Communications in Contemporary Mathematics
Vol. 1, No. 1 (1999) 00–00

THETA CORRESPONDENCE I—SEMISTABLE RANGE: CONSTRUCTION AND IRREDUCIBILITY

HONGYU HE (JACOB LIVINGSTONE)
Depatment of Mathematics & Statistics
Georgia State University
Atlanta, GA 30303

(Leave 1 inch blank space for publisher.)

The main purpose of this paper is to study theta correspondence from representation theoretic point of view. There are two problems we have in mind. One is the construction of unipotent representations of semisimple Lie group. The other is the parametrization of unitary dual of semisimple Lie group. In the first paper of this series, we define semistable range in the domain of theta correspondence. Roughly speaking, semistable range is a range where one can define certain averaging operator analytically. In this paper, we prove that if the averaging operator is not vanishing, then it produces the theta correspondence. This paper pave the way to study theta correspondence using analytic machinery.

In Memory of My Grandmother

1. Introduction

Let G be a real classical group. Let $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ be its complexified Lie algebra. Let π be an irreducible unitary representation of G. Then there is a nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ associated with π, often call the associated variety of π. Since the classification of irreducible unitary representation is not known for the real classical groups of type I, one wish to study the inverse problem, that is, to classify the irreducible unitary representations whose associated variety is $\mathcal{O}$. For most of the nilpotent orbit, it is not known how to attach unitary representations to it. Finally, one may tentatively define unipotent representation to be one type of the most fundamental subclass of unitary representation (see [2], [3]). At this time, unipotent representation is not well-understood except for a few cases.

In the current research, theta correspondence is one of the main techniques used in constructing small unipotent representations (see [8]) of G. Jian-Shu Li studied the theta correspondence in *stable range* (see [9]). He showed that irreducible unitary representations of lower rank all occur in the theta correpondence in the stable range. In [12], the author gave a characterization of associated varieties attached to irreducible lower rank representations. Roughly speaking, the associated varieties attached to lower rank representations are a class of small nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$. In [12], we concluded that the set of irreducible lower rank representations is only a small class of unipotent representations. The remaining problem is to construct the unipotent representation associated with the rest of the nilpotent orbits. I must emphasis that Li's construction more of less relies on the geometric structure within the dual pair in stable range.

Jian-Shu Li's works partly revealed a strong connection between unipotent representation and theta correspondence. Nevertheless, we do not know how to generalize his results outside the stable range. This paper is our first attemp to generalize some results of Li. In particular, we will give an affirmitive answer to a conjecture of Li annuounced in the IAS/Park City Mathematics Institute on Representation Theory of Reductive Groups, 1998 (see [8]). Our method is analytic and algebraic.

Let (G_1, G_2) be a dual reductive pair in a real symplectic group G. Let V_i be the underlying linear space for G_i. Let ω be the oscillator representation of MG, the double cover of G. Let MG_1 and MG_2 be the preimage of G_1 and G_2 under the double covering. Let $\mathcal{R}(MG_i, \omega)$ be the set of equivalence classes of irreducible representations of MG_i which occur as a quotient of ω (in a proper category). Let

$$\theta : \mathcal{R}(MG_1, \omega) \rightarrow \mathcal{R}(MG_2, \omega)$$

be the theta correspondence as defined in [5].

Let π be an irreducible representation of MG_1. Let $\mathcal{P}$ be the Harish-Chandra module of ω and V be the Harish-Chandra module of π. We formally define a complex linear pairing $(,)_\pi$

$$(\mathcal{P} \otimes V^*, \mathcal{P}^c \otimes V)_\pi \rightarrow \mathbb{C}$$

as follows: for $\phi \in \mathcal{P}, \psi \in \mathcal{P}^c, \eta \in V^*, u \in V$

$$(\phi \otimes \eta, \psi \otimes u)_\pi = \int_{MG_1} (\phi, \omega(g)\psi)\eta(\pi(g)u)dg$$

Here V^* is the dual Harish-Chandra module of π and $\mathcal{P}^c$ is $\mathcal{P}$ with the complex conjugate linear structure. In this paper, we examine the convergence of this integral and define the semistable range in terms of the growth condition of the matrix coefficients of π. We show that in semistable range, $(,)_\pi$ is well-defined. To illustrate how big the semistable range is, we look at two sets of examples. Without loss of generality, we assume

$$\dim_D V_1 \leq \dim_D V_2.$$

Then all the tempered ireducible representations of MG_1 are in the semistable range of (G_1, G_2) (see [8], [13]). For (G_1, G_2) in the stable range with G_1 the smaller member, i.e.,

$$\dim_D V_2 \geq \dim_D V_1$$

any irreducible unitary representation of MG_1 (except the trivial representation for $(Sp(n,\mathbb{R}), O(2n,2n))$ is in the semistable range (see [8], [13]).

Notice that $\mathcal{P}^c \otimes V$ here is equipped with a MG_2-module structure in a proper category. Let $\mathcal{R}_\pi$ be the radical of $(,)_\pi$ defined as the maximal linear subspace of $\mathcal{P}^c \otimes V$ such that

$$(\mathcal{P} \otimes V^*, \mathcal{R}_\pi) = 0$$

Then $\mathcal{P}^c \otimes V/\mathcal{R}_\pi$ is also a MG_2-module in a proper sense. In [8], Jian-Shu Li asked the following questions based on what was understood in the stable range of theta correspondence:

1. Does $\mathcal{P}^c \otimes V/\mathcal{R}_\pi$ give us something close to $\theta(\pi)$;
2. Suppose $\pi \in \mathcal{R}(MG_1, \omega)$, must $(,)_\pi$ be nonvanishing?

In the paper, we prove the following theorem:

Theorem 1 *Let (G_1, G_2) be an irreducible dual pair. Suppose (π, V) is in the semistable range of (MG_1, MG_2) and $(,)_\pi$ is nonzero. Then*

- *π occurs in $\mathcal{R}(MG_1, \omega)$;*
- *$\mathcal{P}^c \otimes V/\mathcal{R}_\pi$ is irreducible;*
- *$\mathcal{P}^c \otimes V/\mathcal{R}_\pi$ is isomorphic to $\theta(\pi)^*$.*

Nevertheless, it is not known whether the 2nd conjecture of Li is true. However, in his thesis [12], the author showed that under a very mild condition, $(,)_\pi$ is nonvanishing. Some of the nonvanishing theorems are announced in AMS Eastern Sectional Conference in Providence, 1999 and others will be published elsewhere. Therefore, for the purpose of constructing representations, nonvanishing of $(,)_\pi$ is sufficient for now.

In the light of unipotent representation and the conjectures formulated by Arthur ([2]) and Adams ([1]), we formulate the following **conjectures**:

- Unitarity is preserved under theta correspondence in semistable range. In other words, if $\pi \in \mathcal{U}(MG_1)$ is in the semistbale range of (MG_1, MG_2) such that $(,)_\pi$ is nonvanishing, then $\theta(\pi) \in \mathcal{U}(MG_2)$;
- Suppose the previous conjecture is true. The theta correspondence in the semistable range descends into a correspondence of nilpotent orbits as specified in [15].

The first conjecture is difficult due to the fact that we do not understand much about the theory of **positive definite functions** on noncompact classical groups. The main reference in this direction is Godement's thesis [4] half century ago.

I shall remark that both conjectures are true for theta correspondence in stable range (see [9], [10], [14]). These two conjectures if proved would not only provide enough techniques to attack the conjectures of Arthur and Adams, but also furnish us with powerful tools to construct unipotent representations. In the subsequent paper [13], we prove the preservence of unitarity under a mild assumption.

The following is what is covered in this paper. In Chapter 2, we set up the notation and review the basic theory of admissible representations. In chapter 3, we define the oscillator representation and establish some estimation concerning its matrix coefficients. In Chapter 4, we review the basic theory of theta correspondence established in ([5]). In Chapter 5, we define the semistable range of theta correspondence based on the convergence of $(,)_\pi$. For the rest of this paper, we assume semistability. In chapter 6, we define the averaging operator $\mathcal{L}_\pi$ and we prove it is an intertwinning operator. I should mention a quite interesting (folk) lemma we used in this chapter.

Lemma 1 (Triangular Lemma) *For G a reductive group with compact center. Let $k \exp H(g) k'$ be a KAK decomposition of g with $H(g)$ in the positive Weyl chamber of $\mathfrak{a}$. Let $(,)$ be an invariant real quadratic form on $\mathfrak{g}$ which is negative definite on $\mathfrak{k}$. Let $\|.\|$ be the norm on $\mathfrak{a}$ induced from $(,)$. We have*

$$\|H(g_1 g_2)\| \leq \|H(g_1)\| + \|H(g_2)\|$$

In Chapter 7, we show that the image of this intertwining operator is irreducible. We conclude that theta correspondence in semistable range can be obtained by the averaging operator.

2. Preliminary

In this paper, all representations are complex representations and all homomorphisms are complex homomorphisms unless stated otherwise. We will use capital letter to denote Lie groups and "mathfrak" letters to denote the corresponding (real) Lie algebra. Let G be a real reductive Lie group with compact center. Let K be a maximal compact subgroup of G. Throughout this paper, a representation π of G would mean a $(\mathfrak{g}, K)$-module, considered the most fundamental object in representation theory of G. We use $\mathcal{R}(G)$ to denote the equivalence classes of irreducible $(\mathfrak{g}, K)$-modules and $\mathcal{U}(G)$ the equivalence classes of irreducible unitarizable $(\mathfrak{g}, K)$-modules. For G compact, $\mathcal{R}(G) = \mathcal{U}(G)$. A Hilbert representation (π, H) of G would mean a Hilbert space H and a group homomorphism π from G into $\mathcal{B}(H)$, the space of bounded linear operators on H, such that the canonical group action

$$G \times H \to H$$

is continuous. Throughout this paper, we will assume that the Hilbert norm is preserved by the maximal compact subgroup K. Then each Hilbert representation $\mathcal{H}$ produces a $(\mathfrak{g}, K)$-module $\mathcal{H}_K$ by taking the K-finite vectors in H. In particular, we may define the σ-isotypic subspace H_σ for each $\sigma \in \mathcal{U}(K)$. By an admissible representation, we mean a (g, K)-module which is the space of K-finite vectors in a Hilbert representation whose K-types have finite multiplicities. The classification of $\mathcal{R}(G)$ was successfully carried out by Langlands (see [7]). Throughout this paper, "overline"over a set means the closure under the norm specified with its context and "overline"over a number (over a vector) means complex conjugation.

