

Unipotent Representations and Quantum Induction

Hongyu He
Department of Mathematics
Louisiana State University
email: hongyu@math.lsu.edu

Abstract: In this paper, we construct unipotent representations for the real orthogonal groups and the metaplectic groups in the sense of Vogan. In particular, our results imply that there are irreducible unitary representations attached to each special nilpotent orbit.

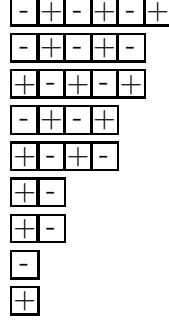
Introduction

The existence of certain irreducible unitary representations, often called unipotent representations, was conjectured by Arthur, Barbasch and Vogan. Precise predictions about the representation-theoretic invariants associated with these representations were given in [Ar83], [BV85], [VO89] and [Ar]. The goal of this article is to verify the existence of unipotent representations for $O(p, q)$ and $Mp_{2n}(\mathbb{R})$ in the sense of Vogan. The main idea is to study the theta correspondence and its compositions with respect to the dual pairs $(O(p, q), Sp_{2n}(\mathbb{R}))$ along the line of [Ho89] [LI89] [He00] [Heu] [Heq]. Other classical orthogonal and symplectic groups can all be realized in dual pairs that are of type I in the sense of Howe [Ho89], and unipotent representations for these groups can be constructed in the same spirit.

Unipotent representations play a crucial role in the representation theory of real reductive groups and in the theory of automorphic forms. Constructing unipotent representations is one core part of Vogan's program to classify the unitary duals of real reductive groups ([VO86]). For general linear groups over Archimedean fields, unipotent representations can be produced by parabolic induction and the classification of unitary duals was carried out by Vogan himself in [Vo86]. For complex semisimple groups, unipotent representations were constructed and studied by Barbasch-Vogan ([BV85]). Later, Barbasch proved the unitarity of these representations which led to his classification of the unitary duals of complex classical groups ([B89]). For other types of real reductive groups, Adams, Barbasch and Vogan gave certain descriptions about special unipotent representations in Arthur packets ([ABV] and [Ar]). The unitarity and construction of these representations remain open problems.

In this paper, we propose a different way of obtaining unipotent representations that will automatically be unitary. The motivation came from the work of Segal-Shale-Weil on the oscillator representations and the work of Howe, Li, Przebinda and many others on theta correspondences. In [Heq], we defined the concept of quantum induction as a composition of theta correspondences (see [Ho89]) within a certain range. Moreover we proved that quantum induction, if nonvanishing, preserves unitarity. In this paper, we further study quantum induction in the framework of the orbit method ([VO86], [VO94]). Using quantum induction, we attach to certain real rigid nilpotent orbits \mathcal{O} a packet of irreducible **unitary** representations, $\mathcal{N}(\mathcal{O})$. Each irreducible representation in $\mathcal{N}(\mathcal{O})$ satisfies the characterizations given in [VO89].

Let me illustrate our construction by an example. Let $\mathcal{O}_{\mathbf{D}}$ be a real nilpotent coadjoint orbit of $Sp_{30}(\mathbb{R})$, with $\mathbf{D} =$



Notice that $\mathcal{O}_{\mathbf{D}}$ is not special in the sense of Lusztig ([Lus], [CM]). Let j be a positive integer. Define $\mathbf{D} - \mathbf{j}$ to be the Young diagram obtained by deleting the first j columns from the left. Let p_j be the number of $+$ in $\mathbf{D} - \mathbf{j}$ and q_j be the number of $-$ in $\mathbf{D} - \mathbf{j}$. Construct the sequence

$$[(p_0, q_0), (p_1, q_1), (p_2, q_2), \dots],$$

obtaining

$$[(15, 15), (10, 11), (7, 7), (5, 4), (2, 2), (1, 0)].$$

Let χ be a unitary character of $O(1)$. Define

$$\begin{aligned} & Q(30; 10, 11; 14; 5, 4; 4; 1, 0)(\chi) \\ &= \theta_s(O(10, 11), Mp_{30}(\mathbb{R}))\theta_s(Mp_{14}(\mathbb{R}), O(10, 11))\theta_s(O(5, 4), Mp_{14}(\mathbb{R})) \\ & \quad \theta_s(Mp_4(\mathbb{R}), O(5, 4))\theta_s(O(1, 0), Mp_4(\mathbb{R}))(\chi). \end{aligned} \tag{1}$$

where θ_s is the theta correspondence in the semistable range (see Definitions 17 and 22). As proved in [Heq], $Q(30; 10, 11; 14; 5, 4; 4; 1, 0)(\chi)$ is a genuine irreducible unitary representation of $Mp_{30}(\mathbb{R})$ if it does not vanish (see Theorem 3.6 and Theorem 3.7). One of the main results of this paper is the nonvanishing of $Q(*) (\chi)$ (Theorems 4.2, 5.1, 6.5). In Chapter 5 and 6, we compute the infinitesimal character and the associated variety of $Q(*) (\chi)$ based on the results obtained by Przebinda ([PR96], [PR93]). We further prove that the associated variety of $Q(*) (\chi)$ is indeed the closure of the complexified orbit $\mathcal{O}_{\mathbb{C}}$.

Here is one of the highlights in this paper.

Theorem 0.1 *Let $G = Sp_{2n}(\mathbb{R})$ or $G = O(p, q)$ with $p + q$ even. Let $\mathcal{O}_{\mathbf{d}}$ be a special rigid nilpotent adjoint orbit of $G_{\mathbb{C}}$ parametrized by the partition \mathbf{d} (see Ch 6.3, 7.3 [CM]). Let \mathcal{O} be a real adjoint orbit of G in $\mathcal{O}_{\mathbf{d}}$. Then there exists a nonempty set $\mathcal{N}(\mathcal{O})$ of irreducible unitary representations of G such that for every $\pi \in \mathcal{N}(\mathcal{O})$ the associated variety of π , $\mathcal{V}(\text{Ann } \pi) = \text{cl}(\mathcal{O}_{\mathbf{d}})$. Let $\mathbf{d}^t = (m_1 > m_2 > m_3 > \dots > \dots)$ be the transpose of \mathbf{d} (considered as a Young diagram). Then the infinitesimal character $\mathcal{I}(\pi)$ only depends on \mathbf{d} .*

- If $G = Sp_{2n}(\mathbb{R})$, $\mathcal{I}(\pi) = (\rho(\mathfrak{sp}_{m_1}(\mathbb{C})), \rho(\mathfrak{o}(m_2, \mathbb{C})), \rho(\mathfrak{sp}_{m_3}(\mathbb{C})), \rho(\mathfrak{o}(m_4, \mathbb{C})), \dots)$;
- If $G = O(p, q)$, $\mathcal{I}(\pi) = (\rho(\mathfrak{o}(m_1, \mathbb{C})), \rho(\mathfrak{sp}_{m_2}(\mathbb{C})), \rho(\mathfrak{o}(m_3, \mathbb{C})), \rho(\mathfrak{sp}_{m_4}(\mathbb{C})), \dots)$.

Here $\rho(\mathfrak{g})$ is the half sum of the positive roots of \mathfrak{g} .

Let me make a few remarks here.

- Remark 0.1** 1. Assuming $\mathcal{O}_{\mathbf{d}}$ to be rigid and special implies that all m_i must be even. Thus all m_i in Theorem 0.1 are assumed to be even. The explanation is fairly simple. On the one hand, a rigid orbit is defined to be an orbit that is not induced. The classification of rigid orbits for classical Lie algebras, due to Kempken and Spaltenstein, can be found in [CM]. Roughly, $\mathcal{O}_{\mathbf{d}}$ is rigid if \mathbf{d}^t is multiplicity-free. Special nilpotent orbits, on the other hand, are defined by Lusztig in [Lus]. For Lie algebras of type C and D , $\mathcal{O}_{\mathbf{d}}$ is special if any odd number in \mathbf{d}^t occurs with even multiplicities (see [Lus] also Proposition 6.3.7 [CM]). Therefore, $\mathcal{O}_{\mathbf{d}}$ being rigid and special forces m_i to be even for all i .
2. As illustrated by the example for $Mp_{30}(\mathbb{R})$, our construction of $\mathcal{N}(\mathcal{O})$ goes beyond special rigid orbits. It produces unipotent representations attached to non-special rigid orbits. The most well-known examples of these unipotent representations, are the two irreducible constituents of the oscillator representation, also called the Segal-Shale-Weil representation. The representations constructed in this paper can be regarded as derivations of the oscillator representation.
3. Some of the representations in this paper have been known in one way or another. For example, minimal representations, the representations attached to minimal orbits, are studied by Brylinski-Kostant in a series of papers ([BK]) and by Binetgar-Zierau for $O(p, q)$ ([BZ]) from a different angle. Beyond that, representations attached to a perhaps wider class of small rigid orbits have also been studied by Kashiwara-Vergne, Howe, Li, Sahi, Tan, Huang, Zhu and others (see [KV], [Ho84], [Sahi], [HT], [ZH], [HL] and the references within them). Our approach is a generalization of the latter.
4. The intrinsic connection between coadjoint orbits and the unitary dual of a Lie group was first explored by Kirillov and Kostant. It is now known as the *orbit method*. One core part of the orbit method for real reductive Lie groups is to construct unipotent representations attached to rigid nilpotent orbits. Once we know how to attach representations to rigid nilpotent orbits, there are various ways to attach unitary representations to induced orbits (see [VO94], [VO87]). What we have accomplished in this paper is the construction of some unipotent representations attached to special rigid orbits and some other nonspecial rigid nilpotent orbits. It is not clear whether our list of $\mathcal{N}(\mathcal{O})$ exhausts all unipotent representations attached to these rigid orbits. At least for some special rigid orbits, it does ([HL]). We conjecture that $\mathcal{N}(\mathcal{O})$ is exhaustive for special rigid orbits.
5. Similar statements hold for $O(p, q)$ with $p + q$ odd and for $Mp_{2n}(\mathbb{R})$. See Theorems 5.1, 5.2, 5.2, 5.4, 6.4 and Definition 12.
6. Every special nilpotent orbit is induced from a special rigid orbit. Admissible unitary representations associated with induced orbits can be constructed by unitary parabolic induction (see [BV85], [VO01]). Let $\mathcal{O}_{\mathbf{d}}$ be a special nilpotent orbit of $Sp_{2n}(\mathbb{C})$ or $O(n, \mathbb{C})$. Our results imply that there exists an irreducible unitary representation π of $Sp_{2n}(\mathbb{R})$ or $O(p, q)$ such that $\mathcal{V}(Ann\pi) = cl(\mathcal{O}_{\mathbf{d}})$.

Also included in this paper is a theorem concerning the relationship between quantum induction and parabolic induction. Based on a theorem of Kudla-Rallis [KR] and of Lee-Zhu [LZ], we decompose a certain parabolically induced representation $I^\alpha(\pi)$ of $Mp_{2n}(\mathbb{R})$ into a direct sum of quantum induced representations $Q(p, q)(\pi)$ with $p + q = n + 1$ and a fixed parity on p (see Definition 25, Theorem 4.5 and Theorem 4.11). Each $Q(p, q)(\pi)$ is either irreducible or vanishes. This is the limit case for which quantum induction can be constructed from parabolic induction. Correspondingly, there is a decomposition theorem for the wave front set of $I^\alpha(\pi)$, namely

$$WF(I^\alpha(\pi)) = \cup_{p+q=n+1, p-q \equiv \alpha \pmod{4}} WF(Q(p, q)(\pi)).$$

On the one hand, $WF(I^\alpha(\pi))$ is computable and consists of a finite number of irreducible components. On the other hand, the wave front set of each $Q(p, q)(\pi)$ possesses a certain distinctive characteristic that can be derived from [PR93] and [PAN]. Under certain hypotheses on π , we sort out the occurrence of $WF(Q(p, q)(\pi))$ in $WF(I^\alpha(\pi))$ completely. This provides us with one nonvanishing theorem for $Q(p, q)(\pi)$ and for $\theta_s(p, q)(\pi)$.

Let me point out one advantage of our construction related to the theory of automorphic forms. Theta correspondence originated from the studies of theta series (see [Si], [WE65], [We65], [Ho79]). As pointed out to me by Jian-Shu Li, theta correspondence should map automorphic representations to automorphic representations (see [Ho79], [Ra87], [Li94]). Thus representations in $\mathcal{N}(\mathcal{O})$ should all be automorphic. In the final part of this paper, along the lines of [VO86], we make some conjectures regarding the automorphic dual in the sense of Burger-Li-Sarnak [BLS].

I wish to acknowledge my gratitude to Monica Nevins, Shu-Yen Pan, Tomasz Przebinda and David Vogan for some very helpful e-mail communications and to Bill Graham, Chen-Bo Zhu, Gestur Olafsson and Ray Fabec for reading the manuscript and for their comments. I also wish to thank the referee for his very valuable suggestions and comments, in particular his suggestions concerning Definition 31 and the proof of Theorem 4.11. In the midst of revising this paper, my father Decai He, passed away. My father had always encouraged and inspired me in my mathematical endeavors. I would like to dedicate this paper to him.

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Chapter 1

Invariants

In mathematics, classification problems are often approached by constructing invariants. In this chapter, we attach invariants to the equivalence classes of irreducible admissible Hilbert representations of a reductive group G . One hopes that these invariants could shed some light on the classification and construction of irreducible unitary representations. Our purpose here is not to give a historical account of these invariants, but rather, to review some basic facts we need concerning these invariants. We adopt the standard notations from [KN] and [Wallach].

1.1 Notation

Reductive groups and semisimple groups in this paper are assumed to have at most a finite number of components. Let G be a Lie group. We adopt the following notation:

1. G_0 —the identity component of G ;
2. \mathfrak{g} —the real Lie algebra of G_0 ;
3. \mathfrak{g}^* —the dual space of \mathfrak{g} ;
4. $\mathfrak{g}_{\mathbb{C}}$ —the complex Lie algebra of G_0 ;
5. $U(\mathfrak{g})$ —the complex universal enveloping algebra;
6. $Z(\mathfrak{g})$ —the center of $U(\mathfrak{g})$;

Let G be a real reductive group. We adopt the following notation:

1. K —a maximal compact subgroup of G ;
2. $\Pi(G)$ —the set of equivalence classes of irreducible (\mathfrak{g}, K) -modules;
3. $\Pi_u(G)$ —the set of equivalence classes of unitarizable (\mathfrak{g}, K) -modules, or equivalently, the set of equivalence classes of irreducible unitary representations of G ;
4. $(\ , \)$ —an invariant quadratic form on \mathfrak{g} such that $(\ , \)|_{\mathfrak{k}}$ is positive definite;
5. \mathfrak{p} —the orthogonal complement of \mathfrak{k} with respect to $(\ , \)$;

6. \mathfrak{a} —a maximal Abelian Lie subalgebra of \mathfrak{p} ;
7. A —the connected Abelian group generated infinitesimally by \mathfrak{a} ;
8. KAK —a Cartan decomposition.

Let V be a finite dimensional vector space over \mathbb{F} . We use V^* to denote $\text{Hom}_{\mathbb{F}}(V, \mathbb{F})$. Let G be a real reductive group.

Notation 1 *Let (π, \mathcal{H}) be a Hilbert representation of G . Fix a maximal compact subgroup K . Let $V(\pi)$ be the space of smooth K -finite vectors in \mathcal{H} . To emphasize π , \mathcal{H} is often denoted by \mathcal{H}_π .*

The space $V(\pi)$ is a (\mathfrak{g}, K) -module. π is said to be admissible if each K -type occurs in $V(\pi)$ with finite multiplicity. If, in addition, $V(\pi)$ is finitely generated as a $U(\mathfrak{g})$ -module, then $V(\pi)$ is often called a Harish-Chandra module.

If π is unitary and irreducible, then the Hilbert norm is unique up to a scalar multiplication. So, equivalence classes of irreducible unitary representations are in one-to-one correspondence with irreducible unitarizable Harish-Chandra modules.

All unitary representations in this paper, unless otherwise stated, are taken as suitable unitarized Harish-Chandra modules.

This convention is consistent with the way we construct irreducible unitary representations. First, we will construct a (\mathfrak{g}, K) -module (π, V) . Then we will show that V is an irreducible (\mathfrak{g}, K) -module. Hence V is a Harish-Chandra module. Finally, we will prove the existence of an invariant inner product on V . Then π is an irreducible unitary representation of G . For simplicity, we use π to denote the group action and the Lie algebra action. We define three involutions in the category of Harish-Chandra modules:

1. π^* , the contragredient representation;
2. π^c , the representation π equipped with the conjugate complex linear structure;
3. π^h , the Hermitian dual representation of π .

We have $(\pi^*)^c = \pi^h$. If π is unitary, then $\pi^h \cong \pi$.

