-2j-1} = (\frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \ldots, \frac{q-p}{2}, \frac{q-p}{2} - 2j - 1)$$

   for all integer $j \in [0, \frac{q-p-2}{4})$.

If $p = q$, then there is no discrete spherical spectrum. $\mathcal{H}_{\lambda_{it}}$ will decompose into $\mathcal{H}_{\lambda_{it}}$ and $\mathcal{H}_{\lambda_{-it}}$ with respect to the group $SO_0(p,q)$. If $q - p = 1, 2$, $L^2(\mathbb{R}^{p+q})$ does not have any discrete spherical spectrum. By Theorem 2.1, we have

**Theorem 7.1.** *Suppose that $q \geq p$ and $pq > 1$. Let $\lambda_t = (\frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \ldots, \frac{q-p}{2}, t)$. Let $u, v$ be two $K$-invariant vectors in $L^2(\mathcal{R}^{p+q})$. Then for any $H \in \mathfrak{a}^+$, $k_1, k_2 \in K$, we have*

$$|(L(k_1 \exp H k_2)u, v)| \leq \exp \lambda_{\frac{q-p}{2}-1}(H) \Xi(\exp H) \|u\| \|v\| \qquad (q - p > 2)$$

$$|(L(g)u, v)| \leq \phi_{\lambda_0}(g) \|u\| \|v\| \qquad (q - p = 0, 1, 2).$$

By Theorem 5.2, we have the following

**Theorem 7.2.** *Suppose that $q \geq p$ and $pq > 1$. Let $G = O(p,q)$ and $K = O(p) \times O(q)$. Let $\mathcal{C}(\mathfrak{k})$ be the Casimir operator. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation that is weakly contained in $L^2(\mathbb{R}^{p+q})$. Let $u, v$ be two $\mathfrak{k}$-smooth vectors in $\mathcal{H}_\pi$. Let $\lambda_t = (\frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \ldots, \frac{q-p}{2}, t)$. Then for any $H \in \mathfrak{a}^+$, $k_1, k_2 \in K$, we have for $q - p > 2$*

$$|(\pi(k_1 \exp H k_2)u, v)| \leq$$

$$C \exp \lambda_{\frac{q-p}{2}-1}(H) \Xi(\exp H) \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{p^2+q^2} u\| \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{p^2+q^2} v\| \tag{7.1}$$

*for $q - p = 0, 1, 2$*

$$|(\pi(g)u, v)| \leq C\phi_{\lambda_0}(g) \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{p^2+q^2} u\| \|(\mathcal{C}(\mathfrak{k}) - 2\|\rho_K\|^2 - 1)^{p^2+q^2} v\|.$$

*Proof.* It is clear that $r_K = [\frac{p}{2}] + [\frac{q}{2}] < p + q$ and $l_K < p^2 - p + q^2 - q$. Our assertion follows from Theorem 5.2. □

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