Let V be a (unitary) representation of a compact group K. Let $\sigma \in \mathcal{U}(K)$. We will use V_σ to denote the σ-isotypic subspace of V. We will use V_K to denote the subspace of K-finite vectors in V. Obviously, we have

$$V_K = \oplus_{\sigma \in \mathcal{U}(K)} V_\sigma.$$

2.1. *Setup*

Let G be a reductive Lie group with compact center. We fix the following data and notation:

1. $G = K \exp \mathfrak{p}$—Cartan decomposition, θ—Cartan involution;
2. $(,)$—a invariant real bilinear form on $\mathfrak{g}$ to be specified later in this chapter, positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$;
3. $\mathfrak{a}$—a maximal Abelian Lie subalgebra in $\mathfrak{p}$; $A = \exp \mathfrak{a}$;
4. $\mathfrak{a}_{\mathbb{R}}^*$—real dual space of $\mathfrak{a}$;
5. For $\alpha \in \mathfrak{a}_{\mathbb{R}}^*$, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \forall\ a \in \mathfrak{a}, [a, x] = \alpha(a)x\}$ (if $\mathfrak{g}_\alpha$ is nonempty, α is called a root, $\mathfrak{g}_\alpha$ is called a root space);
6. Δ—simple roots; Σ_r^+—positive restricted roots;
7. $\mathfrak{n}$—direct sum of positive (restricted) root spaces; $N = \exp \mathfrak{n}$;
8. $\mathfrak{a}^+ = \{a \in \mathfrak{a}, \mid \alpha(a) > 0\ \forall\ \alpha \in \Delta\}$; $A^+ = \exp(\mathfrak{a}^+)$;
9. $\rho = \frac{1}{2} \sum_{\lambda \in \Sigma_r^+} \dim(\mathfrak{g}_\lambda)\lambda$;
10. $\alpha \succeq \beta$ if $\alpha = \beta + \Sigma_{\alpha_i \in \Delta} c_i \alpha_i$ for c_i nonnegative; $\alpha \succ \beta$ if at least one $c_i > 0$ (in such case, we say α dominates β);
11. $M = Z_K(\mathfrak{a})$—the centralizer of $\mathfrak{a}$ in K;
12. $\mathfrak{b}$— maximal torus in $\mathfrak{m}$;
13. $W(\mathfrak{a}, \mathfrak{g})$—Weyl group;

2.2. *Decomposition*

Theorem 2 (KAK Decomposition) *Every $g \in G$ has a decomposition of the form $k_1 a k_2$ with $k_1, k_2 \in K, a \in \overline{A^+}$. The element a is often written as $a(g)$ or $\exp(H(g))$, and it is unique up to conjugation by Weyl group $W(\mathfrak{a}, \mathfrak{g})$. When $a \in A^+$, g is called regular. Under the KAK decomposition, the Haar measure*

$$dg = \prod_{\lambda \in \Sigma_r^+} (\exp(\lambda(H)) - \exp(-\lambda(H)))^{\dim \mathfrak{g}_\lambda} dk_1 dH dk_2$$

In short we will write

$$dg = \Delta(H) dk_1 dH dk_2$$

If $H \in \mathfrak{a}^+$, then $\Delta(H) \neq 0$. The Haar measure of the singular (non-regular) elements is zero.

The KAK decomposition can be computed as follows. According to Cartan decompostion, for any $g \in G$, we may decompose it uniquely as $k(g)\exp(p(g))$. And $p(g) \in \mathfrak{p}$ can be written as $Ad(k)H(g)$, where $H(g) \in \mathfrak{a}$. Thus, we obtain the KAK decomposition of g:

$$g = k(g)k\exp(H(g))k^{-1}$$

One advantage of this definition is that we can define a norm on $g \in G$, such that

$$\|g\| = (p(g), p(g))^{\frac{1}{2}} = (\exp H(g), \exp H(g))^{\frac{1}{2}}$$

Now we may speak of $g_n \to \infty$ if $\|g_n\| \to \infty$. Finally we give a rough estimation on $\Delta(H)$. For $H \in \overline{\mathfrak{a}^+}$, it can be shown that

$$\Delta(H) \leq c\exp(2\rho(H))$$

We will use this fact in Chapter 5.

2.3. *General Theory of Matrix Coefficient*

Let π be an admissible representation of G. We will always fix a K-invariant inner product on π.

Definition 1 *For $(u, v) \in \pi$, we define*

$$\pi_{u,v}(g) = (\pi(g)u, v)$$

$\pi_{u,v}$ is called a matrix coefficient of π. Equivalent, let π^* be the dual of π in the category of $(\mathfrak{g}, K)$-modules. Let $\delta \in \pi^*$. Then

$$\pi_{\delta,u}(g) = \delta(\pi(g)u)$$

is also called a matrix coefficient.

Notice these two definitions coincide for admissible representations. Let π be an irreducible admissible representation of G with infinitesimal character λ. Here λ will be in the complex dual of $\mathfrak{a}\oplus\mathfrak{b}$. Then roughly speaking, $\pi_{u,v}$ possesses an asymptotic expansion at infinity. In particular, there exist a finite number of exponents $v \in \mathfrak{a}_{\mathbb{C}}^*$, often called leading exponents, attached to $\pi_{u,v}$. Finally, the leading exponents of π are those leading exponents that can not be integrally dominated by other exponents. We cite Theorem 8.33 from [6]

Theorem 3 . *Let π be an irreducible representation of G with infinitesimal character λ. Then every leading exponent v satisfies*

$$v + \rho = (w\lambda)|_{\mathfrak{a}}$$

for some $w \in W(\mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$.

The reason we introduce the leading exponent here is that we want to gain control over the growth of matrix coefficients of π. Theorem 8.47 from [6] is the one we will use in our estimations:

Theorem 4 *Suppose π is an irreducible admissible representation of G. Then the following conditions on a weight $v_0 \in \mathfrak{a}_{\mathbb{R}}^*$ are equivalent:*

1. *We have $\Re(v) \preceq v_0$ for every leading exponent v of π;*
2. *There is an integer $q \geq 0$ such that for every $u, v \in \pi$, there exists a constant C such that*

$$\pi_{u,v}(\exp(H)) \leq C \exp((v_0)H)(1 + \|H\|)^q, \qquad (\forall\ H \in \overline{\mathfrak{a}^+})$$

3. *There is an integer $q \geq 0$ such that for every $u, v \in \pi$, there exists a constant C such that*

$$\pi_{u,v}(g) \leq C \exp((v_0)(H(g)))(1 + \|H(g)\|)^q, \qquad (\forall\ g \in G)$$

From now on, C or c will be used as symbolic constant. Let L and R represent left and right regular representation of G. According to [6] Theorem 8.47, we have

Theorem 5 . *Suppose π is an irreducible admissible representation of G and v_0 satisfies one of the conditions in Theorem 4. Let $x \in U(\mathfrak{g})$. Then the estimates of Theorem 4 hold for every $L(x)\pi_{u,v}$ and $R(x)\pi_{u,v}$.*

From now on, we will denote the space of matrix coefficients of π by $\mathcal{M}(\pi)$.

3. Oscillator Representation

3.1. *Schrödinger model*

Let G be the standard symplectic group of a $2n$ dimensional vector space V. Let $K = Sp_{2n}(\mathbb{R}) \cap O_{2n}(\mathbb{R})$ be the maximal compact subgroup. It is the unitary group if we properly equip V with a complex structure. It is well-known that $SU(n)$ is simply connected. Therefore, $\pi(G) = \pi(U(n)) = \mathbb{Z}$. Thus there exists a unique double covering of G, and we denote it by MG. MG is often called the metaplectic group. Let H be a subgroup of G. We denote the preimage of H under the metaplectic covering by MH.

The most interesting property of MG is its presentation as unitary operators on $L^2(\mathbb{R}^n)$. This presentation is often called the Schrödinger model. The exact form of this presentation is hard to write down. But the infinitesimal generators in $\mathfrak{g}$ can be written down explicitly. We write

$$e(i,j)_{k,l} = \delta_i^k \delta_j^l$$

where δ_i^k is the Kronecker symbol. We define a basis in $\mathfrak{g}$:

$$X_{k,-j} = e(k,j) - e(n+j, n+k)$$

$$X_{k,j} = e(k, n+j) + e(j, n+k)$$

$$X_{-k,-j} = e(n+k, j) + e(n+j, k)$$

Definition 2 (Schrödinger Model) *There exists a unitary representation ω of MG on $L^2(\mathbb{R}^n)$ such that*

$$\omega(X_{k,-j}) = x_k \partial x_j + \frac{1}{2}\delta_k^j;$$

$$\omega(X_{k,j}) = i x_k x_j \ (k \neq j) \qquad \omega(X_{k,k}) = \frac{i}{2} x_k^2$$

$$\omega(X_{-k,-j}) = i \partial x_k \partial x_j \ (k \neq j) \qquad \omega(X_{-k,-k}) = \frac{i}{2} \partial^2 x_k$$

Let $GL(n)$ be the subgroup of $Sp_{2n}(\mathbb{R})$ of the following form

$$\begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix}$$

Then

$$MGL(n) = \{(g, \xi) \mid g \in GL(n), \xi^2 = \det g\}$$

$$\omega(g,\xi)(f)(x) = \xi f(g^t x) \qquad ((g,\lambda) \in MGL(n))$$

See [17] Ch 5.3 and 5.5 for more details.

Now, we look at the Harish-Chandra module attached to the Schrödinger model. Since MK is connected, the MK-finite vectors are those $\mathfrak{k}$-finite vectors. From now on, $\frac{\partial}{\partial x_i}$ will be denoted as ∂_i.