Notation 2 *Suppose $a, b \in \mathbb{R}^n$. Write $a \preceq b$ if and only if for every $1 \leq k \leq n$,*

$$\sum_{j=1}^k a_j \leq \sum_{j=1}^k b_j;$$

write $a \prec b$ if and only if for every $1 \leq k \leq n$,

$$\sum_{j=1}^k a_j < \sum_{j=1}^k b_j.$$

The ordering \preceq is a partial ordering.

1.2 Infinitesimal Character and Harish-Chandra Homomorphism

Let \mathfrak{g} be a complex reductive Lie algebra. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . Let $W(\mathfrak{g}, \mathfrak{h})$ be the Weyl group generated by the root system $\Sigma(\mathfrak{g}, \mathfrak{h})$. Let $U(\mathfrak{h})^{W(\mathfrak{g}, \mathfrak{h})}$ be the space of $W(\mathfrak{g}, \mathfrak{h})$ -invariant vectors in $U(\mathfrak{h})$. Then the Harish-Chandra homomorphism

$$\mathcal{HC} : Z(\mathfrak{g}) \rightarrow U(\mathfrak{h})^{W(\mathfrak{g}, \mathfrak{h})}$$

is an algebra isomorphism (see [KN], Chapter VIII. 5 or [Wallach]). Identify $U(\mathfrak{h})$ with the symmetric algebra of \mathfrak{h} . For each vector Λ in the complex dual space of \mathfrak{h} , define a character χ_Λ of $Z(\mathfrak{g})$ by

$$\chi_\Lambda(x) = \Lambda(\mathcal{HC}(x)) = \mathcal{HC}(x)(\Lambda) \quad (x \in Z(\mathfrak{g}))$$

It is well-known that every character of $Z(\mathfrak{g})$ can be obtained this way and that Λ is unique up to the action of $W(\mathfrak{g}, \mathfrak{h})$. In short, $\text{Spec}(Z(\mathfrak{g})) \cong \mathfrak{h}^* // W(\mathfrak{g}, \mathfrak{h})$. Here the categorical quotient $\mathfrak{h}^* // W(\mathfrak{g}, \mathfrak{h})$ coincides with the geometric quotient since $W(\mathfrak{g}, \mathfrak{h})$ is finite.

Let G be a connected real reductive Lie group with Lie algebra \mathfrak{g} . Let K be a maximal compact subgroup of G . Let (π, \mathcal{H}) be an irreducible admissible Hilbert representation of G . Let $V(\pi)$ be the Harish-Chandra module of (π, \mathcal{H}) , consisting of all the K -finite vectors in the Hilbert space \mathcal{H} . We retain π for the infinitesimal action of $U(\mathfrak{g})$ on the smooth vectors of (π, \mathcal{H}) . Since π is irreducible, $V(\pi)$ is an irreducible $(U(\mathfrak{g}), K)$ -module. Since G is connected, $Z(\mathfrak{g}) = U(\mathfrak{g})^G$. By Schur's lemma, $Z(\mathfrak{g})$ acts on $V(\pi)$ by a character

$$\chi : Z(\mathfrak{g}) \rightarrow \mathbb{C}.$$

Thus there exists a Λ such that $Z(\mathfrak{g})$ acts on $V(\pi)$ by χ_Λ . For simplicity, we call Λ an **infinitesimal character** of π .

Let G be a real reductive group with a finite number of components. Let \mathfrak{h} be a complex Cartan subalgebra in $\mathfrak{g}_{\mathbb{C}}$. Then $U(\mathfrak{g})^{G_0} = Z(\mathfrak{g})$. Let $U(\mathfrak{g})^G$ be the G -invariant vectors in $U(\mathfrak{g})$. Consider the adjoint action of G/G_0 on $Z(\mathfrak{g})$. Each element in G/G_0 acts on $Z(\mathfrak{g})$ as an algebra automorphism. Furthermore, $U(\mathfrak{g})^G$ is precisely the subalgebra of $Z(\mathfrak{g})$ invariant under the action of G/G_0 . By the Harish-Chandra homomorphism, G/G_0 acts on

$$U(\mathfrak{h})^{W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})}$$

as algebra automorphisms. This action induces an action of G/G_0 on

$$\text{Spec}(U(\mathfrak{h})^{W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})}).$$

Since G/G_0 is finite, by invariant theory

$$\text{Spec}(\mathcal{HC}(U(\mathfrak{g})^G))$$

is precisely in one-to-one correspondence with the G/G_0 -orbits in

$$\text{Spec}(U(\mathfrak{h})^{W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})}).$$

Let π be an irreducible (\mathfrak{g}, K) -module. By Schur's lemma, $U(\mathfrak{g})^G$ must act by a character ξ .

Definition 1 Let \mathfrak{h} be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We say that $\Lambda \in \mathfrak{h}^*$ is an infinitesimal character of π if $\chi_{\Lambda}|_{U(\mathfrak{g})^G} = \xi$.

Λ is unique up to the action of G/G_0 and $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$. For semisimple Lie group G , the action of G/G_0 on $\mathfrak{h}^*/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ is not difficult to understand. Let ι be the action of G/G_0 on $\mathfrak{h}^*/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$, induced from the action of G/G_0 on \mathfrak{g}^*/G_0 . Then ι induces an action of G/G_0 on a closed Weyl chamber, consequently on the positive root system. Hence $\iota(G/G_0)$ can be regarded as automorphisms of the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$. The automorphisms of the Dynkin diagram are not difficult to classify.

The infinitesimal character is one of the main tools used in the literature to study irreducible admissible representations. In fact, there is only a finite number of infinitesimal equivalence classes of irreducible representations with a fixed infinitesimal character.

Notation 3 Let $\Pi_{\Lambda}(G)$ be the set of infinitesimal equivalence classes of irreducible admissible Hilbert representations of G with infinitesimal character Λ .

1.3 Leading Exponents of Irreducible Representations

Let G be a real reductive Lie group. Fix a maximal compact subgroup K . Unless otherwise stated, matrix coefficients in this paper are assumed to be K -finite. Fix a nondegenerate real invariant bilinear form $(\ , \)$ on \mathfrak{g} and a maximal Abelian Lie subalgebra \mathfrak{a} of \mathfrak{p} as in (1.1). Let r be the real dimension of \mathfrak{a} . We call r the *real rank* of G . Let $\Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system.

Fix a positive root system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. Let $\rho(G)$ be the half sum of all positive roots in $\Sigma(\mathfrak{g}, \mathfrak{a})$. Let M be the centralizer of \mathfrak{a} in K . Let $W(G, \mathfrak{a})$ be the normalizer of \mathfrak{a} in K modulo M . We call $W(G, \mathfrak{a})$ the real Weyl group. Let $W(\mathfrak{g}, \mathfrak{a})$ be the Weyl group generated by the root system. Clearly, $W(\mathfrak{g}, \mathfrak{a}) \subseteq W(G, \mathfrak{a})$.

For $Sp_{2n}(\mathbb{R})$, \mathfrak{a} is isomorphic to \mathbb{R}^n . The real Weyl group $W(Sp_{2n}(\mathbb{R}), \mathfrak{a})$ is generated by permutations and sign changes on n variables. For $O(p, q)$, \mathfrak{a} is isomorphic to $\mathbb{R}^{\min\{p, q\}}$. The real Weyl group $W(O(p, q), \mathfrak{a})$ is also generated by permutations and sign changes on $\min\{p, q\}$ variables. See 1.1.1 and 1.3.2.

Attached to each irreducible admissible representation π of a connected G , is a finite number of vectors in $\mathfrak{a}_{\mathbb{C}}^*$, called leading exponents. Leading exponents are the main data used to produce the Langlands classification (see [LA], [KN], [Wallach]). For connected G , leading exponents depend on the choice of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. For G with finite number of connected components, we define the leading exponents of $\pi \in \Pi(G)$ to be the leading exponents of the irreducible subrepresentations of $\pi|_{G_0}$.

Leading exponents are closely related to the infinitesimal character. For an irreducible finite dimensional representation π of a connected real reductive group G with real rank equal to complex rank, leading exponent is just the highest weight of π with respect to a maximally split Cartan subgroup. In this situation, finite dimensional representation theory says that the highest weight

v is related to the infinitesimal character Λ by the following equation

$$v + \rho = w\Lambda$$

for some w in $W(G, \mathfrak{h})$. A similar statement holds for leading exponents for any irreducible admissible representation of a real reductive group. By Theorem 8.33 from [KN], we have

Theorem 1.1 *Let \mathfrak{b} be a Cartan subalgebra of \mathfrak{m} . Take $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b}$. Suppose π is an irreducible admissible representation of a real reductive group G . Let v be a leading exponent of π . Then there exists an infinitesimal character Λ of π in $\mathfrak{h}_{\mathbb{C}}^*$ such that*

$$v + \rho(G) = \Lambda|_{\mathfrak{a}}.$$

Notice that for non-split groups, v is in $\mathfrak{a}_{\mathbb{C}}^*$ and Λ is in $(\mathfrak{a} \oplus \mathfrak{b})_{\mathbb{C}}^*$. By Theorem 1.1, for any $\pi \in \Pi_{\Lambda}$, the set of leading exponents is finite.

Notation 4 *For $v \in \mathfrak{a}_{\mathbb{C}}^*$, the complex dual of \mathfrak{a} , we denote the real part of v by $\Re(v)$.*

Leading exponents are extracted from a certain asymptotic expansion of the matrix coefficients at ∞ . Therefore, they control the growth of matrix coefficients. We cite the following estimate from [KN].

Theorem 1.2 *Let π be an irreducible admissible representation of a real reductive group G . Let v_0 be in the real dual of \mathfrak{a} . Let \mathfrak{a}^+ be the closed Weyl Chamber associated with $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ (see Prop. 5.14 [KN]). Let A^+ be the closed Weyl Chamber in A obtained by exponentiating \mathfrak{a}^+ . Let $a(g)$ be the middle term of the KA^+K decomposition of g . If every leading exponent v of π satisfies*

$$(v_0 - \Re(v))(H) \geq 0 \quad (\forall H \in \mathfrak{a}^+), \quad (1.1)$$

then there is an integer $q \geq 0$ such that each K -finite matrix coefficient of π is dominated by a multiple of

$$\exp(v_0(\log a(g)))(1 + (\log a(g), \log a(g))^{\frac{1}{2}})^q.$$

The converse also holds.

When $G = O(p, p)$, due to the action of G/G_0 , A^+ and \mathfrak{a}^+ can be made smaller (see 1.3.1). Theorem 1.2 remains valid for this generalized Weyl chamber A^+ .

Theorem 1.2 provides a uniform bound for matrix coefficients for all $\pi \in \Pi_{\Lambda}(G)$. Let π be a unitary representation in $\Pi_{\Lambda}(G)$. If Λ is “small”, we can easily find a v_0 to dominate the set

$$\{w\Lambda|_{\mathfrak{a}} - \rho(G) \mid w \in W(G, \mathfrak{h}_{\mathbb{C}})\}.$$

By Theorem 1.2, we gain fairly good control over the growth of the matrix coefficients for all $\pi \in \Pi_{\Lambda}$. If Λ is large, then the convex cone spanned by the set

$$\{w\Lambda|_{\mathfrak{a}} - \rho(G) \mid w \in W(G, \mathfrak{h}_{\mathbb{C}})\}$$

is widely spread. The uniform bound from Theorem 1.2 does not yield any useful information.

For unitary representations in $\Pi(G)$, the matrix coefficients are all bounded by constant functions. Theorem 1.2 then implies

Corollary 1.1 *Let G be a real reductive group. If $\pi \in \Pi_u(G)$, then every leading exponent v of π satisfies*

$$\Re(v)(H) \leq 0 \quad (\forall H \in \mathfrak{a}^+).$$

In this paper, we will mostly deal with small Λ . To give a broader picture of the importance of $\Pi_u(G)$ with small infinitesimal character, we recall one definition from [VO00] and [SV].

Definition 2 *A unitary representation $\pi \in \Pi_\Lambda(G)$ is called “unitarily small” if Λ is in the convex hull spanned by*

$$\{w(\rho(\mathfrak{g}_\mathbb{C})) \mid w \in W(\mathfrak{g}, \mathfrak{h})\}.$$

Salamanca-Vogan conjectured that any irreducible unitary representation of G not small may be constructed by parabolic or cohomological induction from an irreducible unitarily small representation of a reductive subgroup of G ([SV], [VO00]).

It is precisely for “unitarily small” representations that Theorem 1.2 will produce useful information about matrix coefficients beyond what is stated in the corollary. One goal of this paper is to construct and study some unitarily small representations far away from the tempered unitary representations.

1.3.1 Example I: The groups $Mp_{2n}(\mathbb{R})$ and $Sp_{2n}(\mathbb{R})$

Let G be either $Mp_{2n}(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$. Fix

$$\mathfrak{a} = \{ \text{diag}(a_1, a_2, \dots, a_n, -a_1, -a_2, \dots, -a_n) \}.$$

Then the Weyl group $W(G, \mathfrak{a})$ is generated by permutations and sign changes on $\{a_i\}_{i=1}^n$. Fix positive roots $\Sigma^+ = \{e_i \pm e_j \mid i \leq j; i, j \in [1, n]\}$. Then

$$\rho(G) = (n, n-1, \dots, 1)$$

and

$$\mathfrak{a}^+ = \{ \text{diag}(a_1, a_2, \dots, a_n, -a_1, -a_2, \dots, -a_n) \mid a_1 \geq a_2 \geq \dots \geq a_n \geq 0 \}.$$

Clearly,

$$A^+ = \{ \text{diag}(A_1, A_2, \dots, A_n, A_1^{-1}, A_2^{-1}, \dots, A_n^{-1}) \mid A_1 \geq A_2 \geq \dots \geq A_n \geq 1 \}.$$

Finally, the Inequality (1.1) is equivalent to

$$v_0 \succeq \Re(v)$$

(see Notation 2 and 4).

1.3.2 Example II: The Groups $O(p, q)$

Let $G = O(p, q)$ with $q \geq p$. Here $O(p, q)$ is the isometry group of

$$(x, y) = \sum_{i=1}^p (x_i y_{p+i} + x_{p+i} y_i) + \sum_{j=1}^{q-p} x_{2p+j} y_{2p+j}.$$

Fix

$$\mathfrak{a} = \{ \text{diag}(a_1, a_2, \dots, a_p, -a_1, -a_2, \dots, -a_p, 0, 0, \dots, 0) \mid a_i \in \mathbb{R} \}.$$

Then the Weyl group $W(G, \mathfrak{a})$ is again generated by permutations and sign changes on $\{a_i\}_{i=1}^p$. Notice for $p = q$, $W(G, \mathfrak{a})$ is bigger than $W(\mathfrak{g}, \mathfrak{a})$.

Fix restricted positive roots

$$\Sigma^+ = \{e_i \pm e_j \mid i < j; i, j \in [1, p]\} \cup \{e_i \mid i \in [1, p]\} \quad \text{if } p < q$$

and

$$\Sigma^+ = \{e_i \pm e_j \mid i < j; i, j \in [1, p]\} \quad \text{if } p = q.$$

Then

$$\rho(G) = \left(\overbrace{\frac{p+q-2}{2}, \frac{p+q-4}{2}, \dots, \frac{q-p}{2}}^p \right).$$

Fix a (generalized) Weyl chamber

$$\mathfrak{a}^+ = \{ \text{diag}(a_1, a_2, \dots, a_p, -a_1, -a_2, \dots, -a_p, 0, 0, \dots, 0) \mid a_1 \geq a_2 \geq \dots, a_n \geq 0 \}.$$

Then

$$A^+ = \{ \text{diag}(A_1, A_2, \dots, A_p, A_1^{-1}, A_2^{-1}, \dots, A_p^{-1}) \mid A_1 \geq A_2 \geq \dots, A_p \geq 1 \}.$$

The Inequality (1.1) is again equivalent to $v_0 \succeq \Re(v)$. Notice that for $O(p, p)$, due to the action of $O(p, p)/SO_0(p, p)$, A^+ is half of the Weyl Chamber determined by Σ^+ .

1.4 Global Characters

The global character is also known as the Harish-Chandra character. For each irreducible admissible Hilbert representation π of G and a compactly supported smooth function f on G , define $\Theta(\pi)(f)$ to be the trace of the operator

$$\pi(f) = \int f(g)\pi(g)dg.$$

A theorem of Harish-Chandra states that Θ_π is a distribution on G which can be identified with a locally integrable function (still denoted by Θ_π) and Θ_π is real analytic on the set of regular semisimple elements of G (see [HC]).

For π unitary, Milićić defined a notion of the rate of growth of Θ_π which we will not recall here. We will state one theorem relating γ to the matrix coefficients of π (see [MI]).

Theorem 1.3 (Thm. 1 [MI]) *Let $\pi \in \Pi_u(G)$. Let $\Xi(g)$ be Harish-Chandra's Ξ function. Then the following are equivalent:*

1. Θ_π has the rate of growth $\gamma \in \mathbb{R}$;
2. every K -finite matrix coefficient of π is bounded by

$$C\Xi^{1-\gamma}(g)(1 + \|\log(a(g))\|)^s \quad (C, s \geq 0, a(g) \in A^+);$$

3. every leading exponent v of π satisfies

$$\Re(v)(H) \leq (\gamma - 1)\rho(G)(H) \quad (\forall H \in \mathfrak{a}^+).$$

So if $\gamma = 0$ then π is tempered and $\gamma = 1$ if π is trivial. For $G = Mp_{2n}(\mathbb{R})$ or $G = O(p, q)$, the last statement is equivalent to

$$\Re(v) \preceq (\gamma - 1)\rho(G).$$

1.5 Associated Variety, Asymptotic Cycle and Wave Front Set

Recall that $U(\mathfrak{g})$ has a natural filtration

$$\mathbb{C} = U_0(\mathfrak{g}) \subseteq U_1(\mathfrak{g}) \subseteq U_2(\mathfrak{g}) \subseteq U_3(\mathfrak{g}) \subseteq \dots \subseteq U_n(\mathfrak{g}) \subseteq \dots$$

1.5.1 Annihilator and Associated Variety

Let π be an admissible irreducible representation of a reductive group G . Let $V(\pi)$ be the Harish-Chandra module of π . Then $V(\pi)$ is a $U(\mathfrak{g})$ module. Consider the annihilator of $V(\pi)$,

$$\text{Ann}(V(\pi)) = \{D \in U(\mathfrak{g}) \mid \pi(D)V(\pi) = 0\}.$$

$\text{Ann}(V(\pi))$ is an ideal of $U(\mathfrak{g})$ and inherits a filtration from the standard filtration of $U(\mathfrak{g})$. It follows that the induced graded algebra $gr(\text{Ann}(V(\pi)))$ is an ideal of $gr(U(\mathfrak{g})) = S(\mathfrak{g})$ and necessarily commutative.

Notation 5 Let $\mathcal{V}(\text{Ann } \pi)$ be the associated variety of $gr(\text{Ann}(V(\pi)))$.

Theorem 1.4 (Borho-Brylinski [BB], Joseph [Jo]) Suppose G is a connected semisimple group and π is an irreducible representation of G . Let \mathfrak{g}^* be the dual space of $\mathfrak{g}_{\mathbb{C}}$. Then $\mathcal{V}(\text{Ann } \pi)$ is the closure of a single nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}^*$.

For G having a finite number of components, $\mathcal{V}(\text{Ann } \pi)$ will be the closure of a finite number of nilpotent coadjoint orbits of equal dimension. In fact, G/G_0 acts on the set of nilpotent coadjoint orbits. The associated variety $\mathcal{V}(\text{Ann } \pi)$ will be the closure of the union of one nilpotent orbit with its translations under $Ad(G/G_0)$. For a detailed account of associated varieties of Harish-Chandra modules, see [VO89].

1.5.2 Asymptotic Cycle and Wave Front Set

Let G be a semisimple Lie group with finitely many components. Let us consider the global character Θ_{π} . One can lift Θ_{π} to an invariant distribution D on \mathfrak{g} . There exists a Taylor expansion of D near 0,

$$D(f(tx)) \cong \sum_{i=-r}^{\infty} t^i D_i(f(x)).$$

In [BV80], Barbasch-Vogan proved that the Fourier transform \widehat{D}_i is supported on the nilcone of \mathfrak{g}^* .

Notation 6 Let $\text{supp}(AS(\pi))$ be the closure of the union of supports of \widehat{D}_i . We call $\text{supp}(AS(\pi))$ the support of the asymptotic cycle of π .

Theorem 1.5 (Barbasch-Vogan) Suppose G is a connected semisimple Lie group. Then the real dimension of $\text{supp}(AS(\pi))$ is equal to the complex dimension of $\mathcal{V}(Ann \pi)$. $\text{supp}(AS(\pi))$ is a union of nilpotent orbits contained in $\mathfrak{g} \cap \mathcal{V}(Ann \pi)$. Let $r = \frac{\dim(AS(\pi))}{2}$. Then D_{-r} is the lowest nonzero term of the Taylor expansion of D and $\text{supp}(\widehat{D_{-r}})$ is of maximal dimension in $\text{supp}(AS(\pi))$.

This theorem holds for G with a finite number of components. See [BV80].

Another notion similar to $\text{supp}(AS(\pi))$ is the wave front set of π defined by Howe ([Ho81]). Originally, $WF(\pi)$ was defined as a closed subset of the cotangent bundle T^*G . Because of the G -action, $WF(\pi)$ can be regarded as a closed subset of \mathfrak{g}^* . For G semisimple and π irreducible, $WF(\pi)$ is in the nilpotent cone of \mathfrak{g}^* . In [Ho81], Howe studied the behavior of wave front sets $WF(\pi)$ under restrictions to certain subgroups.

Suppose that π is irreducible and G is semisimple. Rossmann proved that $WF(\pi)$ and $\text{supp}(AS(\pi))$ are identical (see Theorem C, [Ro]). In what follows, we will not distinguish between $WF(\pi)$ and $\text{supp}(AS(\pi))$ as long as π is irreducible. There are two basic facts the reader should keep in mind. The first fact is that $WF(\pi)$ lies in the real space \mathfrak{g}^* . The second fact is that the algebraic closure of $WF(\pi)$ is exactly the associated variety $\mathcal{V}(Ann \pi) \subset \mathfrak{g}_{\mathbb{C}}^*$.

Notation 7 From now on, we will identify \mathfrak{g}^* with \mathfrak{g} using a fixed invariant bilinear form on \mathfrak{g} . We will also identify $\mathfrak{g}_{\mathbb{C}}^*$ with $\mathfrak{g}_{\mathbb{C}}$ as complexifications of \mathfrak{g}^* and \mathfrak{g} .

Chapter 2

Nilpotent Orbits

An element x in a Lie algebra \mathfrak{g} is called nilpotent if $ad(x)$ is nilpotent. Let G be a Lie group with Lie algebra \mathfrak{g} . If x is nilpotent and $g \in G$, then $Ad(g)x$ is also nilpotent. It follows that under the adjoint action of G , nilpotent elements are grouped into G -orbits. Each G -orbit is called a nilpotent adjoint orbit. For semisimple Lie groups, there are finitely many nilpotent orbits and the classification of nilpotent adjoint orbits is completely known. We cite Collingwood-McGovern's book [CM] as the main reference for this chapter. Unlike in [CM], nilpotent orbits in this paper depend on the Lie groups, not just on the Lie algebra. In particular, for the orthogonal groups, a nilpotent orbit may not be connected. For our convenience, we state the Springer-Steinberg theorem slightly differently than in [CM]. The reason for doing this is given in Theorem 2.6. The main results are Theorems 2.3, 2.6, 2.7.

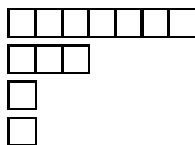
2.1 Young Diagrams and Complex Nilpotent Orbits

A sequence of positive integers

$$\mathbf{d} = (d_1 \geq d_2 \geq \dots \geq d_{r-1} \geq d_r > 0 = d_{r+1})$$

is said to be a partition of n if $n = \sum_{j=1}^r d_j$. Write $\|\mathbf{d}\| = n$.

Notation 8 *In this paper, a partition of n will be represented by a Young diagram of n boxes, arranged as follows:*



with the i -th row of length d_i . The transpose of \mathbf{d} is denoted by \mathbf{d}^t .

In our notation, the Young diagram above can be written as $\mathbf{d} = (7, 3, 1, 1)$ and its transpose $\mathbf{d}^t = (4, 2, 2, 1, 1, 1, 1)$.

Definition 3 *If d_j is odd for every $j \leq r$, we say \mathbf{d} is very odd. If d_j is even for every $j \leq r$, we say \mathbf{d} is very even. If the nonzero d_j are all distinct, we say \mathbf{d} is multiplicity free. A row of even*

length is called an even row. A row of odd length is called an odd row.

2.1.1 Complex Nilpotent Orbits of $Sp_{2n}(\mathbb{C})$

Definition 4 A Young diagram \mathbf{d} is said to be a symplectic Young diagram of size $2n$ if odd rows occur with even multiplicity and $\|\mathbf{d}\| = 2n$. We use $\mathcal{Y}D_-(2n)$ to denote the set of symplectic Young diagrams of size $2n$.

A symplectic group is the linear group that preserves a nondegenerate skew-symmetric form.

Theorem 2.1 Nilpotent adjoint orbits of $Sp_{2n}(\mathbb{C})$ are in one to one correspondence with $\mathcal{Y}D_-(2n)$.

We denote the nilpotent adjoint orbit corresponding to \mathbf{d} by $\mathcal{O}_{\mathbf{d}}(-)$. The subscript $-$ is used to indicate that the group G is defined with respect to a skew-symmetric form. If $\mathcal{O}_{\mathbf{d}}(-)$ is known to be symplectic, we will simply write $\mathcal{O}_{\mathbf{d}}$. For our convenience, we denote the symplectic group $Sp_{2n}(\mathbb{C})$ by $G(\mathcal{O}_{\mathbf{d}}(-))$ or $G(\mathcal{O}_{\mathbf{d}})$. The group $Sp_{2n}(\mathbb{C})$ is thus attached to the orbit implicitly.

2.1.2 Complex Nilpotent Orbits of $O(n, \mathbb{C})$

Definition 5 A Young diagram \mathbf{d} is said to be an orthogonal Young diagram of size n if even rows occur with even multiplicity and $\|\mathbf{d}\| = n$. We use $\mathcal{Y}D_+(n)$ to denote the set of all orthogonal Young diagrams of size n .

An orthogonal group is a linear group preserving a nondegenerate symmetric form.

Theorem 2.2 Nilpotent adjoint orbits of $O(n, \mathbb{C})$ are in one to one correspondence with $\mathcal{Y}D_+(n)$.

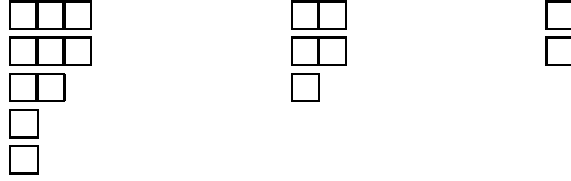
We denote the nilpotent adjoint orbit corresponding to \mathbf{d} by $\mathcal{O}_{\mathbf{d}}(+)$. The script $+$ is used to indicate that the group G preserves a symmetric form. If the orbit $\mathcal{O}_{\mathbf{d}}(+)$ is known to be orthogonal, we will simply denote it by $\mathcal{O}_{\mathbf{d}}$. In this context, the orthogonal group $O(n, \mathbb{C})$ is denoted by $G(\mathcal{O}_{\mathbf{d}}(+))$ or $G(\mathcal{O}_{\mathbf{d}})$.

Depending on the context, $\mathcal{O}_{\mathbf{d}}$ can refer to either $\mathcal{O}_{\mathbf{d}}(+)$ or $\mathcal{O}_{\mathbf{d}}(-)$.

2.1.3 Operator -1

Definition 6 Let \mathbf{d} be a Young diagram of size n . Define $\mathbf{d} - \mathbf{1}$ to be the new Young diagram obtained by deleting the first column of \mathbf{d} . Define $\mathbf{d} - \mathbf{i}$ to be the new Young diagram obtained by deleting the first i columns of \mathbf{d} .

Consider $\mathbf{d} = [3^2, 2^1, 1^2]$. The Young diagrams \mathbf{d} , $\mathbf{d} - \mathbf{1}$ and $\mathbf{d} - \mathbf{2}$ are listed as follows



Notice that \mathbf{d} is a symplectic Young diagram, $\mathbf{d} - \mathbf{1}$ is an orthogonal Young diagram, and $\mathbf{d} - \mathbf{2}$ is a symplectic Young diagram.

Theorem 2.3 *If \mathbf{d} is a symplectic Young diagram, then $\mathbf{d} - \mathbf{1}$ is an orthogonal Young diagram. If \mathbf{d} is an orthogonal Young diagram, then $\mathbf{d} - \mathbf{1}$ is a symplectic Young diagram.*

2.2 Signed Young Diagrams and Real Nilpotent Orbits

Type I classical groups are subgroups of the general linear groups that preserve certain sesquilinear forms (see [LI89] for the definition). Let G be a real classical group of type I. Real nilpotent orbits of G are simply nilpotent G -orbits in the real Lie algebra \mathfrak{g} . For type I classical groups, the real nilpotent orbits for G are parameterized by equivalence classes of signed Young diagrams.

Definition 7 (Ch. 9.3 [CM]) *Signed Young diagrams are Young diagrams with $+$ or $-$ labeling the boxes in such a way that signs alternate across rows. Two signed Young diagrams are considered to be equivalent if one signed diagram can be obtained from the other by interchanging the rows of same lengths.*

Let \mathbf{D} be a signed Young diagram. We will use D^+ to denote the number of positive boxes in \mathbf{D} and D^- to denote the number of negative boxes in \mathbf{D} . We call (D^+, D^-) the signature of \mathbf{D} .

In this chapter, we are only interested in the nilpotent orbits of $Sp_{2n}(\mathbb{R})$ and $O(p, q)$.

Notation 9 *For the sake of our discussion, we fix a matrix realization for each G . So the group $O(p, q)$ and $Sp_{2n}(\mathbb{R})$ in this paper, will come with a fixed sesquilinear form.*

The correspondence between signed Young diagrams and real nilpotent orbits depends on the sesquilinear forms. I would like to thank the referee for pointing this to me.

2.2.1 Real Orbits of $Sp_{2n}(\mathbb{R})$

Theorem 2.4 (Springer and Steinberg) *Nilpotent adjoint orbits of $Sp_{2n}(\mathbb{R})$ are parametrized by the equivalence classes of signed Young diagrams with the following properties:*

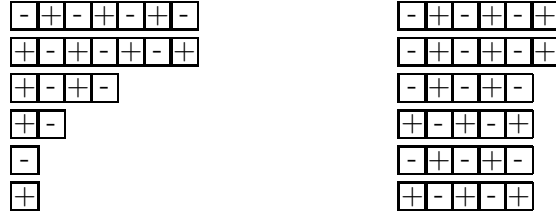
- the signature of \mathbf{D} is (n, n) ;
- for every s , rows of length $2s + 1$ must occur with even multiplicity and must have their leftmost boxes labeled

$$-, +, -, +, \dots, -, +$$

from the highest row to the lowest row.

Clearly, the signed Young diagrams that satisfy our second condition must have signature (n, n) . So, the first condition is redundant. In the second condition, the choice of the sign pattern of rows of odd lengths is artificial. In fact, it suffices that there are same number of boxes labeled $+$ and $-$ in the rows of length $2s + 1$. Nevertheless, I have included this in Theorem 2.4 for two purposes. The first is to maximize the analogy with Theorem 2.5. The second is to make the signed young diagrams easy to manipulate.

We denote the set of signed Young diagrams in Theorem 2.4 by $\mathcal{YD}_-(n, n)$. The following signed Young diagrams are in $\mathcal{YD}_-(n, n)$:



Notice that $Mp_{2n}(\mathbb{R})$ -adjoint orbits coincide with $Sp_{2n}(\mathbb{R})$ - adjoint orbits.

Notation 10 We denote the nilpotent orbit corresponding to \mathbf{D} by $\mathcal{O}_{\mathbf{D}}(-)$ or simply $\mathcal{O}_{\mathbf{D}}$ if \mathbf{D} is specified to be symplectic. We denote the group $Mp_{2n}(\mathbb{R})$ by $G(\mathcal{O}_{\mathbf{D}})$.

The reader should notice that $G(\mathcal{O}_{\mathbf{d}})$ is a complex group of which $G(\mathcal{O}_{\mathbf{D}})$ is a real form.

Definition 8 Define

$$\tau(x) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} x \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (x \in \mathfrak{sp}_{2n}(\mathbb{R})).$$

Notice that

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = - \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and changing the symplectic form \langle , \rangle to $-\langle , \rangle$ results in the same group $Sp_{2n}(\mathbb{R})$. Therefore τ defines an involution on $\mathfrak{sp}_{2n}(\mathbb{R})$ and on $Sp_{2n}(\mathbb{R})$. By sorting out the signs in the proof of Lemma 9.3.1 in [CM], we obtain

Lemma 2.2.1 1. τ defines an involution on the real Lie algebra $\mathfrak{sp}_{2n}(\mathbb{R})$.