Theorem 6 *Let $\mu(x) = \exp(-\frac{1}{2}\|x\|^2)$. Then the Harish Chandra module in the Schrödinger model is given by $\mu(x)\mathcal{P}$, where $\mathcal{P}$ is the polynomial algebra of n real variables.*

Proof: Let $\mathcal{P}^{\leq m}$ be the space of polynomials with degree less or equal to m. We want to show that $X_{i,-j} - X_{j,-i}, X_{i,j} - X_{-i,-j}$ act on $\mu(x)\mathcal{P}^{\leq m}$ for all $i, j \in [1, n]$. For $p(x) \in \mathcal{P}$, and $i \neq j$, we have

$$\begin{aligned}(x_i x_j - \partial_i\partial_j)(\mu(x)p(x)) =& x_i x_j \mu(x)p(x) - x_i x_j \mu(x)p(x) + x_i\mu(x)\partial_j p(x)) \\ &+ x_j\mu(x)\partial_i p(x) - \mu(x)\partial_i\partial_j p(x) \\ =&(x_i\partial_j p(x) + x_j\partial_i p(x) - \partial_i\partial_j p(x))\mu(x)\end{aligned} \tag{3.1}$$

Thus $\mu(x)\mathcal{P}^{\leq m}$ is preserved by the action of $\{X_{i,j} - X_{-i,-j}\}_{i\neq j}$. For $i = j$, we have,

$$\begin{aligned}(x_i^2 - (\partial_i)^2)(\mu(x)p(x)) =& x_i^2\mu(x)p(x) + \mu(x)p(x) - x_i^2\mu(x)p(x) + x_i\mu(x)\partial_i p(x) \\ &- \mu(x)\partial_i^2 p(x) + x_i\mu(x)\partial_i p(x) \\ =&(2x_i\partial_i p(x) + p(x) - \partial_i^2 p(x))\mu(x)\end{aligned} \tag{3.2}$$

Again, $\mu(x)\mathcal{P}^{\leq m}$ is preserved by $X_{i,i} - X_{-i,-i}$. Therefore, $\mu(x)\mathcal{P}^{\leq m}$ is preserved by $\{X_{i,j} - X_{-i,-j}\}$ for all $i, j \in [1, n]$. Notice that $X_{i,-j} - X_{j,-i}$ acts homogeneously on $\mathcal{P}$. Thus $\mu(x)\mathcal{P}^{\leq m}$ is preserved by $\mathfrak{k}$. Hence $\mu(x)\mathcal{P}$ is contained in the MK-finite subspace of $L^2(\mathbb{R}^n)$. One also can prove easily that $\mu(x)\mathcal{P}$ is dense in $L^2(\mathbb{R}^n)$. Therefore $\mu(x)\mathcal{P}$ is the Harish-Chandra module of the Schrödinger model. □

There are other models for oscillator representation, notably the Fock model which we will not introduce here. Even though the Harish-Chandra modules of the Schrödinger model and the Fock model can be identified with the polynomials of n variables, the modules structures are quite different. For example, it is difficult to write down exactly the MK-types in Schrödinger model, since the MK action does not preserve homogeneity.

3.2. *Matrix coefficients of the oscillator eepresentation*

Let g be an element of G. For simplicity, we use $\widetilde{g}$ to denote any preimage of g under the metaplectic covering when it causes no confusion. Now we can compute the matrix coefficients of the oscillator representation in terms of the Schrödinger model. We follow the multi-index convention.

Theorem 7 *Let $\tilde{a} = (a, \xi)$. Let $a = diag(a_1, a_2, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}) \in A^+ \subseteq Sp_{2n}$, i.e.,*

$$a_1 > a_2 > \ldots > a_n > 1$$

and let

$$(a, \xi) \in MG \qquad (\xi^2 = \prod_1^n a_i)$$

Then $MA \cong (A, \pm 1)$. *We write* $sgn(a, \xi) = sgn(\xi)$. *Then*

$$\begin{aligned}\omega_{\alpha,\beta}(\widetilde{a}) &= (\omega(\widetilde{a})x^{\alpha}\mu(x), x^{\beta}\mu(x)) \\ &= sgn(\widetilde{a})c_{\alpha,\beta}\prod_{i=1}^{n} a_i^{-\beta_i - 1/2}(1 + a_i^{-2})^{-(\alpha_i+\beta_i+1)/2} \\ &= sgn(\widetilde{a})c_{\alpha,\beta}\prod_{i=1}^{n} a_i^{\alpha_i + 1/2}(1 + a_i^{2})^{-(\alpha_i+\beta_i+1)/2}\end{aligned} \tag{3.3}$$

These formulae yield the asymptotic expansions for $\{(a_1, \ldots, a_n) \mid a_i > 1\}$ *and* $\{(a_1, \ldots, a_n) \mid a_i < 1\}$ *respectively. Both domains contain* $n!$ *Weyl chambers of* A. *Moreover, the leading exponent of the oscillator representation is*

$$(-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2})$$

and the infinitesimal character is given by

$$(n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2})$$

Proof: Let $c_{\alpha} = \int x^{\alpha}\mu(x)dx$. Then

$$\begin{aligned}\omega_{\alpha,\beta}(\widetilde{a}) =& sgn(\xi)(\prod_{i=1}^{n} a_i)^{\frac{1}{2}}(a^{\alpha}x^{\alpha}\exp(-\frac{1}{2}\sum a_i^2 x_i^2), x^{\beta}\mu(x)) \\ =& sgn(\xi)(\prod_{i=1}^{n} a_i)^{\frac{1}{2}} a^{\alpha}\int x^{\alpha+\beta}\exp(-\frac{1}{2}\sum(a_i^2+1)x_i^2)dx \\ =& sgn(\xi)(\prod_{i=1}^{n}(a_i))^{\frac{1}{2}} a^{\alpha}\prod_{i=1}^{n}(a_i^2+1)^{-\frac{\alpha_i+\beta_i+1}{2}}\int x^{\alpha+\beta}\mu(x)dx \\ =& sgn(\xi)c_{\alpha+\beta}\prod_{i=1}^{n}(a_i)^{\alpha_i+\frac{1}{2}}(a_i^2+1)^{-\frac{\alpha_i+\beta_i+1}{2}} \\ =& sgn(\xi)c_{\alpha+\beta}\prod_{i=1}^{n}(a_i)^{-\beta_i-\frac{1}{2}}(a_i^{-2}+1)^{-\frac{\alpha_i+\beta_i+1}{2}}\end{aligned} \tag{3.4}$$

Thus the unique leading exponent of the oscillator representation is

$$v = (-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2})$$

Therefore the infinitesimal character is given by

$$v + \rho = (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2})$$

□

Finally, we will give a theorem on the growth condition of the matrix coefficients of the Schrödinger model.

Theorem 8 *Let $g = k_1 \exp(H(g)) k_2$ with $H(g) \in \overline{\mathfrak{a}^+}$ be the KAK decomposition of G. Then for every u, v in the Harish-Chandra module of the Schrödinger model, there exists a constant c such that*

$$|\omega_{u,v}(\tilde{g})| \leq c \exp(-\frac{1}{2}\sum_{i=1}^{n} H_i(g)) \qquad (H(g) = diag(H_1(g), \dots, H_n(g), -H_1(g), \dots -H_n(g)))$$

The same estimates hold for every $L(x)\omega_{u,v}(\tilde{g})$ and $R(x)\omega_{u,v}(\tilde{g})$, where $x \in \mathfrak{g}$ and L and R represent the left and right regular actions.

Proof: We only sketch a proof here. For an arbitrary $w \in \mu(x)\mathcal{P}$, since $\{x^\alpha \mu(x)\}_{\alpha \in \mathbb{N}^n}$ is a basis for $\mu(x)\mathcal{P}$, we write

$$w = \sum w_\alpha x^\alpha \mu(x)$$

Notice that

$$\omega_{u,v}(\tilde{g}) = (\omega(\tilde{g})u, v) = (\omega(\exp(H(g)))\omega(\tilde{k}_2)u, \omega(\tilde{k}_1^{-1})v)$$

Suppose the finite dimensional MK-subspace spanned by u and v is contained in $\mu(x)\mathcal{P}^{\leq m}$. According to the previous theorem, for every pair $(\omega(\tilde{k}_2)u, \omega(\tilde{k}_1^{-1})v)$, there exists a constant $C(\tilde{k}_2, \tilde{k}_1)$ such that

$$|(\omega(\exp(H(g)))\omega(\tilde{k}_2)u, \omega(\tilde{k}_1^{-1})v)| \leq C(\tilde{k}_2, \tilde{k}_1) \exp(-\frac{1}{2}\sum_{i=1}^{n} H_i(g))$$

Here we may choose

$$C(\tilde{k}_2, \tilde{k}_1) = \sum_{\{\alpha,\beta \mid \sum \alpha_i \leq m, \sum \beta_j \leq m\}} |(\omega(\tilde{k}_2)u)_\alpha||(\omega(\tilde{k}_1^{-1})v)_\beta| c_{\alpha,\beta}$$

Now since $(\tilde{k}_2, \tilde{k}_1) \in MK \times MK$ is compact, and $C(\tilde{k}_2, \tilde{k}_1)$ is a continuous function, $c = \max\{C(\tilde{k}_2, \tilde{k}_1) \mid \tilde{k}_2 \in MK, \tilde{k}_1 \in MK\}$ exists. Then we obatin

$$|(\omega(\exp(H(g)))\omega(\tilde{k}_2)u, \omega(\tilde{k}_1^{-1})v)| \leq c \exp(-\frac{1}{2}\sum_{i=1}^{n} H_i(g))$$

The first statement is proved. Since ω is unitary,

$$L(x)(\omega(\tilde{g})u, v) = -(\omega(x)\omega(\tilde{g})u, v) = (\omega(\tilde{g})u, \omega(x)v) = \omega_{u,\omega(x)v}(\tilde{g})$$

$$R(x)(\omega(\tilde{g})u, v) = (\omega(\tilde{g})\omega(x)u, v) = \omega_{\omega(x)u,v}(\tilde{g})$$

Then the first statement implies the second statement. $\square$

For oscillator representation, the growth condition estimate in this theorem is in fact stronger that the growth condition in Theorem 4.

4. Basics of Theta Correspondence

4.1. *Dual pairs*

Let (G_1, G_2) be a pair of subgroups of the symplectic group $Sp(W)$. We say (G_1, G_2) is a reductive dual pair if

1. G_1 and G_2 act reductively on W;

2. G_1 is the centralizer of G_2, G_2 is the centralizer of G_1 in $Sp(W)$

If (G_1, G_2) is a reductive dual pair in $Sp(V, \Omega)$, and if $V = V_1 \oplus V_2$ is a direct sum decomposition such that $\Omega(V_1, V_2) = 0$ and V_1, V_2 are $G_1 \cdot G_2$ invariant, then we say that (G_1, G_2) is reducible. Here $\Omega|_{V_i}$ is automatically nondegenerate. Then $(G_1|_{V_i}, G_2|_{V_i})$ is a reductive dual pair in $Sp(V_i, \Omega|_{V_i})$. Now we see that (G_1, G_2) can be regarded as the direct sum of $(G_1|_{V_i}, G_2|_{V_i})$. Essentially, every reductive dual pair can be decomposed as the direct sum of irreducible ones. All the irreducible dual pairs are classified in [5].