2. τ induces an involution on the set of nilpotent orbits, namely

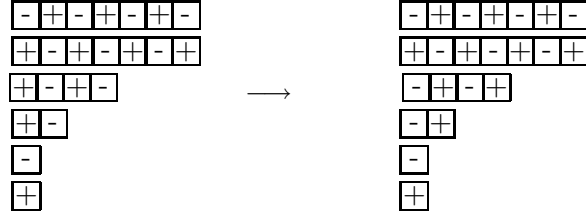
$$\tau(\mathcal{O}_{\mathbf{D}}) = \mathcal{O}_{\tau(\mathbf{D})}.$$

Here $\tau(\mathbf{D})$ is the signed Young diagram obtained by switching the signs ($+$ \leftrightarrow $-$) for the even rows of \mathbf{D} .

3. The involution τ defines a diffeomorphism from $\mathcal{O}_{\mathbf{D}}$ onto $\mathcal{O}_{\tau(\mathbf{D})}$.

The proof is left to the reader.

Notice that the odd rows of $\tau(\mathbf{D})$ remain the same as those of \mathbf{D} . For example, τ :



One can further explore the involution τ on the Lie group level.

Definition 9 1. Define τ on $Sp_{2n}(\mathbb{R})$ by

$$\tau(g) = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} g \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

Clearly τ defines an involution on $Sp_{2n}(\mathbb{R})$. τ is a topological homeomorphism and an isometry with respect to a certain left invariant Riemannian metric.

2. Lift τ to an involution on $Mp_{2n}(\mathbb{R})$. The lift τ exists and is unique. By abusing notation, denote this involution by τ .

3. Let $\pi \in \Pi(Mp_{2n}(\mathbb{R}))$. Define a new representation $(\pi^\tau, \mathcal{H}_\pi)$ by

$$\pi^\tau(g) = \pi(\tau(g)).$$

Lemma 2.2.2 $\pi \rightarrow \pi^\tau$ defines an involution on $\Pi(Mp_{2n}(\mathbb{R}))$ and on $\Pi_u(Mp_{2n}(\mathbb{R}))$.

2.2.2 Real Orbits of $O(p, q)$

Theorem 2.5 (Springer & Steinberg) Nilpotent adjoint orbits of $O(p, q)$ are parametrized by equivalence classes of signed Young diagrams with the following properties

- the signature of \mathbf{D} is (p, q) ;
- for every s , rows of even length $2s$ must occur with even multiplicity and must have their leftmost boxes labeled

$$+, -, +, -, \dots, +, -$$

from the highest row to the lowest row.

Again, the choice of sign patterns for rows of even length is artificial. We denote the set of signed Young diagrams in Theorem 2.5 by $\mathcal{YD}_+(p, q)$. The following signed Young diagrams are in $\mathcal{YD}_+(7, 9)$:



Notation 11 We use $\mathcal{O}_{\mathbf{D}}$ to denote the nilpotent orbit corresponding to $\mathbf{D} \in \mathcal{YD}_+(p, q)$. We use $G(\mathcal{O}_{\mathbf{D}})$ to denote the group $O(p, q)$. Any \mathbf{D} in this paper is interpreted as a signed Young diagram in a previously chosen set \mathcal{YD}_- or \mathcal{YD}_+ . The orbit $\mathcal{O}_{\mathbf{D}}$ refers to either $\mathcal{O}_{\mathbf{D}}(+)$ or $\mathcal{O}_{\mathbf{D}}(-)$ with the understanding that $+$ or $-$ is implicitly known once \mathbf{D} is given.

Of course, if \mathbf{D} is specified to be in \mathcal{YD}_+ or \mathcal{YD}_- , the notation $\mathcal{O}_{\mathbf{D}}$ causes no confusion.

2.3 Nilpotent Orbits of Class \mathcal{U}

Notation 12 Let $\mathcal{O}_{\mathbf{D}}$ be a real nilpotent orbit. We use \mathbf{d} to denote the Young diagram obtained from \mathbf{D} by removing the signs.

Every real nilpotent orbit \mathcal{O} induces a complex nilpotent orbit $\mathcal{O}_{\mathbb{C}}$ by considering the complex group G_{ad} acting on \mathcal{O} in \mathfrak{g} . This map is simply

$$\mathcal{O}_{\mathbf{D}} \rightarrow \mathcal{O}_{\mathbf{d}}$$

for orthogonal groups and symplectic groups.

Definition 10 Let \mathbf{D} be a signed Young diagram. Define $\mathbf{D} - \mathbf{1}$ to be the signed Young diagram obtained from \mathbf{D} by deleting the first column.

Theorem 2.6 $-\mathbf{1}$ defines an operation from $\mathcal{YD}_+(p, q)$ to the disjoint union

$$\cup_{n \leq \min(p, q)} \mathcal{YD}_-(n, n)$$

$-\mathbf{1}$ also defines an operation from $\mathcal{YD}_-(n, n)$ to the disjoint union

$$\cup_{\max(p, q) \leq n} \mathcal{YD}_+(p, q)$$

By Theorem 2.4 and Theorem 2.5, $-\mathbf{1}$ induces an orbital correspondence from real nilpotent orbits of symplectic groups to real nilpotent orbits of orthogonal groups, and conversely.

Definition 11 We say that $\mathcal{O}_{\mathbf{D}}$ (or $\mathcal{O}_{\mathbf{d}}$) is pre-rigid if \mathbf{d}^t is multiplicity free.

This amounts to saying that every integer between 1 and d_1 appears in the partition \mathbf{d} . In other words, the partition \mathbf{d} is of the following form

$$([d_1]^*, [d_1 - 1]^*, [d_1 - 2]^*, \dots, [1]^*)$$

with each multiplicity greater than zero.

Definition 12 (Nilpotent orbits of Class \mathcal{U})

1. \mathcal{U} consists of a class of real nilpotent orbits $\mathcal{O}_{\mathbf{D}}$. For technical reasons, we **exclude** from \mathcal{U} those \mathbf{D} whose last 2 columns are of the same length with at least two rows and are of the following forms



2. Let $\mathbf{D} \in \mathcal{YD}_-(n, n)$ and $\mathbf{d}^t = (m_1 \geq m_2 \geq \dots \geq m_{d_1})$. The nilpotent orbit $\mathcal{O}_{\mathbf{D}}$ of $Mp_{2n}(\mathbb{R})$ is said to be in $\mathcal{U}(Mp_{2n}(\mathbb{R}))$ if \mathbf{d}^t satisfies the following three conditions:

- (a) For every i , $m_{2i} > m_{2i+1}$ and $m_{2i+1} \geq m_{2i+2}$;
- (b) (1);
- (c) \mathbf{d}^t is either very even or very odd.

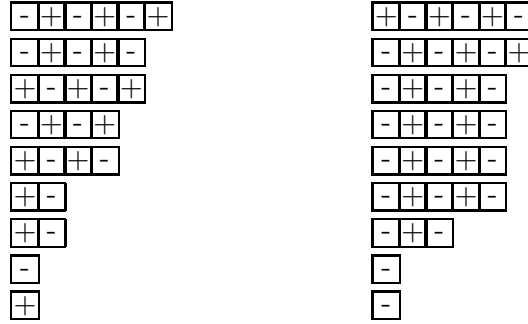
3. Let $\mathbf{D} \in \mathcal{YD}_+(p, q)$ and

$$\mathbf{d}^t = (m_1 \geq m_2 \geq \dots \geq m_{d_1}).$$

The nilpotent orbit $\mathcal{O}_{\mathbf{D}}$ is said to be in $\mathcal{U}(O(p, q))$ if \mathbf{d}^t satisfies the following three conditions:

- (a) For every i , $m_{2i} \geq m_{2i+1}$ and $m_{2i+1} > m_{2i+2}$;
- (b) (1);
- (c) \mathbf{d}^t is either very even or very odd.

Here are two \mathbf{D} , one in $\mathcal{U}(Mp_{30}(\mathbb{R}))$, the other in $\mathcal{U}(O(15, 22))$.



The condition (a) is weaker than saying \mathbf{d}^t is multiplicity free. The condition (c) amounts to saying that all multiplicities of \mathbf{d} except perhaps the first are even. Recall

Lemma 2.3.1 (Thm. 7.3.5 and Prop. 6.3.7 [CM]) *If $\mathcal{O}_{\mathbf{d}}$ is rigid, then $\mathcal{O}_{\mathbf{d}}$ is pre-rigid. If $\mathcal{O}_{\mathbf{d}}$ is special and rigid, then \mathbf{d}^t must be either very even or very odd and must be multiplicity free.*

Corollary 2.1 *Special rigid orbits of $O(p, q)$ and $Sp_{2n}(\mathbb{R})$ are contained in \mathcal{U} .*

By Theorem 2.6, we have

Corollary 2.2 *If $\mathcal{O}_{\mathbf{D}}$ is in $\mathcal{U}(Mp)$, then $\mathcal{O}_{\mathbf{D}-1}$ is in $\mathcal{U}(O)$. If $\mathcal{O}_{\mathbf{D}}$ is in $\mathcal{U}(O)$, then $\mathcal{O}_{\mathbf{D}-1}$ is in $\mathcal{U}(Mp)$.*

2.4 Induced Orbits

2.4.1 Complex Induced Orbits

Let \mathfrak{g} be either $\mathfrak{o}(n, \mathbb{C})$ or $\mathfrak{sp}_{2n}(\mathbb{C})$. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ be a parabolic subalgebra. Let $\mathcal{O}_\mathfrak{l}$ be a nilpotent orbit in \mathfrak{l} .

Definition 13 Define $\text{Ind}_\mathfrak{l}^\mathfrak{g} \mathcal{O}_\mathfrak{l}$ to be the nilpotent G -orbit that intersects $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$ in an open dense set.

Our definition differs slightly from the definition in Ch. 7.1 of [CM] as we require $\text{Ind}_\mathfrak{l}^\mathfrak{g} \mathcal{O}_\mathfrak{l}$ be a G -orbit and G may not be connected. In any case, $\mathcal{O}_\mathfrak{g}$ is the G -orbit of the "generic" elements in $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$. In addition, $\text{Ind}_\mathfrak{l}^\mathfrak{g} \mathcal{O}_\mathfrak{l}$ only depends on \mathfrak{l} , not on the choice of \mathfrak{p} (see Ch. 7.1 [CM]).

Lemma 2.4.1 (Proposition 7.1.4 [CM]) Let \mathfrak{l}_1 and \mathfrak{l}_2 be two Levi subalgebras of \mathfrak{g} and $\mathfrak{l}_1 \subset \mathfrak{l}_2$. Then

$$\text{Ind}_{\mathfrak{l}_2}^\mathfrak{g} (\text{Ind}_{\mathfrak{l}_1}^{\mathfrak{l}_2} \mathcal{O}_{\mathfrak{l}_1}) = \text{Ind}_{\mathfrak{l}_1}^\mathfrak{g} \mathcal{O}_{\mathfrak{l}_1}.$$

Thus orbital induction is "associative".

If $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$, then any Levi subalgebra \mathfrak{l} is of the form

$$\mathfrak{l} = \mathfrak{sp}_{2n_0}(\mathbb{C}) \times \mathfrak{gl}(n_1, \mathbb{C}) \times \dots \times \mathfrak{gl}(n_r, \mathbb{C})$$

where n_0, n_1, \dots, n_r are nonnegative integers summing to n . Similarly, if $\mathfrak{g} = \mathfrak{o}(m, \mathbb{C})$, then any Levi subalgebra \mathfrak{l} is of the form

$$\mathfrak{l} = \mathfrak{o}(m_0, \mathbb{C}) \times \mathfrak{gl}(m_1, \mathbb{C}) \times \dots \times \mathfrak{gl}(m_s, \mathbb{C})$$

where m_0, m_1, \dots, m_s are nonnegative integers and $m_0 + 2m_1 + \dots + 2m_s = m$.

Recall that nilpotent orbits of $SL(n, \mathbb{C})$ are parametrized by partitions of n in terms of the Jordan form.

Lemma 2.4.2 Let $n = n_1 + n_2$. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Let \mathfrak{l} be the block-diagonal matrices of size (n_1, n_2) in \mathfrak{g} . Let $\mathcal{O}_\mathfrak{s}$ and $\mathcal{O}_\mathfrak{t}$ be nilpotent orbits in $\mathfrak{sl}(n_1, \mathbb{C})$ and $\mathfrak{sl}(n_2, \mathbb{C})$ respectively. Then

$$\text{Ind}_\mathfrak{l}^\mathfrak{g} \mathcal{O}_\mathfrak{s} \times \mathcal{O}_\mathfrak{t} = \mathcal{O}_\mathfrak{d}$$

with

$$d_j = s_j + t_j \quad (\forall j).$$

Definition 14 We call \mathfrak{d} the merging of \mathfrak{s} and \mathfrak{t} .

For $G = O(n, \mathbb{C})$ or $G = Sp_{2n}(\mathbb{C})$, computation of induced orbits is slightly more complicated than simply a "merging". One needs the concept of "collapse" (see Ch. 6 [CM]). For the sake of simplicity, we will not introduce the concept of "collapse". We only state one special result concerning $Sp_{2n}(\mathbb{C})$. The general result can be found in [CM] (Lemma 6.3.3) and is due to Gerstenhaber.

Lemma 2.4.3 *Let $G = Sp_{2n}(\mathbb{C})$ and $L = GL(n - m, \mathbb{C}) \times Sp_{2m}(\mathbb{C})$. Let $\mathcal{O}_{\mathbf{s}}$ be a nilpotent orbit of $Sp_{2m}(\mathbb{C})$ such that the number of rows in \mathbf{s} is less than or equal to $n - m$. Then*

$$\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\{0\} \times \mathcal{O}_{\mathbf{s}} = \mathcal{O}_{\mathbf{d}}$$

with

$$d_j = s_j + 2 \quad (\forall 1 \leq j \leq n - m).$$

So under the assumption that the number of the rows in \mathbf{s} is less or equal to $n - m$, \mathbf{d} is the merging of \mathbf{s} with two copies of 1^m . \mathbf{d} remains symplectic since \mathbf{s} is symplectic.

2.4.2 Induced Real Orbits

Let G be a reductive Lie group and L be a Levi subgroup. Let $P = LN$ be a parabolic subgroup of G . Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit in \mathfrak{l} .

Definition 15 *Define $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\mathcal{O}_{\mathfrak{l}}$ to be the union of the nilpotent G -orbits $\mathcal{O}_{\mathfrak{g}}^{(i)}$ that contain an open subset of $\mathcal{O}_{\mathfrak{l}} + \mathfrak{n}$.*

Implicit in the notation is the fact that $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}$ does not depend on the choice of \mathfrak{n} . The proof of this fact is essentially the same as the proof for the complex orbits (see Theorem 7.1.3 of [CM]). $\text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\mathcal{O}_{\mathfrak{l}}$ for a classical group of type I may no longer be a single nilpotent orbit. Nevertheless, it is contained in a single complex nilpotent orbit.

Lemma 2.4.4 *We have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{l}} & \xrightarrow{\text{complex orbit}} & \mathcal{O}_{\mathfrak{l}_{\mathbb{C}}} \\ \downarrow \text{Ind} & & \downarrow \text{Ind} \\ \text{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\mathcal{O}_{\mathfrak{l}} & \xrightarrow{\text{complex orbit}} & \text{Ind}_{\mathfrak{l}_{\mathbb{C}}}^{\mathfrak{g}_{\mathbb{C}}}\mathcal{O}_{\mathfrak{l}_{\mathbb{C}}} \end{array} \quad (2.1)$$

Let $\mathcal{O}_{\mathbf{s}}$ be a nilpotent orbit in $\mathfrak{sp}_{2m}(\mathbb{R})$. By the lemma above,

$$\text{Ind}_{\mathfrak{gl}(n-m, \mathbb{R}) \oplus \mathfrak{sp}_{2m}(\mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})}\{0\} \times \mathcal{O}_{\mathbf{s}}$$

must be contained in the complex induced orbit

$$\text{Ind}_{\mathfrak{gl}(n-m, \mathbb{C}) \oplus \mathfrak{sp}_{2m}(\mathbb{C})}^{\mathfrak{sp}_{2n}(\mathbb{C})}\{0\} \times \mathcal{O}_{\mathbf{s}}.$$

Under the assumption in Lemma 2.4.3, this complex induced orbit is parametrized by \mathbf{d} with

$$d_j = s_j + 2 \quad (\forall 1 \leq j \leq n - m).$$

So

$$\text{Ind}_{\mathfrak{gl}(n-m, \mathbb{R}) \oplus \mathfrak{sp}_{2m}(\mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})}\{0\} \times \mathcal{O}_{\mathbf{s}}$$

is contained in the intersection of $\mathcal{O}_{\mathbf{d}}$ with $\mathfrak{sp}_{2n}(\mathbb{R})$. We have

Sketch of the proof: (1), (2) are evident by the commutative diagram 2.1 and (5) follows from our convention (see Theorem 2.4). Let

$$\mathbf{d} = (s_1 + 2, s_2 + 2, \dots, s_r + 2, \overbrace{2, \dots, 2}^{n-m-r}).$$

An odd number $2l + 1$ must occur even times. **We group $2l + 1$'s into pairs.**

We adopt the notation from Ch 5. 2 of [CM] and the main reference is Ch 5.2 and 9.3 of [CM]. A signed Young diagram can be expressed as

$$a_1^\pm \oplus a_2^\pm \oplus \dots \oplus a_s^\pm$$

with a_i^\pm symbolizing a row of length a_i beginning with the sign \pm . Notice that odd number $2l + 1$ always comes in pairs as $((2l + 1)^+, (2l + 1)^-)$ according to our convention.

Fix a symplectic space $(\mathbb{R}^{2n}, \Omega)$ and a nondegenerate symplectic subspace \mathbb{R}^{2m} . Decompose \mathbb{R}^{2n} into

$$\mathbb{R}^{2m} \oplus V_0 \oplus V'_0$$

such that $V_0 \oplus V'_0$ is the orthogonal complement of \mathbb{R}^{2m} with respect to Ω and V_0 and V'_0 are both isotropic subspaces. Then $\Omega|_{V_0 \oplus V'_0}$ is nondegenerate. Let

$$\mathfrak{n} = \{u \in \mathfrak{sp}_{2n}(\mathbb{R}) \mid uV_0 = 0, u(\mathbb{R}^{2m} \oplus V_0) \subseteq V_0\}.$$

It is easy to see that for any $u \in \mathfrak{n}$ we have $\ker(u) \supseteq V_0$ and $u\mathbb{R}^{2n} \subseteq V_0 \oplus \mathbb{R}^{2m}$.

Let $X_{\mathbf{S}}$ be an element in $\mathcal{O}_{\mathbf{S}} \subset \mathfrak{sp}_{2m}(\mathbb{R})$. Fix an arbitrary $u \in \mathfrak{n}$ such that $X_{\mathbf{S}} + u \in \mathcal{O}_{\mathbf{D}}$. Then $X_{\mathbf{S}} + u \in \mathcal{O}_{\mathbf{D}}$ for some \mathbf{D} . Our main task is to determine all possible \mathbf{D} . Since the signs for odd rows of \mathbf{D} are prefixed, we only need to determine the signs for even rows of \mathbf{D} .

Let $\{H, X_{\mathbf{S}} + u, Y\}$ be a standard triple. Then \mathbb{R}^{2n} decomposes into a direct sum of subspaces

$$V_1 \oplus V_2 \oplus \dots \oplus V_r \oplus V_{r+1} \oplus \dots \oplus V_{n-m},$$

such that

- $\dim V_i = d_i = s_i + 2$ for $i \in [1, r]$ and $\dim V_i = 2$ for $i > r$.
- For even d_i , $\Omega|_{V_i}$ is nondegenerate.
- For an odd pair $d_i = d_{i+1}$, $\Omega|_{V_i \oplus V_{i+1}}$ is nondegenerate, V_i and V_{i+1} are both isotropic.
- All V_i for even d_i and $V_i \oplus V_{i+1}$ for (d_i, d_{i+1}) an odd pair, are perpendicular to each other with respect to the symplectic form Ω . In other words,

$$\Omega = [\oplus_i \text{ even } \Omega|_{V_i}] \oplus [\oplus_{(d_i, d_{i+1}) \text{ an odd pair}} \Omega|_{V_i \oplus V_{i+1}}].$$

- $X_{\mathbf{S}} + u$ decomposes accordingly as

$$X_1 \oplus X_2 \oplus \dots \oplus X_r \oplus X_{r+1} \dots \oplus X_{n-m},$$

where X_i is a principal nilpotent element in $\mathfrak{sp}(V_i, \Omega|_{V_i})$ for even dimensional V_i and $X_i \oplus X_{i+1}$ is a nilpotent element in $\mathcal{O}_{d_i^+, d_{i+1}^-} \subset \mathfrak{sp}(V_i \oplus V_{i+1}, \Omega|_{V_i \oplus V_{i+1}})$ for the odd pair ($d_i = d_{i+1}$).

This is essentially the abstract version of Jordan decomposition for a nilpotent element in a real symplectic Lie algebra. For even dimensional V_i , there are two principal nilpotent classes parametrized by d_i^+ and d_i^- respectively. We now need to determine the sign attached to X_i for each even d_i .

Let d_i be an even integer. Generally speaking, the sign attached to X_i can be determined as follows. There exists a basis of V_i

$$\{e_i^{(1)}, e_i^{(2)}, \dots, e_i^{(d_i)}\}$$

such that

$$X e_i^{(j)} = e_i^{(j+1)} \quad (\forall j \in [1, d_i - 1]),$$

and $X_i e_i^{(d_i)} = 0$. Then the signature of

$$\Omega(X^{d_i-1} e_i^{(1)}, e_i^{(1)})$$

gives the sign attached to d_i for X_i (see Page 139 [CM]). For the standard triple $\{h, x, y\} \subset \mathfrak{sl}_2(\mathbb{R})$, \mathcal{O}_x will correspond to 2^+ and \mathcal{O}_{-x} will correspond to 2^- .

Suppose that d_i is an even integer greater than 2. Now we need to connect the signature attached to X_i with its “projection” to \mathbb{R}^{2m} . Notice that $X_{\mathbf{S}} V_0 = 0$ and $u V_0 = 0$. So $V_0 \subseteq \ker(X_{\mathbf{S}} + u)$. Since $X_{\mathbf{S}} + u \in \mathcal{O}_{\mathbf{d}}$, the kernel of $X_{\mathbf{S}} + u$ is of dimension $n - m$ which is equal to the dimension of V_0 . We see that $\ker(X_{\mathbf{S}} + u) = V_0$. Hence

$$(X_{\mathbf{S}} + u)(\mathbb{R}^{2n}) = V_0^\perp = V_0 \oplus \mathbb{R}^{2m}.$$

In particular,

$$\text{span}\{e_i^{(2)}, \dots, e_i^{(d_i)}\} = X_i(V_i) = V_i \cap (V_0 \oplus \mathbb{R}^{2m}).$$

For $j \in [2, d_i]$, put $e_i^{(j)} = f_i^{(j)} + v_i^{(j)}$ where $f_i^{(j)} \in \mathbb{R}^{2m}$ and $v_i^{(j)} \in V_0$. We have for $j \in [2, d_i - 1]$,

$$(X_{\mathbf{S}} + u)(f_i^{(j)} + v_i^{(j)}) = (X_{\mathbf{S}} + u)e_i^{(j)} = X_i(e_i^{(j)}) = e_i^{(j+1)} = f_i^{(j+1)} + v_i^{(j+1)}.$$

Since $u(\mathbb{R}^{2m} \oplus V_0) \subseteq V_0$ and $X_{\mathbf{S}} V_0 = 0$, we have

$$X_{\mathbf{S}} f_i^{(j)} = f_i^{(j+1)}, \quad u e_i^{(j)} = v_i^{(j+1)}.$$

Let $W_i = \text{span}\{f_i^{(j)} \mid j \in [2, d_i - 1]\}$. Notice that Ω restricted to $\text{span}\{e_i^{(2)}, \dots, e_i^{(d_i-1)}\}$ is nondegenerate. Since V_0 is isotropic and $W_i \perp V_0$, we see that Ω restricted onto W_i is nondegenerate. Furthermore, W_i are mutually perpendicular with respect to Ω since V_i 's are perpendicular to each other. So $X_{\mathbf{S}}|_{W_i}$ is a Jordan block of size $d_i - 2 = s_i$. The sign attached to it, is the signature of

$$\Omega(X_{\mathbf{S}}^{s_i-1} f_i^{(2)}, f_i^{(2)}) = \Omega((X_{\mathbf{S}} + u)^{s_i-1} e_i^{(2)}, e_i^{(2)}).$$

Observe that

$$\Omega((X_{\mathbf{S}} + u)^{s_i-1} e_i^{(2)}, e_i^{(2)}) = \Omega((X_{\mathbf{S}} + u)^{s_i} e_i^{(1)}, (X_{\mathbf{S}} + u) e_i^{(1)}) = -\Omega((X_{\mathbf{S}} + u)^{s_i+1} e_i^{(1)}, e_i^{(1)}).$$

Hence the sign attached to d_i for $X_{\mathbf{S}} + u$ differs from the sign attached to s_i for $X_{\mathbf{S}}$. (3) is proved. We have shown that

$$\text{Ind}_{\mathfrak{gl}(n-m, \mathbb{R}) \oplus \mathfrak{sp}_{2m}(\mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})} \{0\} \times \mathcal{O}_{\mathbf{S}} \subseteq \bigcup_{j=0}^{n-m-r} \mathcal{O}_{\mathbf{D}^{(j)}}.$$

Lastly, we need to show that each $\mathcal{O}_{\mathbf{D}^{(j)}}$ appears in

$$\text{Ind}_{\mathfrak{gl}(n-m, \mathbb{R}) \oplus \mathfrak{sp}_{2m}(\mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})} \{0\} \times \mathcal{O}_{\mathbf{S}}.$$

It suffices to construct an element $X_{\mathbf{D}^{(j)}} \in Y_{\mathbf{S}} + \mathfrak{n}$ for an element $Y_{\mathbf{S}} \in \mathcal{O}_{\mathbf{S}}$. Let $\{e_1, e_2, \dots, e_{n-m}\}$ be a basis for V_0 and $\{e_{n-m+1}, e_{n-m+2}, \dots, e_{2n-2m}\}$ be the dual basis for V'_0 such that the natural bilinear form coincides with the restriction of Ω onto $V_0 \otimes V'_0$. Fix a standard basis $\{e_{2n-2m+1}, \dots, e_{2n}\}$ for $(\mathbb{R}^{2m}, \Omega|_{\mathbb{R}^{2m}})$. Let $E_{i,j}$ be the $2n \times 2n$ matrix with (i, j) -th entry equal to 1 and the rest equal to zero. Without loss of generality, suppose that

$$Y_{\mathbf{S}} = Y_1 \oplus Y_2 \oplus \dots \oplus Y_r$$

is a Jordan decomposition such that

- For even s_i ,

$$Y_i = \pm E_{j_i + \frac{s_i}{2}, m+j_i + \frac{s_i}{2}} + \sum_{k=1}^{\frac{s_i}{2}-1} E_{j_i+k, j_i+k+1} - E_{m+j_i+k+1, m+j_i+k}$$

for some $j_i \geq 2n - 2m$;

- For a pair of odd integer $(s_i = s_{i+1})$,

$$Y_i = \sum_{k=1}^{s_i-1} E_{j_i+k, j_i+k+1} \quad Y_{i+1} = - \sum_{k=1}^{s_i-1} E_{m+j_i+k+1, m+j_i+k},$$

for some $j_i \geq 2n - 2m$.

For even s_i , define

$$X_i = E_{i, j_i+1} + Y_i - E_{m+j_i+1, n-m+i}.$$

For odd pair $(s_i = s_{i+1})$, define

$$X_i = E_{i, j_i+1} + Y_i + E_{j_i+s_i, i+1}, \quad X_{i+1} = -E_{m+j_i+1, n-m+i} + Y_{i+1} - E_{n-m+i+1, j_i+s_i+m}.$$

For $i \in [r+1, n-m]$, define $X_i = \pm E_{i, n-m+i}$ such that the number of $-$ is j and number of $+$ is $n-m-r-j$. Let $X = \sum_{i=1}^{n-m} X_i$. It is then easy to check that $X \in \mathcal{O}_{\mathbf{D}^{(j)}}$. \square .

Our method can be generalized to compute induced real nilpotent orbits. We shall do this in a subsequent paper.

Corollary 2.3 Let $\mathbf{t} = [1^n]$. Then

$$\text{Ind}_{\mathfrak{gl}(n, \mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})} \mathcal{O}_{\mathbf{t}}$$

is the union of $n + 1$ orbits consisting of all $\mathcal{O}_{\mathbf{D}^{(j)}}$ with $\mathbf{d}^{(j)} = [2^n]$.

Theorem 2.8 Let $\mathcal{O}_{\mathbf{S}}$ be a nilpotent orbit of $Sp_{2m}(\mathbb{R})$. Let $\mathbf{s} = (s_1 \geq s_2 \geq \dots \geq s_r > 0)$. Suppose $n - m \geq r$. Then

$$\text{Ind}_{\mathfrak{sp}_{2m}(\mathbb{R}) \oplus \mathfrak{gl}(n-m, \mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})} \tau(\mathcal{O}_{\mathbf{S}}) \times \{0\}$$

is a union of nilpotent orbits $\mathcal{O}_{\mathbf{D}^{(j)}}$ and each $\mathbf{D}^{(j)}$ can be obtained by

1. merging two copies of $[1^{n-m}]$ to \mathbf{S} from the left;
2. extending the signs of \mathbf{S} ;
3. filling the $n - m - r$ rows of length 2 in $\mathbf{D}^{(j)}$ with j $\boxed{-\oplus}$ and $n - m - r - j$ $\boxed{\oplus-}$.

Sketch of the proof: Recall that $\tau(\mathcal{O}_{\mathbf{S}}) = \mathcal{O}_{\tau(\mathbf{S})}$ and $\tau(\mathbf{S})$ is obtained by switching the signs of even rows in \mathbf{S} . Suppose that s_i^{\pm} represents an even row in \mathbf{S} . Then s_i^{\mp} represents the corresponding even row in $\tau(\mathbf{S})$. By Theorem 2.7, d_i^{\pm} will represent the corresponding even row in $\mathbf{D}^{(j)}$. This is the same as adding two boxes to the even rows of \mathbf{S} and extend the signs of \mathbf{S} . (3) follows directly from Thm. 2.7. \square

Chapter 3

Theta Correspondences and Quantum Induction

Notation 13 *Let $Mp_{2n}(\mathbb{R})$ be the metaplectic covering of $Sp_{2n}(\mathbb{R})$. Let $\{1, \epsilon\}$ be the preimage of $1 \in Sp_{2n}(\mathbb{R})$. For any subgroup G of $Sp_{2n}(\mathbb{R})$, let MG be the preimage of G under the metaplectic covering.*

Fix a dual pair $(O(p, q), Sp_{2n}(\mathbb{R}))$ in $Sp_{2n(p+q)}(\mathbb{R})$ (see [Ho79]). Fix a maximal compact subgroup K for $Mp_{2n(p+q)}(\mathbb{R})$ such that $K \cap MO(p, q)$ is a maximal compact subgroup of $MO(p, q)$ and $K \cap MSp_{2n}(\mathbb{R})$ is a maximal compact subgroup of $MSp_{2n}(\mathbb{R})$. Let $\omega(p, q; 2n)$ be the oscillator representation of the group $Mp_{2n(p+q)}(\mathbb{R})$. If the parameters $(p, q; 2n)$ are apparent, we will write ω . Let $V(\omega)$ be the Harish-Chandra module. Unless otherwise stated, any representation π of MG in this paper satisfies $\pi(\epsilon) = -1$. Very often, this assumption reduces our discussion and computation to representations of the linear group.

Notation 14 *Let G be any reductive subgroup of Sp . Let $\mathcal{R}(MG, \omega)$ be the equivalence classes of irreducible Harish-Chandra modules of MG that occur as quotients of $V(\omega)$.*

Theorem 3.1 (Howe, [Ho89]) *Let (G_1, G_2) be a dual pair in Sp . Then $\mathcal{R}(MG_1G_2, \omega)$ yields a one-to-one correspondence between $\mathcal{R}(MG_1, \omega)$ and $\mathcal{R}(MG_2, \omega)$.*

This correspondence is often called (local) theta correspondence, Howe's correspondence or duality correspondence. Let $\theta(p, q; 2n)$ be the theta correspondence from $\mathcal{R}(MO(p, q), \omega)$ to $\mathcal{R}(MSp_{2n}(\mathbb{R}), \omega)$. Let $\theta(2n; p, q)$ be its inverse. If the parameters $(p, q; 2n)$ are apparent, we will just write θ . The description of the sets $\mathcal{R}(MO(p, q), \omega)$ and $\mathcal{R}(MSp_{2n}(\mathbb{R}), \omega)$ is not known in general.

If G is a subgroup of Sp , we say that MG splits if $MG \simeq G \times \{\pm 1\}$. For the dual pair $(O(p, q), Sp_{2n}(\mathbb{R}))$, $MO(p, q)$ always splits and $MSp_{2n}(\mathbb{R})$ splits if and only if $p + q$ is even. If $p + q$ is odd, $MSp_{2n}(\mathbb{R})$ is isomorphic to the metaplectic group $Mp_{2n}(\mathbb{R})$. In all cases, θ can be regarded as a one-to-one correspondence between a certain subset of $\Pi(Mp_{2n}(\mathbb{R}))$ and a certain subset of $\Pi(O(p, q))$. Since this is the viewpoint in some literature (for example [KR]), we will also take this viewpoint if necessary.

3.1 Theta Correspondence in Semistable Range

We adopt the notation from (1.3.1) and (1.3.2). Let \mathbf{n} be the constant vector (n, n, n, \dots, n) of a fixed dimension.