Theorem 9 *There are two types of irreducible reductive dual pairs. (G_1, G_2) is a dual pair of type I means that there exist*

1. *a division algebra D over F,with involution $\sharp$;*

2. *D-modules V_1 and V_2, with non-degenerate $\sharp$-sesqulinear form $(,)_1$ and $(,)_2$; one $\sharp$-hermitian, and the other $\sharp$-skew-Hermitian; G_1 and G_2 are the isometry groups respectively;*

3. *$W = V_1 \otimes_D V_2$, $< a \otimes b, c \otimes d >= Tr_{D/F}((a,c)_1 (b,d)_2^\sharp)$, then $<,>$ is a symplectic form on W, and $Sp = Sp(W)$ is the isometry group of $<,>$.*

(G_1, G_2) is a dual pair of type II means that there exist two $G_1 G_2$-stable Langragian subspaces X and Y such that $W = X \oplus Y$ and there exist

1. *a division algebra D over F;*

2. *left D-module V_1 and right D-module V_2;*

3. *$X = V_1 \otimes_D V_2$;*

4. *(G_1, G_2) is identified with $GL_D(V_1), GL_D(V_2)$ in $Sp(W)$*

For real classical groups, all the dual pairs were made explicit in [5].

Theorem 10 (Howe) *There are seven families of irreducible reductive dual pairs.*

1. *$D = \mathbb{R}$, $\sharp = id$, $(O(p,q), Sp_{2n}(\mathbb{R})) \subseteq Sp_{2(p+q)n}$*

2. *$D = \mathbb{C}$, $\sharp = id$, $(O(p,\mathbb{C}), Sp_{2n}(\mathbb{C})) \subseteq Sp_{4pn}$*

3. $D = \mathbb{C}$, *$\sharp$ = conjugation,* $(U(p,q), U(r,s)) \subseteq Sp_{2(p+q)(r+s)}$

4. $D = \mathbb{H}$,*$\sharp$ = conjugation,* $(Sp(p,q), O^*(2n)) \subseteq Sp_{4n(p+q)}$

5. $D = \mathbb{R}$, $(GL(m,\mathbb{R}), GL(n,\mathbb{R})) \subseteq Sp_{2nm}$

6. $D = \mathbb{C}$, $(GL(m,\mathbb{C}), GL(n,\mathbb{C})) \subseteq Sp_{4mn}$

7. $D = \mathbb{H}$, $(GL(m,\mathbb{H}), GL(n,\mathbb{H})) \subseteq Sp_{8mn}$

The first four are type I (irreducible) dual pairs , and the last three are type II (irreducible) dual pairs.

Let G be a symplect group. Let MG be the metaplectic covering of G. Let H be a subgroup of G. Let MH be the preimage of H under the metaplectic covering. Let $\{1, \epsilon\}$ be the preimage of the identity in G. Let ω be the oscillator representation. Let $\mathcal{P}$ be the Harish-Chandra module of ω. Then we will always have

$$\omega(\epsilon) = -1$$

Thus we may use $\mathcal{U}(MH)(\epsilon)$ $(\mathcal{R}(MH)(\epsilon))$to denote the unitary (admissible) dual of MH such that $\pi(\epsilon) = -1$ holds. We shall keep in mind that this equation is always true in this paper.

4.2. *Theta correspondence*

Let H be a reductive subgroup in G, and K a maximal compact subgroup of H. Let $\mathcal{R}(MH, \mathcal{P})$ be the set of isomorphism classes of irreducible $(\mathfrak{h}, MK)$-modules which occurs as a quotient $\mathcal{P}/\mathcal{N}$, for some $(\mathfrak{h}, MK)$-submodule $\mathcal{N}$. For $\pi \in \mathcal{R}(MH, \mathcal{P})$, let $\mathcal{N}_\pi$ be the intersection of all such $\mathcal{N}$ satisfying $\mathcal{P}/\mathcal{N} \cong \pi$.

Definition 3 *Let (G_1, G_2) be a dual pair in G. Suppose $\pi \in \mathcal{R}(MG_1, \mathcal{P})$. Then $\mathcal{P}/\mathcal{N}_\pi$ is a $(\mathfrak{g}_1 \times \mathfrak{g}_2, M(K_1 \times K_2))$-module. We write*

$$\mathcal{P}/\mathcal{N}_\pi \cong \pi \otimes \theta_0(\pi)$$

where $\theta_0(\pi)$ is a $(\mathfrak{g}_2, MK_2)$-module.

I must make a comment here. Strictly speaking, the tensor product $\pi \otimes \theta_0(\pi)$ is NOT a reprsentation of $MG_1 \times MG_2$, but a representation of $M(G_1G_2)$. It is a twisted tensor product in the following sense. Since π and $\theta_0(\pi)$ are quotients of $\mathcal{P}$, we must have

$$\pi(\epsilon) = -1 \qquad \theta_0(\pi)(\epsilon) = -1$$

We recall that we have the natural fibration

$$\mathbb{Z}/2 \to MG_1 \times MG_2 \to M(G_1G_2)$$

The projection map is just the multiplication in MG. Then $\pi \otimes \theta_0(\pi)$ descends into a representation of $M(G_1G_2)$. This is going to be our viewpoint in interpreting our tensor products.

Theorem 11 (Howe) *The $(\mathfrak{g}_2, MK_2)$-module $\theta_0(\pi)$ is finitely generated, admissible and quasisimple. It has a composition series of finite length. $\theta_0(\pi)$ has a unique irreducible quotient $\theta(\pi)$. The correspondence*

$$\theta : \pi \to \theta(\pi)$$

defines a bijection from $\mathcal{R}(MG_1, \mathcal{P})$ to $\mathcal{R}(MG_2, \mathcal{P})$.

A proof of this theorem can be found in [5]. This correspondence is often called theta correspondence, dual pair correspondence or Howe's correspondence. Sometimes we will write $\mathcal{R}(MG_i, \mathcal{P})$ as $\mathcal{R}(MG_i, \omega)$. In the stances when we want to distinguish different ω's, we will use subscript like ω_j.

The most well-known version of theta correspondence is stated as follows.

Theorem 12 *Let (G_1, G_2) be a reductive dual pair in a symplectic group G. Let ω be the oscillator representation of MG. The set $\mathcal{R}(M(G_1G_2), \omega)$, when its element is decomposed as a (twisted) tensor product of MG_1 and MG_2 representations, produces a one-to-one correspondence between $\mathcal{R}(MG_1, \omega)$ and $\mathcal{R}(MG_2, \omega)$.*

4.3. *Useful Results*

We dig into Howe's original paper "transcending Classical Invariant Theory"for some detailed results. We will let $(\omega, \mathcal{P})$ be the Fock model. $\mathcal{P}$ will be the polynomial ring of complex variables $(z_1, z_2, \dots)$. The Lie algebra $\mathfrak{g}_{\mathbb{C}}$ can be decomposed as

$$\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+_{\mathbb{C}} \oplus \mathfrak{p}^-_{\mathbb{C}}$$

The compact component $\mathfrak{k}_{\mathbb{C}}$ acts on $\mathcal{P}$ homogeneously. $\mathfrak{p}^+_{\mathbb{C}}$ acts by multiplication of 2nd degree polynomials and $\mathfrak{p}^-_{\mathbb{C}}$ acts by 2nd order differential operators with constant coefficient. We begin with the following definitions.

Definition 4 *For any compact subgroup ML of MK, let $\sigma \in \mathcal{U}(ML)$. We define $\mathcal{P}_\sigma$ to be the σ-isotypic subspace of $\mathcal{P}$. We define the degree of σ to be the minimal degree of $\mathcal{P}_\sigma$.*

The degree of σ is well-defined since $\mathcal{P}_\sigma$ is homogeneous.

Definition 5 *For a fixed compact subgroup ML of MK, let $\mathcal{D}(ML)$ be the space of ML-invariant differential operators with constant coefficients on $\mathcal{P}$ (without constant terms). We define the ML-harmonic functions to be those functions in $\mathcal{P}$ which are annihilated by $\mathcal{D}(ML)$. We denote this space by $\mathcal{H}(ML)$.*

We can now get back to our dual pair setting from our previous section. We listed some results from [5].

Lemma 2 *Let $\mathcal{N} \in \mathcal{P}$ be an arbitrary invariant $(\mathfrak{g}_1 \times \mathfrak{g}_2, M(K_1K_2))$ submodule of $\mathcal{P}$. Let $\mathcal{P}^d$ be the space of polynomials of degree at most d. Let $\deg(\mathcal{N})$ be the largest integer d such that $\mathcal{P}^d \subseteq \mathcal{N}$. Then*

1. *The number* $\deg(\mathcal{N})$ *is determined solely by the structure of* $\mathcal{P}/\mathcal{N}$ *as a* MK_1*-module, or as a* MK_2*-module. One has*

$$1+\deg(\mathcal{N}) = \min\{\deg\sigma_1 : \sigma_1 \in \mathcal{R}(MK_1, \mathcal{P}/\mathcal{N})\} = min\{\deg\sigma_2 : \sigma_2 \in \mathcal{R}(MK_2, \mathcal{P}/\mathcal{N})\}$$

   Where $\mathcal{R}(MK_i, \mathcal{P}/\mathcal{N})$ *is the set of irreducible representations of* MK_i *occurring in* $\mathcal{P}/\mathcal{N}$*;*

2. *If* $\sigma_1 \in \mathcal{R}(MK_1, \mathcal{P}/\mathcal{N})$*, then there is at least one* $\sigma_2 \in \mathcal{R}(MK_2, \mathcal{P}/\mathcal{N})$ *such that*

$$\mathcal{H}(MK_1)_{\sigma_1} \cap \mathcal{P}_{\sigma_2} \neq \{0\};$$

3. *If* $\sigma_1 \in \mathcal{R}(MK_1, \mathcal{P}/\mathcal{N})$ *and* $\deg(\sigma_1) = \deg(\mathcal{N})+1$*, then* $\mathcal{H}(MK_1)_{\sigma_1} \cap \mathcal{H}(MK_2)$ *is nontrivial and it is equivalent to* $\sigma_1 \otimes \sigma_2$ *for some* $\sigma_2 \in \mathcal{R}(MK_2, \mathcal{P}/\mathcal{N})$*. Furthermore, no other representation of* MK_2 *occurring in* $\mathcal{H}(MK_1)_{\sigma_1}$ *occurs in* $\mathcal{P}/\mathcal{N}$*. Finally,* $\mathcal{H}(MK_1)_{\sigma_1} \cap \mathcal{H}(MK_2)$ *generates* $(\mathcal{P}/\mathcal{N})_{\sigma}$*.*

Now we apply the above lemma to the case where $\mathcal{N} = \mathcal{N}_\pi$. The following theorem is only different from the lemma because of the fact that $\mathcal{P}/\mathcal{N}_\pi \cong \pi\otimes\theta_0(\pi)$. Therefore, we have a little bit more to say.

Theorem 13

1. *Let* σ_1 *be a* MK_1*-type in* π *with the lowest degree. Let* σ_2 *be a* MK_2*-type in* $\theta_0(\pi)$ *with the lowest degreee. Then* $\deg(\sigma_1) = \deg(\sigma_2)$*.*

2. *Let* σ_1 *be a* MK_1*-type in* π *with the lowest degree. Then there exists a unique* MK_2*-type* σ_2 *in* $\theta_0(\pi)$*, such that*

$$\mathcal{H}(MK_1)_{\sigma_1} \cap \mathcal{H}(MK_2)_{\sigma_2} \cong \sigma_1 \otimes \sigma_2$$

   Here the tensor product is again to be understood as a twisted tensor product. We denote $\mathcal{H}(MK_1)_{\sigma_1} \cap \mathcal{H}(MK_2)_{\sigma_2}$ *by* $\mathcal{H}_{\sigma_1,\sigma_2}$*. Now we fix this pair* (σ_1, σ_2)*.*

3. $(\mathcal{P}/\mathcal{N}_\pi)_{\sigma_1}$ *is generated as a* $(\mathfrak{g}_2, MK_2)$*-module by* $\mathcal{H}_{\sigma_1,\sigma_2}$*.*

4. $\mathcal{P}/\mathcal{N}_\pi$ *is generated as a* $(\mathfrak{g}_1 \times \mathfrak{g}_2, M(K_1K_2))$*-module by* $\mathcal{H}_{\sigma_1,\sigma_2}$*.*

5. *The multiplicity of* σ_1 *in* π*, the multiplicity of* σ_2 *in* $\theta(\pi)$*, and the multiplicity of* σ_2 *in* $\theta_0(\pi)$ *are all one.*

The first fours items can be found in [5]. The statement (5) is not that obvious. First of all the multiplicity of σ_1 in π must be 1. Otherwise, $(\mathcal{P}/\mathcal{N}_\pi)_{\sigma_1}$ must contain two copies of $\sigma_1 \otimes \theta_0(\pi)$ which can not be generated by $\mathcal{H}_{\sigma_1,\sigma_2} \cong \sigma_1 \otimes \sigma_2$ as a $(\mathfrak{g}_2, MK_2)$-module.

From the Lemma, we also know that $(\mathcal{P}/\mathcal{N}_\pi)_{\sigma_2}$ is also generated by $\mathcal{H}_{\sigma_1,\sigma_2}$. It follows from the same argument that the multiplicity of σ_2 in $\theta_0(\pi)$ is one.

Since $\mathcal{P}/\mathcal{N}_\pi$ is generated as a $(\mathfrak{g}_1 \times \mathfrak{g}_2, M(K_1K_2))$-module by $\mathcal{H}_{\sigma_1,\sigma_2}$, it must also generates the unique quotient of $\mathcal{P}/\mathcal{N}_\pi$. Therefore σ_2 is a MK_2-type in $\theta(\pi)$. Furthermore, it has the lowest degree in $\theta_0(\pi)$, thus in $\theta(\pi)$. By the same argument applying to $\theta(\pi)$, we see that the multiplicity of σ_2 in $\theta(\pi)$ is 1. $\square$

To summarize the results we have covered in this chapter, the most important fact is that $\mathcal{H}_{\sigma_1,\sigma_2}$ is the unique copy of $\sigma_1 \otimes \sigma_2$ which generate the quotients $\pi \otimes \theta_0(\pi)$ and $\pi \otimes \theta(\pi)$ in $\mathcal{P}$.

5. Semistable Range

Starting from this chapter, we will try to apply analytic tools we laid out in Chapter 2 and 3 to study theta correspondence. Roughly speaking, semistable range is the range where computation of theta correspondence can be effectively carried out. It should also be said that semistable range contains stable range as defined in ([9], [10]).

5.1. *Preliminary*

Let G be a semisimple Lie group. Let K be a maximal compact subgroup of G. Let π be a $(\mathfrak{g}, K)$-module. In this paper, π will represent both the representation and the underlying module. If we feel a notation for $(\mathfrak{g}, K)$-module is absolutely necessary, we will use V_π to denote it. We define π^c to be the same $(\mathfrak{g}, K)$-module but equipped with conjugate linear scalar multiplication (from right). If π is admissible, then the matrix coefficients of π^c will simply be the conjugate of matrix coefficients of π.

We can also define π^*, the dual of π, again in the category of $(\mathfrak{g}, K)$-modules. The underlying vector space will be the K-finite linear functionals on π. For any $x \in \mathfrak{g}$ and $\eta \in \pi^*$, we define

$$(\pi^*(x)(\eta))(u) = -\eta(\pi(x)u) \qquad (u \in \pi)$$

For any $k \in K$, we define

$$(\pi^*(k)(\eta))(u) = \eta(\pi(k^{-1})u) \qquad (u \in \pi)$$

Then π^* is a $(\mathfrak{g}, K)$-module. Again if π is admissible, π^* will also be admissible and it can be equipped with a K-invariant inner product. The matrix coefficients of π and π^* are related in the following fashion

$$f \in \mathcal{M}(\pi) \Leftrightarrow f(g^{-1}) \in \mathcal{M}(\pi^*)$$

Now we can further define the Hermitian dual to be

$$\pi^h = (\pi^*)^c = (\pi^c)^*$$

Then π is unitary admissible implies that π^h is unitarily equivalent to π.

Finally, for any $(\mathfrak{g}, K)$-module π_1 and π_2, let $Hom\,(\pi_1, \pi_2)$ be the space of homomorphisms from π_1 to π_2. We always eqip it with the canonical $(\mathfrak{g}, K)$-module structure. We denote the $(\mathfrak{g}, K)$-invariant subspace of $Hom\,(\pi_1, \pi_2)$ by $Hom\,_{\mathfrak{g},K}(\pi_1, \pi_2)$.

5.2. *Averaging Matrix coefficients*

Let (G_1, G_2) be an irreducible dual pair in a symplectic group G. Let $(\omega, \mathcal{P})$ be the oscillator representation of MG. Let π be an irreducible admissible representation of MG_1. Now we may formally define a complex linear pairing

$$(\mathcal{P} \otimes \pi^*, \mathcal{P}^c \otimes \pi) \rightarrow \mathbb{C}$$

as follows: for $\phi \in \mathcal{P}, \psi \in \mathcal{P}^c, \eta \in \pi^*, u \in \pi$

$$(\phi \otimes \eta, \psi \otimes u)_\pi = \int_{MG_1} (\phi, \omega(g)\psi)\eta(\pi(g)u)dg$$

We fix a KAK decomposition of G such that G_i can be decomposed as $K_iA_iK_i$ where $K_i = K \cap G_i$ is a maximal compact subgroup of G_i and $A_i = A \cap G_i$ is a maximal split subgroup of G_i. We fix a positive restriced root system in $\mathfrak{a}_{\mathbb{R}}^*$. This induces a positive root system on $\mathfrak{a}_i^*$. More precisely, α is a positive root in $\mathfrak{a}_i^*$ if and only if it is the restriction of a positive root in $\mathfrak{a}_{\mathbb{R}}^*$ onto $\mathfrak{a}_i$. We fix a $(MK_1)\overline{A_1^+}(MK_1)$ decomposition of MG_1 which is induced from $K_1\overline{A_1^+}K_1$ decomposition. Notice that A_1^+ remains the same. Therefore, our estimation on the matrix coefficients on MG_1 is pretty much equivalent to the estimation on G_1. Now we compute $(,)_\pi$

$$\begin{aligned}
&(\phi \otimes \eta, \psi \otimes u)_\pi \\
=& \int_{MK_1 \times \mathfrak{a}_1^+ \times MK_1} (\phi, \omega(k_1 \exp Hk_2)\psi)\eta(\pi(k_1 \exp Hk_2)u)\Delta(H)dk_1dHdk_2 \\
=& \int_{MK_1 \times \mathfrak{a}_1^+ \times MK_1} (\omega(k_1^{-1})\phi, \omega(\exp H)\omega(k_2)\psi)(\pi^*(k_1^{-1})\eta)(\pi(\exp H))(\pi(k_2))u)dk_1dHdk_2 \\
=& \int_{\mathfrak{a}_1^+} (\int_{MK_1} \omega(k_1^{-1})\phi \otimes \pi^*(k_1^{-1})\eta dk_1, (\omega \otimes \pi)(\exp H) \int_{MK_1} (\omega \otimes \pi)(k_2)(\psi \otimes u))dk_2)\Delta(H)dH
\end{aligned} \tag{5.5}$$

Since ϕ, ψ, η, u are all MK_1-finite, it follows that

$$\int_{MK_1} (\omega(k_1^{-1})\phi \otimes \pi^*(k_1^{-1})\eta dk_1; \qquad \int_{MK_1} (\omega \otimes \pi(k_2))(\psi \otimes u)dk_2$$

are in $\mathcal{P} \otimes \pi^*$ and $\mathcal{P}^c \otimes \pi$. Therefore, we obtain the following theorem

Theorem 14 . *If for every $\phi \in \mathcal{P}, \psi \in \mathcal{P}^c, \eta \in \pi^*, u \in \pi$*

$$\int_{\mathfrak{a}_1^+} (\phi, \omega(\exp H)\psi)\eta(\pi(\exp H)u)\Delta(H)dH \tag{5.6}$$

converges, then $(,)_{MG_1}$ is a well-defined pairing between $\mathcal{P} \otimes \pi^*$ and $\mathcal{P}^c \otimes \pi$.

Now, from Theorem 4, we can assume that $\eta(\pi(\exp H)u)$ is controlled by

$$c(1+\|\exp H\|)^q \exp(v(H)) \qquad (H \in \mathfrak{a}_1^+)$$

for some $v \in \mathfrak{a}_{1\mathbb{R}}^*$. We also have the following estimation of $\Delta(H)$:

$$\Delta(H) \leq c \exp(2\rho(MG_1)(H)) \qquad (H \in \mathfrak{a}_1^+)$$

Here we use $\rho(MG_1)$ to denote the half of the sum of positive restricted roots in MG_1 with multiplicity. Then $\rho(MG_1)$ is in $\mathfrak{a}_1^*$. We state the following theorem

Theorem 15 . *Let $\lambda(\omega)$ be the leading exponent $(-\frac{1}{2}, \ldots, -\frac{1}{2})$ of the oscillator representation ω. Let π be an admissible irreducible representation of MG_1 such that the matrix coefficients of π is controlled by*

$$c(1+\|\exp H\|)^q \exp(v(H))$$

for some $v \in \mathfrak{a}_{1\mathbb{R}}^*$. If $v + 2\rho(MG_1) + \lambda(\omega)|_{\mathfrak{a}_1}$ is a strictly negative combination of the simple roots in $\Delta(MG_1)$, then $(\mathcal{P} \otimes \pi^*, \mathcal{P}^c \otimes \pi)_\pi$ is well-defined.