Definition 16 (Def. 5.1 [He00], Def. 3.2.1 [Heq], Thm. 8.17, 8.18, Def. 8.3, 8.4 [He98])
An irreducible representation π of $MO(p, q)$ is said to be in the semistable range of $\theta(p, q; 2n)$ if and only if every leading exponent v of π satisfies

$$\Re(v) - \mathbf{n} + 2\rho(O(p, q)) \prec 0.$$

An irreducible representation π of $MSp_{2n}(\mathbb{R})$ is said to be in the semistable range of $\theta(2n; p, q)$ if and only if every leading exponent v of π satisfies

$$\Re(v) - \frac{\mathbf{p} + \mathbf{q}}{2} + 2\rho(Sp_{2n}(\mathbb{R})) \prec 0.$$

We denote the semistable ranges by $R_s(MO(p, q), \omega)$ and $R_s(MSp_{2n}(\mathbb{R}), \omega)$ respectively. The reader should note that a representation π in $R_s(G, \omega)$ may not occur in $\mathcal{R}(G, \omega)$. This is pointed out to me by Chen-Bo Zhu.

Definition 17 ([LI89], [He00]) *Consider $(G_1, G_2) = (O(p, q), Sp_{2n}(\mathbb{R}))$ or $(G_1, G_2) = (Sp_{2n}(\mathbb{R}), O(p, q))$. Let $\pi \in R_s(MG_1, \omega)$. Define a bilinear form $(\ , \)_\pi$ on $V(\omega) \otimes V(\pi^c)$*

$$(\phi \otimes u, \psi \otimes v)_\pi = \int_{MG_1} (\omega(g)\phi, \psi)(v, \pi(g)u) dg \quad (\phi, \psi \in V(\omega), u, v \in V(\pi)).$$

Let \mathcal{R}_π be the radical of $(\ , \)_\pi$. Define

$$\theta_s(MG_1, MG_2)(\pi) = V(\omega) \otimes V(\pi^c) / \mathcal{R}_\pi.$$

$\theta_s(MG_1, MG_2)(\pi)$ inherits an infinitesimal MG_2 action from $\omega(MG_1, MG_2)$. It is a (\mathfrak{g}, K) -module of MG_2 .

Theorem 3.2 (Theorem 1.1 [He00], [Heu]) *Suppose π is a unitarizable Harish-Chandra module in the semistable range of $\theta(MG_1, MG_2)$. Then $(\ , \)_\pi$ is well-defined. If $(\ , \)_\pi \neq 0$, then $\theta_s(MG_1, MG_2)(\pi)$ is an irreducible Hermitian Harish-Chandra module of MG_2 and $\theta_s(MG_1, MG_2)(\pi)$ is equivalent to $\theta(MG_1, MG_2)(\pi)$.*

This theorem basically says that if $(\ , \)_\pi$ is well-defined and nonvanishing, then θ_s is Howe's correspondence ([Ho89]) on the Harish-Chandra module level. By Howe's Theorem, $\theta_s(MG_1, MG_2)(\pi)$ is an irreducible Harish-Chandra module of MG_2 . The notation $(\ , \)_\pi$ is essentially due to Jian-Shu Li ([LI89]).

In [He01], we proved the following nonvanishing theorem.

Theorem 3.3 (Corollary 1.1 [He01]) *Suppose $p + q \leq 2n + 1$. Suppose $\pi \in R_s(MO(p, q), \omega)$. Let \det be the lift of the determinant of $O(p, q)$ to $MO(p, q)$, i.e.,*

$$\ker(\det) = MSO(p, q).$$

Then either $\theta_s(p, q; 2n)(\pi) \neq 0$ or $\theta_s(p, q; 2n)(\pi \otimes \det) \neq 0$.

3.2 Unitarity and Strongly Semistable Range

In [Heu], we proved the unitarity of $\theta_s(\pi)$ for π in the strongly semistable range.

Definition 18 *An irreducible admissible representation π is in $R_{ss}(MO(p, q), \omega)$ if every leading exponent of π satisfies*

$$\Re(v) - \left(\mathbf{n} - \frac{\mathbf{p} + \mathbf{q}}{2}\right) + \rho(O(p, q)) \preceq 0. \quad (3.1)$$

An irreducible admissible representation π of $MSp_{2n}(\mathbb{R})$ is in $R_{ss}(MSp_{2n}(\mathbb{R}), \omega)$ if every leading exponent of π satisfies

$$\Re(v) - \left(\frac{\mathbf{p} + \mathbf{q}}{2} - \mathbf{n} - \mathbf{1}\right) + \rho(Sp_{2n}(\mathbb{R})) \preceq 0. \quad (3.2)$$

We call R_{ss} the strongly semistable range.

Notice that $R_s(MSp_{2n}(\mathbb{R}), \omega)$ and $R_{ss}(MO(p, q), \omega)$ only depend on $p + q$, not on a particular pair (p, q) . Since

$$\Re(v) - \mathbf{n} + 2\rho(O(p, q)) \prec \Re(v) - \left(\mathbf{n} - \frac{\mathbf{p} + \mathbf{q}}{2}\right) + \rho(O(p, q))$$

and

$$\Re(v) - \frac{\mathbf{p} + \mathbf{q}}{2} + 2\rho(Sp_{2n}(\mathbb{R})) \prec \Re(v) - \left(\frac{\mathbf{p} + \mathbf{q}}{2} - \mathbf{n} - \mathbf{1}\right) + \rho(Sp_{2n}(\mathbb{R})),$$

by the definition of semistable range, $R_{ss} \subseteq R_s$. Thus θ_s in the strongly semistable range is the same as the original θ .

Theorem 3.4 (Corollary 5.2 [Heu]) *Suppose*

- $p + q \leq 2n + 1$;
- $\pi \in R_{ss}(MO(p, q), \omega)$;
- π is unitary;

Then $(\ , \)_\pi$ is positive semidefinite. If $\theta_s(p, q; 2n)(\pi) \neq 0$, then $\theta_s(p, q; 2n)(\pi)$ is an irreducible unitary representation of $MSp_{2n}(\mathbb{R})$.

Theorem 3.5 (Corollary 5.1 [Heu]) *Suppose*

- $n < p \leq q$
- $\pi \in R_{ss}(MSp_{2n}(\mathbb{R}), \omega)$;
- π is unitary;

Then $(\ , \)_\pi$ is positive semidefinite. If $\theta_s(2n; p, q)(\pi) \neq 0$, then $\theta_s(2n; p, q)(\pi)$ is an irreducible unitary representation of $MO(p, q)$.

Strictly speaking, $\theta_s(*) (\pi)$ is an irreducible unitarizable Harish-Chandra module. Notice that $(\ , \)_\pi$ can be regarded as an inner product on $\theta_s(*) (\pi)$. We can simply complete $(\ , \)_\pi$ to obtain the Hilbert space of $\theta_s(*) (\pi)$. In general, invariant inner product on an irreducible Harish-Chandra module is unique up to the multiplication of a constant.

Theorem 3.4 and Theorem 3.5 provide us the basis to construct unitary representation through the theta correspondence in the strongly semistable range.

3.3 Quantum Induction \mathcal{Q}

Notation 15 *Suppose that $p + q + p' + q' \equiv 0 \pmod{2}$. Suppose that $p' + q \geq 2n + 1$ and $q' + p \geq 2n + 1$. Let $\mathcal{E}(p' + q, q' + p; 2n)$ be the representation π_n of $O(p' + q, q' + p)$ studied by Zhu-Huang in [ZH]. The representation $\mathcal{E}(p' + q, q' + p; 2n)$ is essentially $\theta(2n; p' + q, q' + p)$ (trivial) restricted to the $O(p' + q, q' + p)$ component in $MO(p' + q, q' + p)$.*

Notice here that $MO(p' + q, q' + p)$ splits into $O(p' + q, q' + p)$ and $O(p' + q, q' + p)\epsilon$.

Definition 19 *Suppose*

$$p' + q \geq 2n + 1, \quad p + q' \geq 2n + 1, \quad p + q \equiv p' + q' \pmod{2}.$$

Let $O(p, q) \times O(p', q')$ be a subgroup diagonally embedded in $O(p' + q, q' + p)$. Fix a maximal compact subgroup $K = O(p' + q) \times O(q' + p)$ such that

$$K \cap O(p, q), \quad K \cap O(p', q')$$

are maximal compact subgroups of $O(p, q)$ and $O(p', q')$ respectively. Let $\pi \in \Pi(O(p, q))$ be such that

$$(u_1 \otimes v_1, u_2 \otimes v_2) = \int_{O(p, q)} (\mathcal{E}(p' + q, q' + p; 2n)(g)u_1, u_2)(\pi(g)v_1, v_2)dg$$

converges absolutely for every $u_1, u_2 \in V(\mathcal{E}(p' + q, q' + p; 2n))$ and every $v_1, v_2 \in V(\pi)$. Let \mathcal{R} be the radical of (\cdot, \cdot) as a Hermitian form on $V(\mathcal{E}(p' + q, q' + p; 2n)) \otimes V(\pi)$. Then

$$(V(\mathcal{E}(p' + q, q' + p; 2n)) \otimes V(\pi))/\mathcal{R}$$

inherits an infinitesimal $O(p', q')$ -action. Define

$$\mathcal{Q}(p, q; 2n; p', q')(\pi) = V(\mathcal{E}(p' + q, q' + p; 2n)) \otimes V(\pi)/\mathcal{R}.$$

$\mathcal{Q}(p, q; 2n; p', q')(\pi)$ is a $(\mathfrak{o}(p', q'), K \cap O(p', q'))$ -module.

Conjecture 1 *Suppose*

$$p' + q \geq 2n + 1, \quad p + q' \geq 2n + 1, \quad p + q \equiv p' + q' \pmod{2}, \quad p' + q' \geq p + q.$$

If $\mathcal{Q}(p, q; 2n; p', q')(\pi) \neq 0$, then $\mathcal{Q}(p, q; 2n; p', q')(\pi)$ is an irreducible admissible representation of $O(p', q')$. If π is unitary, then $\mathcal{Q}(p, q; 2n; p', q')(\pi)$ is also unitary.

Notation 16 *For $p + q \leq n + n' + 1$, put $\mathcal{E}(2n + 2n'; p, q) = \theta(p, q; 2n + 2n')$ (trivial).*

If $p + q \leq n + n'$, $\mathcal{E}(2n + 2n'; p, q)$ is unitary according to Howe-Li's theory on stable range dual pairs ([Ho84], [LI89]). If $p + q = n + n' + 1$, $\mathcal{E}(2n + 2n'; p, q)$ is unitary according to the result of Przebinda on almost stable range dual pairs (see Lemma 8.6 [PR93]) or a Theorem of Kudla and Rallis ([KR]). $\mathcal{E}(2n + 2n'; p, q)$ is a genuine representation of $Mp_{2n+2n'}(\mathbb{R})$ if $p + q$ is odd.

Definition 20 Suppose $n + n' + 1 \geq p + q$. Let $Sp_{2n}(\mathbb{R}) \times Sp_{2n'}(\mathbb{R})$ be diagonally embedded into $Sp_{2n+2n'}(\mathbb{R})$. Let K be a maximal compact subgroup of $Sp_{2n+2n'}(\mathbb{R})$ such that

$$Sp_{2n}(\mathbb{R}) \cap K, \quad Sp_{2n'}(\mathbb{R}) \cap K$$

are maximal compact subgroups of $Sp_{2n}(\mathbb{R})$ and $Sp_{2n'}(\mathbb{R})$ respectively. Let π be an irreducible representation of $Mp_{2n}(\mathbb{R})$ such that the following Hermitian form (\cdot, \cdot) on $V(\mathcal{E}(2n + 2n'; p, q)) \otimes V(\pi)$ converges:

$$(\varphi \otimes u, \varsigma \otimes v) = \int_{MSp_{2n}(\mathbb{R})} (\mathcal{E}(2n + 2n'; p, q)(g)\varphi, \varsigma)(\pi(g)u, v) dg$$

$$\forall \varphi, \varsigma \in V(\mathcal{E}(2n + 2n'; p, q)); u, v \in V(\pi).$$

Define $\mathcal{Q}(2n; p, q; 2n')(\pi)$ to be $V(\mathcal{E}(2n; p, q; 2n')) \otimes V(\pi)$ modulo the radical of (\cdot, \cdot) . $\mathcal{Q}(2n; p, q; 2n')(\pi)$ inherits an infinitesimal $\mathfrak{sp}_{2n'}(\mathbb{R})$ -action from $\mathcal{E}(2n + 2n'; p, q)$. It is a $(\mathfrak{sp}_{2n'}(\mathbb{R}), MK \cap Mp_{2n'}(\mathbb{R}))$ -module.

Conjecture 2 Suppose $n + n' + 1 \geq p + q$ and $n' \geq n$. If $\mathcal{Q}(2n; p, q; 2n')(\pi) \neq 0$, then $\mathcal{Q}(2n; p, q; 2n')(\pi)$ is an irreducible admissible representation of $Mp_{2n'}(\mathbb{R})$. If π is unitary, then $\mathcal{Q}(2n; p, q; 2n')(\pi)$ is also unitary.

$\mathcal{Q}(\cdot)$ defines an induction process. We call $\mathcal{Q}(\cdot)$ quantum induction.

3.4 Unitary Quantum Induction \mathcal{Q}

$\mathcal{Q}(\cdot)$ is closely related to a composition of two θ_s . The way it is defined allows us to bypass several technicalities which we shall address in this section. One difficult problem arising from composing two θ_s is that the range of one θ_s may not be in the domain of the other θ_s . In [Heq], we study the leading exponents of $\theta_s(\cdot)(\pi)$. Under certain assumptions, one can compose θ_s with another θ_s . Surprisingly, the strongly semistable range plays a crucial role in the composability of θ_s . By identifying \mathcal{Q} with a certain composition of θ_s , we established the unitarity and irreducibility of $\mathcal{Q}(\pi)$ for π in the strongly semistable range. Unitary quantum induction then enables us to construct certain irreducible unitary representations whose existence has been conjectured by Arthur and Barbasch-Vogan.

Theorem 3.6 (Theorem 7.1.1 [Heq]) Let π be an irreducible unitary representation in

$$R_{ss}(MO(p, q), \omega(p, q; 2n)).$$

Suppose

- $q' \geq p' > n$;
- $p' + q' - 2n \geq 2n - (p + q) + 2 \geq 1$;
- $p + q \equiv p' + q' \pmod{2}$.

Then

1. $\theta_s(p, q; 2n)(\pi) \in R_{ss}(MSp_{2n}(\mathbb{R}), \omega(p', q'; 2n));$
2. $\mathcal{Q}(p, q; 2n; p', q')(\pi) \cong \theta_s(2n; p', q')\theta_s(p, q; 2n)(\pi);$
3. If $\mathcal{Q}(p, q; 2n; p', q')(\pi) \neq 0$, then $\mathcal{Q}(p, q; 2n; p', q')(\pi)$ is unitary.