Proof: Let r be the real rank of MG_1. Suppose that

$$v + 2\rho(MG_1) + \lambda(\omega)|_{\mathfrak{a}_1} = \sum_{\alpha_i \in \Delta} c_i \alpha_i$$

with $c_i < 0$. Then the integral 5.6 is bounded by

$$c \int_{\mathbb{R}^{+r}} (1 + \sum x_i)^q \exp(\sum_{i=1}^{r} c_i x_i) dx_1 \ldots dx_r < \infty$$

Here $x_i = \alpha_i(H)$.

We may also use spherical coordinates on $\mathfrak{a}^+$. Let s be the radial coordinate, taken to be the norm induced from the invariant form on $\mathfrak{g}$. Then there exists a strictly negative constant d such that

$$(v + 2\rho(MG_1) + \lambda(\omega))(H) \leq d\|H\| \qquad (H \in \mathfrak{a}^+)$$

Then the integral 5.6 is bounded by

$$c \int_{\mathbb{R}+} (1+s)^m \exp(ds) ds < \infty$$

for some $m > 0$. $\square$

Definition 6 *If an irreducible admissible representation π satisfies the assumptions in the theorem above, in other words, if every leading exponent v of π satisfies the condition that $v+2\rho(MG_1)+\lambda(\omega)|_{\mathfrak{a}_1}$ is a strictly negative combination of the positive roots in $\Delta(MG_1)$, we say that π is in the semistable range of (MG_1, MG_2).*

Based on our definition of semistability, we will be able to define the averaging operator $\mathcal{L}_\pi$. I will prove in the next chapter that $\mathcal{L}_\pi$ is an interwinning operator of $(\mathfrak{g}_2, MK_2)$-modules.

6. Averaging Operators

From now on, we assume that π is an irreducible admissible representation in the semistable range of (MG_1, MG_2).

6.1. *Definition*

Let $Hom\,(\pi^*, \mathbb{C})$ be the space of linear functionals on π^* (not necessarily MK_1-finte). We observe that the MK_1-finite linear functionals in $Hom\,(\pi^*, \mathbb{C})$ are exactly π. Here, $k_1 \in MK_1$ acts on $w \in Hom\,(\pi^*, \mathbb{C})$ as follows:

$$k_1 : w \to (\delta \in \pi^* \to w(\pi^*(k^{-1})\delta)$$

Now we consider the map

$$\mathcal{L}_\pi^0 : \mathcal{P}^c \otimes \pi \to Hom\,(\mathcal{P}, Hom\,(\pi^*, \mathbb{C}))$$

defined by

$$(\mathcal{L}_\pi^0(\psi \otimes u)(\phi))(\delta) = (\phi \otimes \delta, \psi \otimes u)_\pi$$
$$\forall \quad \phi \in \mathcal{P}, \delta \in \pi^*, \psi \in \mathcal{P}^c, u \in \pi$$

Lemma 3 *The image of $\mathcal{L}_\pi^0$ are MK_1-equivariant maps from $\mathcal{P}$ to $Hom\,(\pi^*, \mathbb{C})$.*

Proof: For $k \in MK_1$, $\psi \in \mathcal{P}^c$, $\phi \in \mathcal{P}$, $u \in \pi$, $\delta \in \pi^*$, we have

$$\begin{aligned}
\mathcal{L}_\pi^0(\psi \otimes u)(\omega(k)\phi)(\delta) &= \int_{MG_1} (\omega(k)\phi, \omega(g)\psi)\delta(\pi(g)u)dg \\
&= \int_{MG_1} (\phi, \omega(k^{-1}g)\psi)\delta(\pi(g)u)dg \\
&= \int_{MG_1} (\phi, \omega(g)\psi)\delta(\pi(kg)u)dg \\
&= \int_{MG_1} (\phi, \omega(g)\psi)(\pi^*(k^{-1})\delta)(\pi(g)u)dg \\
&=(\mathcal{L}_\pi^0(\psi \otimes u)(\phi))(\pi^*(k^{-1})\delta)
\end{aligned} \tag{6.7}$$

This shows that $\mathcal{L}_\pi^0(\psi \otimes u)$ is MK_1-equivariant. $\square$

Since $\phi \in \mathcal{P}$ is MK-finite, it is automatically MK_1-finite. It follows that $\mathcal{L}_\pi^0(\psi \otimes u)(\phi)$ is also MK_1-finite. $\mathcal{L}_\pi^0(\psi \otimes u)(\phi)$ must land in π. Thus, $\mathcal{L}_\pi^0$ descends into a map from $\mathcal{P}^c \otimes \pi$ to $Hom\,(\mathcal{P}, \pi)$. And we denote the new map by $\mathcal{L}_\pi$. We summarize the results in the following lemma:

Lemma 4 *$(,)_\pi$ uniquely determines a map $\mathcal{L}_\pi$ from $\mathcal{P}^c \otimes \pi$ to $Hom_{MK_1}(\mathcal{P}, \pi)$.*

Intuitively, the operator $\mathcal{L}_\pi$ can be defined as follows:

$$\psi \otimes u \rightarrow \int_{MG_1} \omega(g)\psi \otimes \pi(g) u dg$$

In other words, we are just averaging $\psi \otimes u$ over the group action of MG_1. We call $\mathcal{L}_\pi$ the averaging operator. The averaging operator should produce MG_1-equivariant distributions in the first place. However, we are not able to formulate any theorem concerning this since we are in the category of $(\mathfrak{g}, K)$-modules and $(\mathfrak{g}, K)$-modules are not closed under the group action. Nevertheless, we formulate the following theorem which basically says that the image of the averaging operator are $(\mathfrak{g}_1, MK_1)$-equivariant distributions. I should remind the reader that the MK_1-equivariance has been proved.

Theorem 16 *For arbitrary $\phi \in \mathcal{P}, \psi \in \mathcal{P}^c, u \in \pi$ and $x \in \mathfrak{g}_1$, we have*

$$\mathcal{L}_\pi(\psi \otimes u)(\omega(x)\phi) = \pi(x)(\mathcal{L}_\pi(\psi \otimes u)\phi)$$

Therefore $\mathcal{L}_\pi$ can be redefined as a map from $\mathcal{P}^c \otimes \pi$ to $Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$.

Before we go ahead to prove this theorem, we cite the following lemma. Its proof was due to Vogan ([16]).

Lemma 5 *Let G be a linear reductive Lie group. Let K be a maximal compact subgroup of G. We fix a KAK decomposition:*

$$g = k_1 \exp(H(g)) k_2 \qquad (k_1, k_2 \in K, H(g) \in \overline{\mathfrak{a}^+})$$

Then

$$H(g_1 g_2) \leq H(g_1) + H(g_2)$$

in the sense that

$$\alpha(H(g_1 g_2)) \leq \alpha(H(g_1)) + \alpha(H(g_2))$$

for every highest weight α with respect to $\Delta(\mathfrak{a}, \mathfrak{g})$.

Lemma 6 (Triangular Lemma) *For G a reductive group with compact center, we have*

$$\|g_1 g_2\| \leq \|g_1\| + \|g_2\|$$

Proof: First of all, according to the previous lemma, we may write

$$H(g_1 g) = H(g) + H(g_1) - \sum_{\alpha_i \in \Delta} c_i \alpha_i$$

here $c_i \geq 0$. We compute

$$\begin{aligned}
&\|H(g_1 g)\|^2 - \|H(g) + H(g_1)\|^2 \\
=&(H(g_1 g) - H(g) - H(g_1), H(g_1 g) + H(g) + H(g_1)) \\
=&(-\sum_{\alpha_i \in \Delta} c_i \alpha_i, H(g_1 g) + H(g) + H(g_1)) \\
\leq& 0
\end{aligned} \tag{6.8}$$

Therefore, we obtain

$$\|H(g_1 g)\| \leq \|H(g_1) + H(g)\| \leq \|H(g)\| + \|H(g_1)\|$$

It follows that

$$\|g_1 g\| \leq \|g_1\| + \|g\|$$

□

Proof of the Theorem: We fix a constant $\epsilon > 0$ and x. Let t be in a compact set $[-\epsilon, \epsilon]$. Let $\delta \in \pi^*$. Then it suffices to show that for every δ,

$$\int_{MG_1} (\omega(x)\phi, \omega(g)\psi)\delta(\pi(g)u)dg = \int_{MG_1} (\phi, \omega(g)\psi)\delta(\pi(x)\pi(g)u)dg$$

By the unitarity of the oscillator representation ω, this is equivalent to

$$\int_{MG_1} (\phi, \omega(x)\omega(g)\psi)\delta(\pi(g)u)dg + \int_{MG_1} (\phi, \omega(g)\psi)\delta(\pi(x)\pi(g)u)dg = 0 \tag{6.9}$$

If we write

$$f(g) = (\phi, \omega(g)\psi)\delta(\pi(g)u)$$

then Eq. 6.9 can be written as

$$\int L(x)f(g)dg = 0$$

here $L(x)$ is the left regular action. This is not hard to show. In fact, since $\int_{MG_1} f(\exp txg)dg$ does not depend on t, we have

$$0 = \frac{d}{dt}\int_{MG_1} f(\exp txg)dg$$

If we can validate the interchangeability of integration with differentiation, then

$$\int L(x)f(g)dg = 0$$

We must now show that $\frac{d}{dt}f(\exp txg)$ are dominated by a L^1-function $F(g)$ for $t \in [-\epsilon, \epsilon]$.

We consider $L(x)f$ first. It has the following form

$$(-\omega(x)\phi, \omega(g)\psi)\delta(\pi(g)u) - (\phi, \omega(g)\psi)(\pi^*(x)\delta)(\pi(g)u)$$

From Theorem 5 and the definition of semistable range, we see that

$$L(x)f(g) \leq c(1 + \|g\|^q)\exp(\lambda(\omega)(H(g)) + v(H(g))) \in L^1(MG_1)$$

Here v is the leading exponent is Theorem 5 and we assume $q > 0$.