Let me say a few words about the proof of this theorem. (1) is proved in [Heq] through an estimate on the matrix coefficients of $\theta_s(p, q; 2n)(\pi)$. By (1), $\theta_s(2n; p', q')\theta_s(p, q; 2n)(\pi)$ is well-defined. In fact, by Definition 17, the Harish-Chandra module of $\theta_s(p, q; 2n)(\pi)$ consists of “distributions” of the following form

$$\int_{MO(p,q)} \omega(p, q; 2n)(g_1)\phi \otimes \pi^c(g_1)u dg_1 \quad (\phi \in V(\omega(p, q; 2n)), u \in V(\pi)).$$

For the same reason, the Harish-Chandra module of $\theta_s(2n; p', q')\theta_s(p, q; 2n)(\pi)$ consists of “distributions” of the following form

$$\int_{MSp_{2n}(\mathbb{R})} \omega(p', q'; 2n)(g_2)\psi \otimes \theta_s(p, q; 2n)(\pi)^c(g_2)v dg_2 \quad (\psi \in V(\omega(p', q'; 2n)), v \in V(\theta_s(p, q; 2n)(\pi))).$$

Combining these two statements, $V(\theta_s(2n; p', q')\theta_s(p, q; 2n)(\pi))$ consists of “distributions” of the form

$$\int_{g_2 \in MSp_{2n}(\mathbb{R})} \int_{g_1 \in MO(p,q)} \omega(p', q'; 2n)(g_2)\psi \otimes \omega(p, q; 2n)^c(g_2g_1)\phi \otimes \pi(g_1)u dg_1 dg_2 \quad (3.3)$$

$$(\psi \in V(\omega(p', q'; 2n)), \phi \in V(\omega(p, q; 2n)), u \in V(\pi)).$$

Notice that $\omega(p, q; 2n)^c \cong \omega(q, p; 2n)$ and

$$\omega(p', q'; 2n) \otimes \omega(q, p; 2n) \cong \omega(p' + q, q' + p; 2n).$$

So Equation (3.3) becomes

$$\int_{g_2 \in MSp_{2n}(\mathbb{R})} \int_{g_1 \in MO(p,q)} \omega(p' + q, q' + p; 2n)(i(g_1)g_2)(\psi \otimes \phi) \otimes \pi(g_1)u dg_1 dg_2.$$

Here $i : g_1 \rightarrow i(g_1)$ is a canonical embedding of $MO(p, q)$ into $MO(p' + q, q' + p)$. By the theorems of Howe-Li,

$$\left\{ \int_{g_2 \in MSp_{2n}(\mathbb{R})} \omega(p' + q, q' + p; 2n)(g_2)(\psi \otimes \phi) dg_2 \mid \psi \in V(\omega(p', q'; 2n)), \phi \in V(\omega(q, p; 2n)) \right\}$$

can be identified with the Harish-Chandra module of $\mathcal{E}(p' + q, q' + p; 2n)$. Therefore,

$$V(\theta_s(2n; p', q')\theta_s(p, q; 2n)(\pi))$$

consists of

$$\int_{MO(p,q)} \mathcal{E}(p' + q, q' + p; 2n)(i(g_1))\eta \otimes \pi(g_1)u dg_1 \quad (\eta \in V(\mathcal{E}(p' + q, q' + p; 2n)), u \in V(\pi)).$$

We have assumed that $\pi(\epsilon) = -1$ from the beginning. The integral over $MO(p, q)$ is just twice the integral over $O(p, q)$. It follows that $V(\theta_s(2n; p', q')\theta_s(p, q; 2n)(\pi))$ consists of vectors of the following form

$$\int_{O(p, q)} \mathcal{E}(p' + q, q' + p; 2n)(i(g_1))\eta \otimes \pi(g_1)u \, dg_1 \quad (\eta \in V(\mathcal{E}(p' + q, q' + p; 2n)), u \in V(\pi)).$$

The absolute convergences proved in [Heq] allow us to change the order of the integrals. Thus

$$\theta_s(2n; p', q')\theta_s(p, q; 2n)(\pi) \cong \mathcal{Q}(p, q; 2n; p', q')(\pi).$$

(3) follows easily by Theorems 3.4 and 3.5. \square

Definition 21 *Under the hypotheses in Theorem 3.6, define*

$$Q(p, q; 2n; p', q')(\pi) = \theta_s(2n; p', q')\theta_s(p, q; 2n)(\pi).$$

We call Q *unitary quantum induction*. By Theorem 3.6, $Q(p, q; 2n; p', q')(\pi)$ is well-defined and equivalent to $\mathcal{Q}(p, q; 2n; p', q')(\pi)$. A similar statement holds for $\mathcal{Q}(2n; p, q; 2n')(\pi)$.

Theorem 3.7 (Theorem 7.2.1 [Heq]) *Let π be a unitary representation in $R_{ss}(MSp_{2n}(\mathbb{R}), \omega(p, q; 2n))$. Suppose*

- $2n' - p - q \geq p + q - 2n - 2$;
- $n < p \leq q$.

Then

1. $\theta_s(2n; p, q)(\pi) \in R_{ss}(MO(p, q), \omega(p, q; 2n'))$;
2. $\mathcal{Q}(2n; p, q; 2n')(\pi) \cong \theta_s(p, q; 2n')\theta_s(2n; p, q)(\pi)$;
3. if $\mathcal{Q}(2n; p, q; 2n')(\pi) \neq 0$, then $\mathcal{Q}(2n; p, q; 2n')(\pi)$ is unitary.

Definition 22 *Under the hypotheses of Theorem 3.7, define*

$$Q(2n; p, q; 2n')(\pi) = \theta_s(p, q; 2n')\theta_s(2n; p, q)(\pi).$$

By Theorem 3.7, $Q(2n; p, q; 2n')(\pi)$ is well-defined and equivalent to $\mathcal{Q}(2n; p, q; 2n')(\pi)$.

In summary, \mathcal{Q} is a generalization of $Q = \theta_s \circ \theta_s$. Without the requirement of unitarity, we can loose some restrictions posed in Theorems 3.6 and 3.7. We only state the Theorem for $\theta_s(p, q; 2n')\theta_s(2n; p, q)$ since we will need it in the next Chapter.

Theorem 3.8 *Let π be a unitary representation in $R_{ss}(MSp_{2n}(\mathbb{R}), \omega(p, q; 2n))$. Suppose $2n' - p - q \geq p + q - 2n - 2$. Then $\theta_s(2n; p, q)(\pi) \in R_{ss}(MO(p, q), \omega(p, q; 2n'))$; and*

$$\mathcal{Q}(2n; p, q; 2n')(\pi) \cong \theta_s(p, q; 2n')\theta_s(2n; p, q)(\pi).$$

This theorem follows directly from the estimate established in Theorem 6.3.1 and Definition 18 of [Heq]. The proof is omitted.

The advantage of introducing \mathcal{Q} will be manifested when we begin to relate unitary quantum induction to parabolic induction. Based on the studies on parabolic induction, we will establish the nonvanishing of $\theta_s(2n; p, q)(\pi)$ under some restrictions. The nonvanishing of $\theta_s(2n; p, q)(\pi)$ is otherwise very hard to establish if p or q is less than $2n$.

3.5 Moment Map and $\Theta(G_1, G_2)$

From now on, let $G_1 = O(p, q)$ and $G_2 = Sp_{2n}(\mathbb{R})$. Write

$$G_{1\mathbb{C}} = O(p + q, \mathbb{C}), \quad G_{2\mathbb{C}} = Sp_{2n}(\mathbb{C}).$$

Let $\text{Mat}(p + q; 2n)$ be the set of $p + q$ by $2n$ real matrices. Let

$$W_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Definition 23 *Define the moment maps*

$$m_1 : x \in \text{Mat}(p + q, 2n) \rightarrow I_{p,q}xW_nx^t \in \mathfrak{o}(p, q),$$

$$m_2 : x \in \text{Mat}(p + q, 2n) \rightarrow W_nx^tI_{p,q}x \in \mathfrak{sp}_{2n}(\mathbb{R}).$$

We say that x is nilpotent if $m_1(x)$ is nilpotent.

Observe that $m_1(x)^l = I_{p,q}xm_2(x)^{l-1}W_nx^t$. We have

Lemma 3.5.1 *$m_1(x)$ is nilpotent if and only if $m_2(x)$ is nilpotent.*

The group $Sp_{2n}(\mathbb{R}) \times O(p, q)$ acts on the set of nilpotent element in $\text{Mat}(p + q, 2n)$. Thus we obtain a set of nilpotent $Sp_{2n}(\mathbb{R}) \times O(p, q)$ -orbits in $\text{Mat}(p + q, 2n)$. Clearly, m_i induces a map from the nilpotent orbits in $\text{Mat}(p + q, 2n)$ to nilpotent orbits in \mathfrak{g}_i .

Similarly, we define the complex moment maps, still denoted by m_i . This should not cause any notational problem. For example, the usage of \mathbf{D} usually points to the real moment map and the usage of \mathbf{d} usually points to the complex moment map.

For a complex nilpotent orbit \mathcal{O}_1 in $\mathfrak{g}_{1\mathbb{C}}$, consider the closure of

$$m_2(m_1^{-1}(\text{Closure of } \mathcal{O}_1)).$$

By a Theorem of Daszkiewicz-Kraskiewicz-Przebinda, the closure of

$$m_2(m_1^{-1}(\text{Closure of } \mathcal{O}_1))$$

is the closure of a unique nilpotent orbit in $\mathfrak{g}_{2\mathbb{C}}$ (see [DKP]). This yields a correspondence between certain nilpotent orbits in $\mathfrak{g}_{1\mathbb{C}}$ and certain nilpotent orbits in $\mathfrak{g}_{2\mathbb{C}}$. We denote this correspondence

by $\Theta(G_{1\mathbb{C}}, G_{2\mathbb{C}})$. Similarly, we define $\Theta(G_{2\mathbb{C}}, G_{1\mathbb{C}})$. The reader should be warned that $\Theta(G_{1\mathbb{C}}, G_{2\mathbb{C}})$ is NOT the inverse of $\Theta(G_{2\mathbb{C}}, G_{1\mathbb{C}})$. For the nilpotent orbits we are concerned with, $\Theta(G_{1\mathbb{C}}, G_{2\mathbb{C}})$ and $\Theta(G_{2\mathbb{C}}, G_{1\mathbb{C}})$ are quite easy to describe ([DKP]).

Lemma 3.5.2 (see Thm. 4.2 [DKP]) *Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$ be a nilpotent orbit of either $O(p, q)$ or $Sp_{2n}(\mathbb{R})$ (see Definition 12). Construct an alternating sequence of complex orthogonal orbits and complex symplectic orbits*

$$\mathcal{O}_{\mathbf{d}}, \mathcal{O}_{\mathbf{d}-1}, \mathcal{O}_{\mathbf{d}-2}, \dots, \mathcal{O}_{\mathbf{d}-\mathbf{d}_1+1}$$

and a corresponding alternating sequence of complex orthogonal groups and symplectic groups:

$$G(\mathcal{O}_{\mathbf{d}}), G(\mathcal{O}_{\mathbf{d}-1}), \dots, G(\mathcal{O}_{\mathbf{d}-\mathbf{d}_1+1}).$$

Then $\forall j$,

$$\Theta(G(\mathcal{O}_{\mathbf{d}-j}), G(\mathcal{O}_{\mathbf{d}-j+1}))(cl(\mathcal{O}_{\mathbf{d}-j})) = cl(\mathcal{O}_{\mathbf{d}-j+1}).$$

We define $\Theta(G_2, G_1)$ and $\Theta(G_1, G_2)$ for real nilpotent orbits in the same fashion. Pan showed that $\Theta(G_2, G_1)(\mathcal{O}_{\mathbf{D}})$ is the closure of at most two real nilpotent orbits (see Theorem 8.11 [PAN]). We give two lemmas that can be easily deduced from the descriptions of $\Theta(G_2, G_1)$ and $\Theta(G_1, G_2)$ in [PAN].

Lemma 3.5.3 (see 6.4, 8.6 [PAN]) *Suppose $p + q \leq 2n$. Then the real nilpotent orbit $\mathcal{O}_{\mathbf{D}}$ occurs in the image of m_2 if and only if $G(\mathcal{O}_{\mathbf{D}-1}) = O(r, s)$ with $r \leq p$ and $s \leq q$.*

Lemma 3.5.4 (see 6.4, 8.1, 8.11 [PAN]) *Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$ be a real nilpotent orbit of either $O(p, q)$ or $Sp_{2n}(\mathbb{R})$. Construct an alternating sequence of real nilpotent orbits of symplectic groups and orthogonal groups*

$$\mathcal{O}_{\mathbf{D}}, \mathcal{O}_{\mathbf{D}-1}, \mathcal{O}_{\mathbf{D}-2}, \dots, \mathcal{O}_{\mathbf{D}-\mathbf{d}_1+1}$$

and a corresponding sequence of real groups

$$G(\mathcal{O}_{\mathbf{D}}), G(\mathcal{O}_{\mathbf{D}-1}), G(\mathcal{O}_{\mathbf{D}-2}), \dots, G(\mathcal{O}_{\mathbf{D}-\mathbf{d}_1+1}).$$

Then $\forall j$,

$$\Theta(G(\mathcal{O}_{\mathbf{D}-j}), G(\mathcal{O}_{\mathbf{D}-j+1}))(cl(\mathcal{O}_{\mathbf{D}-j})) = cl(\mathcal{O}_{\mathbf{D}-j+1}).$$

3.6 Przebinda's Results on $\mathcal{V}(\text{Ann } \theta(\pi))$

In [PR93], Przebinda proved that under some conditions, the following diagram commutes:

$$\begin{array}{ccc} \pi & \xrightarrow{\theta} & \theta(MG, MG')(\pi) \\ \downarrow & & \downarrow \\ \mathcal{V}(\text{Ann } \pi) & \xrightarrow{\Theta(G_{\mathbb{C}}, G'_{\mathbb{C}})} & \mathcal{V}(\text{Ann } \theta(MG, MG')(\pi)) \end{array} \quad (3.4)$$

(See Notation 5). We state Przebinda's theorem for $O(p, q)$ and $Mp_{2n}(\mathbb{R})$. Recall from Theorem III.1 [MI] and Theorem 8.48 of [KN] that π has the rate of growth γ if and only if for every leading exponent v of π ,

$$\Re(v) \preceq (\gamma - 1)\rho(G)$$

(see also Theorem 4.5 of [PR93]).

Theorem 3.9 (Theorem 0.9 [PR93]) *Let π be an irreducible unitary representation of $O(p, q)$. Suppose*

1. $2n + 1 \geq p + q$;
2. Θ_π has rate of growth γ with $\gamma + 1 < \frac{2n}{p+q-2}$, equivalently, every leading exponent v of π satisfies

$$\Re(v) < \left(\frac{2n}{p+q-2} - 2\right)\rho(O(p, q));$$
3. $\gamma \geq 0$, equivalently, γ is not a discrete series representation;
4. $(\ , \)_\pi$ converges and does not vanish;
5. $(\ , \)_\pi$ is positive semidefinite;
6. there exists a full rank element $x \in \text{Mat}(p + q, 2n)$ such that the $O(p, q)$ -orbit generated by $m_1(x)$ is of maximal dimension in $AS(\Theta_\pi)$.

Then $\theta(p, q; 2n)(\pi)$ is unitary and its associated variety is the complex orbit

$$\Theta(O(p + q, \mathbb{C}), Sp_{2n}(\mathbb{C}))(\mathcal{V}(\text{Ann } \pi)).$$

Let me make two remarks here. Firstly, the assumption (1) does not appear in Przebinda's original theorem. Conditions (2) and (3) imply $2n > p + q - 2$. We add (1) as an assumption for the sake of clarity. The reader should also notice that condition (1) guarantees the nonvanishing of $(\ , \)_\pi$ or $(\ , \)_{\det \otimes \pi}$ by Theorem 3.3. Secondly, condition (6) is satisfied in the setting of Lemma 3.5.4. More precisely,

Lemma 3.6.1 *Suppose that π is an irreducible unitary representation of $O(p, q)$, that $2n + 1 \geq p + q$, and that \mathbf{D} is a signed Young tableau for $Sp_{2n}(\mathbb{R})$. Assume that $\mathcal{O}_{\mathbf{D}-1}$ is of maximal dimension in $AS(\Theta_\pi)$. Then there is an element $x \in \text{Mat}(p + q, 2n)$ of rank $p + q$ with the properties*

1. $O(p, q)m_1(x) = \mathcal{O}_{\mathbf{D}-1}$;
2. $Sp_{2n}(\mathbb{R})m_2(x) = \mathcal{O}_{\mathbf{D}}$.

Proof: According to Lemma 3.5.3, $\mathcal{O}_{\mathbf{D}}$ occurs in the image of m_2 . By Lemma 3.5.4, there exists x in $\text{Mat}(p + q, 2n)$ such that $m_2(x) \in \mathcal{O}_{\mathbf{D}}$ and $m_1(x) \in cl(\mathcal{O}_{\mathbf{D}-1})$. Hence

$$\text{rank}(W_n x^t I_{p,q} x) = \text{rank}(m_2(x)) = p + q.$$

But $x \in \text{Mat}(p + q, 2n)$. So x must be of full rank. For x of full rank, observe that

$$[m_2(x)]^r = W_n x^t [m_1(x)]^{r-1} I_{p,q} x.$$

By writing out the defining equations for $m_2(x) \in \mathcal{O}_{\mathbf{D}}$, we obtain $m_1(x) \in \mathcal{O}_{\mathbf{d}-1}$. But $m_1(x) \in cl(\mathcal{O}_{\mathbf{D}-1})$. It follows that $m_1(x) \in \mathcal{O}_{\mathbf{D}-1}$. \square

Similarly, Przebinda's theorem for $Mp_{2n}(\mathbb{R})$ can be formulated as follows.

Theorem 3.10 (Theorem 0.9 [PR93]) *Let π be an irreducible unitary representation of $Mp_{2n}(\mathbb{R})$. Suppose*

1. $p + q > 2n$;
2. Θ_π has the rate of growth γ with $\gamma + 1 < \frac{p+q}{2n}$, equivalently, every leading exponent v of π satisfies

$$\Re(v) < \left(\frac{p+q}{2n} - 2\right)\rho(Mp_{2n}(\mathbb{R}));$$
3. $\gamma \geq 0$, equivalently, π is not a discrete series representation;
4. $(\cdot, \cdot)_\pi$ converges and does not vanish;
5. $(\cdot, \cdot)_\pi$ is positive semidefinite;
6. there exists a full rank element $x \in \text{Mat}(p+q, 2n)$ such that the $Sp_{2n}(\mathbb{R})$ -orbit generated from $m_2(x)$ is of maximal dimension in $AS(\Theta_\pi)$.