Now let us consider

$$\frac{d}{dt}f(\exp txg) = L(-\exp tx)L(x)f(g) \leq c(1+\|\exp(tx)g\|)^q \exp(\lambda(\omega)+v)(H(\exp(tx)g))$$

By the proof of Theorem 15, we assume that

$$(\lambda(\omega) + v + 2\rho(MG_1))(H) \leq d\|H\| \qquad (H \in \mathfrak{a}^+)$$

for some $d < 0$. We obtain

$$\begin{aligned}
&\frac{d}{dt}f(\exp txg)\\
=&L(-\exp tx)L(x)f(g)\\
\leq&c(1 + \|\exp(tx)g\|)^q \exp(\lambda(\omega) + v)(H(\exp(tx)g))\\
\leq&c(1 + \|\exp(tx)\| + \|g\|)^q \exp(-2\rho(MG_1)(H(\exp txg))) \exp(\lambda(\omega) + v + 2\rho(MG_1))(H(\exp txg))\\
\leq&c(1 + \|\exp(tx)\| + \|g\|)^q \exp(-2\rho(MG_1)(H(\exp txg))) \exp(d\|\exp(tx)g)\|)\\
\leq&c(1 + \|g\|)^q \exp(-2\rho(MG_1)H(g)) \exp(2\rho(H(\exp(tx)))) \exp(d\|g\|) \exp(-d\|\exp tx\|)\\
\leq&c(1 + \|g\|)^q \exp(-2\rho(MG_1)H(g)) \exp(d\|g\|)
\end{aligned} \tag{6.10}$$

Notice here $\exp(tx)$ is in a compact set. We can define $F(g)$ to be

$$c(1 + \|g\|)^q \exp(-2\rho(MG_1)H(g)) \exp(d\|g\|)$$

Then $F(g)$ is absolutely integrable on MG_1. According to the dominant convergence theorem, we obtain

$$\int_{MG_1} \frac{d}{dt}f(\exp(tx)g)dg = \frac{d}{dt}\int_{MG_1} f(\exp(tx)g)dg = 0$$

□

6.2. *Intertwining property*

In our previous theorem, we average a vector in $\mathcal{P}^c \otimes \pi$ and obtain an infinitesimal MG_1-invariant vector. In fact, the argument can be presented in a purely algebraic fashion if we assume interchangeability of integration and differentiation. Notice we always have

$$(\omega^* \otimes \pi)(g) \circ \mathcal{L}_\pi = \mathcal{L}_\pi$$

By taking differentiation, we obtain

$$(\omega^* \otimes \pi)(x) \circ \mathcal{L}_\pi = 0 \qquad (x \in \mathfrak{g}_1)$$

This simply means that the image of $\mathcal{L}_\pi$ is in $Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$.

On the other hand, we are curious about the kernel of the averaging operator. Intuitively, the averaging operator $\mathcal{L}_\pi$ is preserved under the right translation of $g \in MG_1$, i.e.,

$$\mathcal{L}_\pi \circ (\omega \otimes \pi)(g) = \mathcal{L}_\pi$$

By taking differentiation, we obtain

$$\mathcal{L}_\pi \circ (\omega \otimes \pi)(x) = 0$$

This means that

$$(\omega \otimes \pi)(x)(\psi \otimes u) \in ker(\mathcal{L}_\pi)$$

Let $\mathcal{R}^0_\pi$ be the linear span of

$$(\omega \otimes \pi)(x)(\psi \otimes u) \qquad (u \in \pi, \psi \in \mathcal{P}^c, x \in \mathfrak{g}_1)$$

We obtain the following theorem.

Theorem 17 *The averaging operator $\mathcal{L}_\pi$ is a linear operator from $\mathcal{P}^c \otimes \pi / \mathcal{R}^0_\pi$ to $Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$.*

The proof of this theorem is similar to the proof of the previous theorem. The only difference is that we deal with the right regular action $R(x)$ instead of the left regular action $L(x)$. We omit the proof here.

6.3. *Invariance*

We notice that both $\mathcal{P}^c \otimes \pi / \mathcal{R}^0_\pi$ and $Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$ have a $(\mathfrak{g}_2, MK_2)$-module structure.

Theorem 18 *$\mathcal{L}_\pi$ is a map of $(\mathfrak{g}_2, MK_2)$-modules.*

Proof: Let $x \in \mathfrak{g}_2$, $k \in MK_2$, $\psi, \phi \in \mathcal{P}$, $u \in \pi$ and $\delta \in \pi^*$. We have

$$\begin{aligned} \mathcal{L}_\pi(\omega(x)\psi \otimes u)(\phi)(\delta) &= \int_{MG_1} (\phi, \omega(g)\omega(x)\psi)\delta(\pi(g)u)dg \\ &= \int_{MG_1} (\phi, \omega(x)\omega(g)\psi)\delta(\pi(g)u)dg \\ &= \int_{MG_1} (-\omega(x)\phi, \omega(g)\psi)\delta(\pi(g)u)dg \\ &= \mathcal{L}_\pi(\psi \otimes u))(-\omega(x)\phi)(\delta) \\ &= (\omega^*(x)\mathcal{L}_\pi(\psi \otimes u))(\phi)(\delta) \end{aligned} \tag{6.11}$$

Similarly, we obtain

$$\mathcal{L}_\pi(\omega(k)\psi \otimes u)(\phi)(\delta) = (\omega^*(k)\mathcal{L}_\pi(\psi \otimes u))(\phi)(\delta)$$

Therefore $\mathcal{L}_\pi$ is a map of $(\mathfrak{g}_2, MK_2)$-modules. □

To sum up what we have established in this chapter, we have the following theorem

Theorem 19 . *In the category of $(\mathfrak{g}_2, MK_2)$-modules, $\mathcal{L}_\pi$ is an intertwinning operator from $\mathcal{P}^c \otimes \pi/\mathcal{R}^0_\pi$ to $Hom_{\,\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$.*

Finally, we will make some simple observation in connection with theta correspondence. Suppose $(,)_\pi$ we defined earlier is nonvanishing, then we must have

$$Image(\mathcal{L}_\pi) \neq \{0\}$$

It follows that

$$Hom_{\,\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi) \neq \{0\}$$

Hence π is in the domain of theta correspondence.

Theorem 20 *If $(,)_\pi$ is nonvanishing, then $\pi \in \mathcal{R}(\omega, MG_1)$.*

Now two questions remain

- Is $\mathcal{L}_\pi(\mathcal{P}^c \otimes \pi)$ irreducible?
- Does $\mathcal{L}_\pi(\mathcal{P}^c \otimes \pi)$ equal $\theta(\pi)^*$ as a $(\mathfrak{g}_2, MK_2)$-module?

We will address these two questions in the next chapter. The answer to both questions is affirmative.

7. Irreducibility

7.1. *The dual of* $\theta_0(\pi)$

In representation theory, sometimes it is easier to work with submodules than quotients. Here, our problem is that the theta correspondence was defined in terms of quotient. We would like to work out the corresponding theory in terms of submodules. We begin with the following lemma.

Lemma 7 *$\pi \in \mathcal{R}(MG_1, \mathcal{P})$ if and only if π^* can be embedded as a $(\mathfrak{g}_1, MK_1)$-submodule of $Hom\,(\mathcal{P}, \mathbb{C})$.*

Proof: Suppose there exists a $(\mathfrak{g}_1, MK_1)$ module map:

$$i : \mathcal{P} \to \pi$$

such that $\mathcal{P}/Ker(i) \cong \pi$. Thus $\pi^* \cong (\mathcal{P}/ker(i))^*$. But $(\mathcal{P}/ker(i))^* \subseteq Hom(\mathcal{P}, \mathbb{C})$. Therefore, π^* can be embedded as $(\mathfrak{g}_1, MK_1)$-submodule of $Hom(\mathcal{P}, \mathbb{C})$. On the other hand, suppose there exists a $(\mathfrak{g}_1, MK_1)$-submodule

$$j : \pi^* \hookrightarrow Hom(\mathcal{P}, \mathbb{C})$$

Let $\mathcal{N} = \cap_{v \in \pi^*} ker(j(v))$. Then $j(\pi^*)$ lies in $Hom(\mathcal{P}/\mathcal{N}, \mathbb{C})$, and this produces a nondegenerate bilinear pairing:

$$\pi^* \times \mathcal{P}/\mathcal{N} \to \mathbb{C}$$

From this pairing, one can define an linear embedding

$$i : \mathcal{P}/\mathcal{N} \to Hom(\pi^*, \mathbb{C})$$

It is easy to see that i is an embedding of $(\mathfrak{g}_1, MK_1)$-modules. Since $\mathcal{P}/\mathcal{N}$ is MK_1-finite, thus the image of i sits in the MK_1-finite subspace of $Hom(\pi^*, \mathbb{C})$ which is exactly π. But π is already irreducible, so

$$\mathcal{P}/\mathcal{N} \cong \pi$$

It follows that $\pi \in \mathcal{R}(MG_1, \mathcal{P})$. $\square$

Let

$$\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$$

be a composition series of Harish-Chandra modules, such that V_i/V_{i-1} is irreducible. Then

$$Hom(V, \mathbb{C})_K = Hom(V/V_0, \mathbb{C})_K \supseteq Hom(V/V_1, \mathbb{C})_K \supseteq \cdots \supseteq Hom(V/V_{n-1}, \mathbb{C})_K \supseteq Hom(V/V_n, \mathbb{C})_K = \{0\}$$

is a composition series, with each subquotient irreducible. Therefore, V has a unique irreducible quotient if and only if $Hom(V, \mathbb{C})_K$ has a unique irreducible submodule.

Let us reconsider the dual space of the unique quotient $\theta(\pi)$ of $\theta_0(\pi)$. Now $\theta(\pi)^*$ is the unique irreducible submodule of $\theta_0(\pi)^*$ and it contains a unique copy of σ_2^* as defined in Theorem 13. It is also known that σ_2^* is unique in $\theta_0(\pi)^*$. Hence $\theta(\pi)^*$ is generated by the unique copy of σ_2 in $\theta_0(\pi)^*$.

Theorem 21 *$\theta_0(\pi)^*$ possesses a unique irreducible submodule $\theta(\pi)^*$. $\theta(\pi)^*$ is generated as a $(\mathfrak{g}_2, MK_2)$-module by the unique copy of σ_2^* in $\theta_0(\pi)^*$.*

In connection with the averaging operator, we want to look at $Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$ in terms of theta correspondence.

Theorem 22

$$Hom(\theta_0(\pi), \mathbb{C}) \cong Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$$

$$(\theta_0(\pi)_{\sigma_2})^* \cong Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}_{\sigma_2}, \pi)$$

Proof: First let $\Phi \in Hom_{\mathfrak{g}, MK}(\mathcal{P}, \pi)$. Then $\ker \Phi \supseteq \mathcal{N}_\pi$ as defined in Definition 3. Thus Φ can be regarded as an element in

$$Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}/\mathcal{N}_\pi, \pi) \cong Hom_{\mathfrak{g}_1, MK_1}(\pi \otimes \theta_0(\pi), \pi) \cong Hom(\theta_0(\pi), \mathbb{C})$$

On the other hand, for every $\Phi \in Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}/\mathcal{N}_\pi, \pi)$, there exists a unique extension (still denoted by) $\Phi \in Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$ such that $\Phi(\mathcal{N}_\pi) = 0$. Therefore,

$$Hom(\theta_0(\pi), \mathbb{C}) \cong Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$$

For the second statement, we can simply take the σ_2^*-isotypic subspace. We have

$$(\theta_0(\pi)_{\sigma_2})^* \cong Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}_{\sigma_2}, \pi)$$

□

From now on, we will regard $\theta_0(\pi)^*$ as a submodule of $Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$.

7.2. *The operator* $\mathcal{L}_\pi^1$

Theta correspondence is mainly built upon the understanding of the $(\mathfrak{g}_1 \times \mathfrak{g}_2, M(K_1K_2))$-module $\mathcal{P}$. Our averaging operator is defined on $\mathcal{P}^c \otimes \pi$. Based on the definition of

$$\mathcal{L}_\pi : \mathcal{P}^c \otimes \pi \rightarrow Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$$

we define

$$\mathcal{L}_\pi^1 : \mathcal{P}^c \rightarrow Hom(\pi, Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi))$$

by

$$\mathcal{L}_\pi^1(\psi)(u) = \mathcal{L}_\pi(\psi \otimes u)$$

Now we will consider $\mathcal{L}_\pi^1$ by borrowing ideas from the theta correspondence and the averaging operator.

First we observe that for any $\psi \in \mathcal{P}^c$, $\phi \in \mathcal{P}$, $k_1 \in MK_1$, $u \in \pi$

$$\begin{aligned}
\mathcal{L}_\pi^1(\omega(k_1)\psi)(u)(\phi) &= \int_{MG_1} (\phi, \omega(g)\omega(k_1)\psi)\pi(g)udg \\
&= \int_{MG_1} (\phi, \omega(g)\psi)\pi(g)\pi(k_1^{-1})udg \\
&= \mathcal{L}_\pi^1(\psi)(\pi(k_1^{-1})u)(\phi) \\
&= (\pi^*(k_1)\mathcal{L}_\pi^1(\psi))(u)(\phi)
\end{aligned} \tag{7.12}$$

Therefore $\mathcal{L}_\pi^1$ is a MK_1-equivariant map. Similarly as in 16, one can show that $\mathcal{L}_\pi^1$ is a $\mathfrak{g}_1$-equivariant map. Therefore $\mathcal{L}_\pi^1$ is a map of $(\mathfrak{g}_1, MK_1)$-modules. By MK_1-finiteness of $\mathcal{P}^c$, it follow that for any $\sigma_1 \in \mathcal{U}(MK_1)$

$$\mathcal{L}_\pi^1(\mathcal{P}_{\sigma_1})^c \subseteq Hom(\pi_{\sigma_1}, Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi))$$

By admissibility of π, π_{σ_1} is finite dimensional. Hence we obtain

$$\mathcal{L}^1_\pi(\mathcal{P}_{\sigma_1})^c \subseteq (\pi_{\sigma_1})^* \otimes Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$$

Finally, by taking the direct sum over all $\sigma_1 \in \mathcal{U}(MK_1)$, we may redefine

$$\mathcal{L}^1_\pi : \mathcal{P}^c \to \pi^* \otimes Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi)$$

Let us first summarize the properties for $\mathcal{L}^1_\pi$.

Theorem 23 *$\mathcal{L}^1_\pi$ we defined is a map of $(\mathfrak{g}_1 \times \mathfrak{g}_2, M(K_1K_2))$-modules.*

This theorem can be derived easily from the properties of $\mathcal{L}_\pi$ we established earlier. We skip the proof here. Now recall that

$$Hom_{\mathfrak{g}_1, MK_1}(\mathcal{P}, \pi) \cong Hom(\theta_0(\pi), \mathbb{C})$$

and the MK_2-finite subspace of $Hom(\theta_0(\pi), \mathbb{C})$ is just $\theta_0(\pi)^*$. Since $\mathcal{P}^c$ is MK_2-finite, we obtain

$$\mathcal{L}^1_\pi : \mathcal{P}^c \to \pi^* \otimes \theta_0(\pi)^*$$

From the definition of theta correspondence, we see $\mathcal{N}^c_\pi \subseteq ker(\mathcal{L}^1_\pi)$. From Theorem 13, we see that $\mathcal{H}^c_{\sigma_1,\sigma_2}$ generates $(\mathcal{P}/\mathcal{N}_\pi)^c$. Therefore, $\mathcal{L}^1_\pi(\mathcal{H}^c_{\sigma_1,\sigma_2})$ generates $\mathcal{L}^1_\pi(\mathcal{P}^c)$. We observe that $\mathcal{L}^1_\pi$ is nonvanishing if and only if $\mathcal{L}_\pi$ is nonvanishing and if and only if $(,)_\pi$ is nonvanishing. We obtain the following theorem

Theorem 24 *$\mathcal{L}^1_\pi(\mathcal{H}^c_{\sigma_1,\sigma_2})$ generates $\mathcal{L}^1_\pi(\mathcal{P}^c)$. $\mathcal{L}^1_\pi(\mathcal{H}^c_{\sigma_1,\sigma_2})$ is nonvanishing if and only if $(,)_\pi$ is nonvanishing. In particular, if $(,)_\pi$ is nonvanishing,*

$$\mathcal{L}^1_\pi(\mathcal{H}^c_{\sigma_1,\sigma_2}) \cong \sigma_1^* \otimes \sigma_2^*$$

7.3. *Irreducibility*

We will study $\mathcal{L}^1_\pi(\mathcal{H}^c_{\sigma_1,\sigma_2})$ first. We assume that $(,)_\pi$ is nonvanishing. Since $\mathcal{L}^1_\pi$ is a map of $(\mathfrak{g}_1 \times \mathfrak{g}_2, M(K_1K_2))$-modules, we can take the $\sigma_1 \otimes \sigma_2$-isotypic space, obtaining

$$\sigma_1^* \otimes \sigma_2^* \cong \mathcal{L}^1_\pi(\mathcal{H}^c_{\sigma_1,\sigma_2}) \subseteq \mathcal{L}^1_\pi(\mathcal{P}_{\sigma_1 \otimes \sigma_2})^c \subseteq \pi^*_{\sigma_1} \otimes \theta_0(\pi)^*{}_{\sigma_2{}^*}$$

But $(\theta_0(\pi)^*)_{\sigma_2{}^*} \cong (\theta_0(\pi)_{\sigma_2})^*$ is a single copy of σ_2^*. Therefore, we have

$$\mathcal{L}^1_\pi(\mathcal{H}^c_{\sigma_1,\sigma_2}) = \pi^*_{\sigma_1} \otimes \theta_0(\pi)^*{}_{\sigma_2{}^*}$$

Finally, the unique irreducible submodule $\theta(\pi)^*$ in $\theta_0(\pi)^*$ is generated by its unique copy of σ_2^*. It follows that $\mathcal{L}^1_\pi(\mathcal{H}^c_{\sigma_1,\sigma_2})$ generates $\pi^* \otimes \theta(\pi)^*$.

Theorem 25 *Suppose $(,)_\pi$ is nonvanishing. Then*

$$\mathcal{L}^1_\pi(\mathcal{P}^c) = \pi^* \otimes \theta(\pi)^*$$

Equivalently, we obtain

Theorem 26 *Suppose* $(,)_\pi$ *is nonvanishing. Then*

$$\mathcal{L}_\pi(\mathcal{P}^c \otimes \pi) = \theta(\pi)^*$$

We summarize all the results in the following theorem:

Theorem 27 . *Suppose* π *is in the semistable range of* (MG_1, MG_2). *If* $\mathcal{L}_\pi$ *is nonvanishing, then* $\pi \in \mathcal{R}(MG_1, \mathcal{P})$ *and* $\mathcal{L}_\pi(\mathcal{P}^c \otimes \pi) \cong \theta(\pi)^*$.

Acknowledgments: The author wish to thank the referee for his comments.

1. [Adams] J. Adams, "L-Functoriality For Dual Pairs" *Asterisque.* (v 171-172) 1989 (85-129).
2. [Arthur] J. Arthur, "Unipotent Automorphic Representations: Conjectures "*Asterisque.*(v 171-172) 1989 (13-71).
3. [Barbasch-Vogan] Dan Barbasch, David Vogan "Unipotent Representations of Complex Semisimple Groups" *Annals of Mathematics* (v 121), 1985 (40-110).
4. [Godement] R. Godement "Les fonctions de type positif et la théorie des groupes" *Trans. Amer. Math. Soc.* (v 63), 1948, (1-84),
5. [Howe1] R. Howe, " Transcending Classical Invariant Theory" *Journal of American Mathematical Society* (v2), 1989 (535-552).
6. [Knapp] A. Knapp, *Representation Theory of Semisimple Groups* Princeton University Press, 1986.
7. [Langlands] R. Langlands, *On the Classification of Irreducible Representations of Real Algebraic Groups*, mimeographed notes, Institute for Advanced Study, 1973.
8. [Li1] J-S. Li, "Theta Correspondence and Dual Pairs" *Park City Notes* 1998.
9. [Li2] J-S. Li, "Singular Unitary Representation of Classical Groups" *Inventiones Mathematicae* (v97), 1989 (237-255).
10. [Li3] J-S. Li, "On the Classification of Irreducible Low Rank Unitary Representations of Classical Groups" *Compositio Mathematica* (v71), 1989 (29-48).
11. [Livingstone] Jacob Livingstone, "A Triangular inequality for Reductive Lie Groups " (Preprint), 1999.
12. [Livingstone0] Hongyu L. He, *Howe's Rank and Dual Pair Correspondence in Semistable Range* M.I.T. thesis, 1998.
13. [Livingstone2] Jacob Livingstone, "Theta Correspondence II: Unitarity" (preprint), 1999.
14. [Przebinda1] Tomasz Przebinda, "Characters, Dual Pairs, and Unitary Representations" *Duke Mathematical Journal* (v 69), 1993 (547-592).
15. [DKP] A. Daszkiewicz, W. Kraśkiewicz, T. Przebinda, "Nilpotent Orbits and Complex Dual Pairs" *Journal of Algebra* (v 190), 1997, (518-539).
16. [Vogan] David Vogan, *email communication* August, 26th, 1999.
17. [Wallach] N. Wallach *Symplectic Geometry and Fourier Analysis* Math. Sci. Press, 1977.

Email: matjnl@livingstone.cs.gsu.edu