Then $\theta(2n; p, q)(\pi)$ is unitary and its associated variety is the complex nilpotent orbit

$$\Theta(Sp_{2n}(\mathbb{C}), O(p+q, \mathbb{C}))(\mathcal{V}(\text{Ann } \pi)).$$

It is easy to see that (2) and (3) imply (1). We leave (1) there for the sake of clarity. Similar to Lemma 3.6.1, we have

Lemma 3.6.2 *Suppose that π is an irreducible unitary representation of $Mp_{2n}(\mathbb{R})$, that $p+q > 2n$, and that \mathbf{D} is a signed Young tableau for $O(p, q)$. Assume that $\mathcal{O}_{\mathbf{D}-1}$ is of maximal dimension in $AS(\Theta_\pi)$. Then there is an element $x \in \text{Mat}(p+q, 2n)$ of rank $2n$ with the properties*

1. $O(p, q)m_1(x) = \mathcal{O}_{\mathbf{D}}$;
2. $Sp_{2n}(\mathbb{R})m_2(x) = \mathcal{O}_{\mathbf{D}-1}$.

So (6) in Theorem 3.10 holds if there is an orbit in $AS(\Theta_\pi)$ of the form $\mathcal{O}_{\mathbf{D}-1}$ for $\mathbf{D} \in \mathcal{YD}(p, q)$.

Last let us recall the following result of Przebinda.

Theorem 3.11 (Corollary 2.8, [PR93]) *Consider the dual pair $(O(p, q), Sp_{2n}(\mathbb{R}))$.*

- *If $\pi \in \mathcal{R}(MO(p, q), \omega)$, then $WF(\pi)$ is a subset of $m_1(\text{Mat}(p+q; 2n))$.*
- *If $\pi \in \mathcal{R}(MSp_{2n}(\mathbb{R}), \omega)$, then $WF(\pi)$ is a subset of $m_2(\text{Mat}(p+q; 2n))$.*

Chapter 4

A Nonvanishing Theorem

In [He01], we established a nonvanishing theorem for $\theta_s(p, q; 2n)$ with $p+q \leq 2n+1$. This theorem is cited as Theorem 3.3 in the last Chapter. In this chapter, we will prove a nonvanishing theorem for $\theta_s(2n; p, q)$. These two nonvanishing theorems will then be used to construct a packet of irreducible unitary representations $\mathcal{N}(\mathcal{O})$ attached to a real special rigid orbit \mathcal{O} and more generally any $\mathcal{O} \in \mathcal{U}$ (see Definition 12). We start with a trivial lemma.

Lemma 4.0.3 *Suppose that*

$$\theta_s(p, q; 2n_2)\theta_s(2n_1; p, q)(\pi) = \mathcal{Q}(2n_1; p, q; 2n_2)(\pi).$$

If $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi) \neq 0$, then $\theta_s(2n_1; p, q)(\pi) \neq 0$.

Combining with Theorem 3.8, we obtain

Theorem 4.1 *Suppose $2n_2 - p - q \geq p + q - 2n_1 - 2$. Let π be an irreducible admissible representation in $R_{ss}(MSp_{2n_1}(\mathbb{R}), \omega(p, q; 2n_1))$ (see Def. 18). If $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi) \neq 0$, then $\theta_s(2n_1; p, q)(\pi) \neq 0$.*

At first glance, $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$ seems to be more difficult to treat than $\theta_s(2n_1; p, q)$. This is true with one exception. As predicted in [Heq], when $p + q = n_1 + n_2 + 1$, quantum induction can be obtained from parabolic induction. In this chapter, we will examine $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$ for $n_1 + n_2 + 1 = p + q$. We will prove that precisely for $n_1 + n_2 + 1 = p + q$ and $n_1 \leq n_2$, $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$ can be obtained from parabolically induced representations. As a consequence, we can determine the nonvanishing of $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$.

Notation 17 *Let m, n be two nonnegative integers such that $m + n \neq 0$. Let $P_{m,n}$ be the standard maximal parabolic subgroup of $Sp_{2m+2n}(\mathbb{R})$ with Levi subgroup $GL(m, \mathbb{R}) \times Sp_{2n}(\mathbb{R})$.*

4.1 Nonvanishing of $\mathcal{Q}(2n_1; p, q; 2(p + q - n_1 - 1))(\pi)$: Overview and Main Result

Suppose that $n_2 \geq n_1$. Let $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ be a subgroup of $Sp_{2n_1+2n_2}(\mathbb{R})$. Let $U(n_1+n_2)$ be a maximal compact subgroup of $Sp_{2n_1+2n_2}(\mathbb{R})$ such that $U(n_1+n_2) \cap Sp_{2n_1}(\mathbb{R})$ and $U(n_1+n_2) \cap Sp_{2n_2}(\mathbb{R})$

are maximal compact subgroups of $Sp_{2n_1}(\mathbb{R})$ and $Sp_{2n_2}(\mathbb{R})$ respectively.

First of all, a theorem of Kudla and Rallis says that, for some α , the parabolic induced representation

$$I^\alpha = \text{Ind}_{MP_{n_1+n_2,0}}^{Mp_{2n_1+2n_2}(\mathbb{R})} \chi^\alpha$$

(see Definition 24) decomposes into a direct sum of irreducible representations, namely $\mathcal{E}(2n_1 + 2n_2; p, q)$ for some p and q ([KR], [LZ]). Regard I^α as a representation of $Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})$ by restriction. For $\pi \in \Pi(Mp_{2n_1}(\mathbb{R}))$ satisfying a certain growth condition, I^α induces a representation of $Mp_{2n_2}(\mathbb{R})$, $I^\alpha(\pi)$ (see Definition 25). Using the theorem of Kudla and Rallis, we show that $I^\alpha(\pi)$ decomposes into a direct sum of $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$ for some $p + q = n_1 + n_2 + 1$.

Next, we study the restriction of I^α to $Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})$. Notice that the Harish-Chandra module $V(I^\alpha)$ consists of sections of a homogeneous line bundle over the Lagrangian Grassmannian of $\mathbb{R}^{2n_1+2n_2}$. By analyzing the action of $Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})$ on the Grassmannian, we prove that

$$I^\alpha(\pi) \cong \text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha,$$

whenever the left hand side is well-defined (see Theorem 4.11). In order that $I^\alpha(\pi)$ is well-defined, π must satisfy a certain growth condition. The details are given in Section 4.4.

Now, let us assume that $I^\alpha(\pi)$ is well-defined. Then we know that

$$\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha \simeq \bigoplus \mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$$

for a set of pairs (p, q) . The remaining question is which $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$ is nonvanishing. This turns out to be a hard question. We don't have any answer in general. But for some π , we can detect the nonvanishing of $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$ precisely, thanks to the notion of wave front sets. Notice that the decomposition of $I^\alpha(\pi)$ into $\mathcal{Q}(\ast)(\pi)$ induces a decomposition on the wave front level, namely,

$$WF(\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha) = \bigcup WF(\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)). \quad (4.1)$$

On the one hand, we know that

$$WF(\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha) = \text{Ind}_{\mathfrak{gl}(n_2-n_1)(\mathbb{R}) \times \mathfrak{sp}_{2n_1}(\mathbb{R})}^{\mathfrak{sp}_{2n_2}(\mathbb{R})} WF(\pi^\tau).$$

As we have pointed out earlier, the induced orbit consists of a few irreducible components which can be parametrized easily. For the π 's we are concerned with, Theorems 2.7 and 2.8 provide us with the precise information about the nilpotent orbits contained in the left hand side of Equation 4.1.

On the other hand, for some π , for example, those specified in Theorem 3.8, the right hand side of Equation 4.1 equals

$$\bigcup WF(\theta_s(p, q; 2n_2)\theta_s(2n_1; p, q)(\pi)).$$

Theorem 3.11 of Przebinda gives some control on which nilpotent orbit occurs in the wave front set of $\theta(p, q; 2n_2)(\ast)$ ([PR93]). Lemma 3.5.3 of Pan gives the parametrization of these nilpotent

orbits ([PAN]).

We have now two sets of nilpotent orbits, one from the induced orbit, the other from the moment map m_2 with respect to various $(O(p, q), Sp_{2n_2}(\mathbb{R}))$. By matching these two sets of wave front sets, we obtain a nonvanishing theorem for $WF(\mathcal{Q}(2n_1; p, q; 2n_2))$, which in turn gives us the nonvanishing of $\mathcal{Q}(2n_1; p, q; 2n_2)$ and $\theta_s(2n_1; p, q)$. Some of the results here will be used in Chapter 6. In this Chapter, we will prove the following generic nonvanishing theorem.

Theorem 4.2 *Consider the group $Mp_{2n_1+2n_2}(\mathbb{R})$ with $n_1 < n_2$. Let π be a unitary representation in $R_{ss}(Mp_{2n_1}(\mathbb{R}), \omega(n_1 + n_2 + 1, 0; 2n_1))$ (see Def. 18). Let $\mathcal{O}_{\mathbf{D}}$ be a nilpotent orbit of maximal dimension in*

$$\text{Ind}_{\mathfrak{sp}_{2n_1}(\mathbb{R}) \oplus \mathfrak{gl}(n_2 - n_1, \mathbb{R})}^{\mathfrak{sp}_{2n_2}(\mathbb{R})} WF(\pi^\tau).$$

Let S be the subset of $\{(p, q) \mid p+q = n_1+n_2+1, p \text{ fixed parity}\}$, consisting of those (p, q) such that $\mathcal{O}_{\mathbf{D}}$ is contained the image of the moment map m_2 associated with $(O(p, q), Sp_{2n_2}(\mathbb{R}))$. Then there exists a pair $(p, q) \in S$ such that $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi) \neq 0$. For such a pair, $\theta_s(2n_1; p, q)(\pi) \neq 0$.

Notice that

$$R_{ss}(Mp_{2n_1}(\mathbb{R}), \omega(n_1 + n_2 + 1, 0; 2n_1)) = R_{ss}(Mp_{2n_1}(\mathbb{R}), \omega(p, q; 2n_1))$$

(see Def. 18) for every $p+q = n_1+n_2+1$. The most favorable situation is when the set S , which depends on \mathbf{D} , consists of only one pair (p, q) . By our theorem, we will have $\theta_s(2n_1; p, q)(\pi) \neq 0$. The proof of Theorem 4.2 will be given in Ch. 4.5.

In this chapter, we take θ and θ_s as a correspondence between representations of $Mp_{2n}(\mathbb{R})$ and representations of $O(p, q)$. Throughout this Chapter, we assume $n_2 \geq n_1$.

4.2 Results of Kudla-Rallis and Lee-Zhu

Consider the symplectic group $Sp_{2n}(\mathbb{R})$. Let MAN be the Langlands decomposition of $P_{n,0}$ (Notation 17). The Levi factor $MA \cong GL(n, \mathbb{R})$. Let $Mp_{2n}(\mathbb{R})$ be the metaplectic covering of $Sp_{2n}(\mathbb{R})$. The group $MGL(n, \mathbb{R})$ can be expressed as

$$\{(\xi, x) \mid x \in GL(n, \mathbb{R}), \xi^2 = \det x, \xi \in \mathbb{C}\}.$$

Put $\chi(\xi, x) = \frac{\xi}{|\xi|}$. Then χ is a character of $MGL(n, \mathbb{R})$ of order 4. We extend χ trivially on N . χ becomes a character of $MP_{n,0}$.

Definition 24 $I_n^\alpha = \text{Ind}_{MP_{n,0}}^{Mp_{2n}(\mathbb{R})} \chi^\alpha$.

I_n^α is a unitarily induced representation.

Theorem 4.3 (Kudla-Rallis, [KR]) *Suppose $p+q = n+1$ and $\alpha \equiv p-q \pmod{4}$. Then $\theta(p, q; 2n)(\text{trivial})$ is a subrepresentation of I_n^α .*

By studying the K -types in $\theta(p, q; 2n)(\text{trivial})$ for various $p+q = n+1$, one has

Theorem 4.4 ([KR], [LZ])

$$I_n^\alpha = \bigoplus_{p+q=n+1, \alpha \equiv p-q \pmod{4}} \theta(p, q; 2n)(\text{trivial}).$$

More precisely, for n odd,

$$I_n^0 = \text{Ind}_{P_{n,0}}^{Sp_{2n}(\mathbb{R})} \text{trivial} = \bigoplus_{p+q=n+1, p \equiv q \pmod{4}} \theta(p, q; 2n)(\text{trivial}),$$

$$I_n^2 = \text{Ind}_{P_{n,0}}^{Sp_{2n}(\mathbb{R})} \text{sgn}(\det) = \bigoplus_{p+q=n+1, p-q \equiv 2 \pmod{4}} \theta(p, q; 2n)(\text{trivial}).$$

For n even,

$$I_n^1 = \bigoplus_{p+q=n+1, 1 \equiv p-q \pmod{4}} \theta(p, q; 2n)(\text{trivial}),$$

$$I_n^3 = \bigoplus_{p+q=n+1, 3 \equiv p-q \pmod{4}} \theta(p, q; 2n)(\text{trivial}).$$

This decomposition theorem can be derived directly from the results in [KR]. It was explicitly stated in [LZ].

Notation 18 We use I^α to denote $I_{n_1+n_2}^\alpha$ which is the main object of study in this section.

Definition 25 Let $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ be embedded diagonally into $Sp_{2n_1+2n_2}(\mathbb{R})$. Let $U(n_1+n_2)$ be a maximal compact subgroup of $Sp_{2n_1+2n_2}(\mathbb{R})$ such that $U(n_1+n_2) \cap Sp_{2n_1}(\mathbb{R})$ and $U(n_1+n_2) \cap Sp_{2n_2}(\mathbb{R})$ are maximal compact subgroups of $Sp_{2n_1}(\mathbb{R})$ and $Sp_{2n_2}(\mathbb{R})$ respectively. The embedding $Sp_{2n_1}(\mathbb{R}) \rightarrow Sp_{2n_1+2n_2}(\mathbb{R})$ lifts to $Mp_{2n_1}(\mathbb{R}) \rightarrow Mp_{2n_1+2n_2}(\mathbb{R})$ (see for example [Heu]). Consider I^α . Let $\pi \in \Pi(Mp_{2n_1}(\mathbb{R}))$ such that the form $(\ , \)_1$

$$(\phi_1 \otimes u_1, \phi_2 \otimes u_2)_1 = \int_{Mp_{2n_1}(\mathbb{R})} (\pi(g)u_1, u_2)(I^\alpha(g)\phi_1, \phi_2) dg$$

converges for every $u_1, u_2 \in V(\pi)$, $\phi_1, \phi_2 \in V(I^\alpha)$. Define $I^\alpha(\pi)$ to be $V(I^\alpha) \otimes V(\pi)$ modulo the radical of $(\ , \)_1$. $I^\alpha(\pi)$ is a $(\mathfrak{sp}_{2n_2}(\mathbb{R}), MU(n_1+n_2) \cap Mp_{2n_2}(\mathbb{R}))$ -module.

From the definition of quantum induction \mathcal{Q} and Theorem 4.4, we obtain

Theorem 4.5 Suppose $\alpha \equiv n_1 + n_2 + 1 \pmod{2}$. Then as (\mathfrak{g}, K) -modules

$$I^\alpha(\pi) = \bigoplus_{p+q=n_1+n_2+1, p-q \equiv \alpha \pmod{4}} \mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$$

whenever one side is well-defined.

Let me remind the reader that $\theta(p, q; 2n)(\text{trivial})$ is denoted by $\mathcal{E}(2n; p, q)$ in the definition of \mathcal{Q} .

Suppose that $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$. It is known, at least under the assumptions of Theorem 3.7, that $\mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$ is a unitarizable Harish-Chandra module of $Mp_{2n_2}(\mathbb{R})$. Theorem 4.5 then implies that $I^\alpha(\pi)$ is also a unitarizable Harish-Chandra module in this case. Later, in Ch. 4.4, we will show directly that $I^\alpha(\pi)$ is a unitarizable Harish-Chandra module of $Mp_{2n_2}(\mathbb{R})$ in a much more general setting.

4.3 $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ action on Lagrangian Grassmannian $X_{n_1+n_2}$

We are interested in the $Mp_{2n_2}(\mathbb{R}) \times Mp_{2n_1}(\mathbb{R})$ -action on

$$I^\alpha = \text{Ind}_{MP_{n_1+n_2,0}}^{Mp_{2n_1+2n_2,0}(\mathbb{R})} \chi^\alpha$$

(see Notations 13, 17). Recall that representation I^α consists of sections of a certain homogeneous line bundle over the Lagrangian Grassmannian $Sp_{2n_1+2n_2}(\mathbb{R})/P_{n_1+n_2,0}$ (see Notation 17).

Notation 19 Denote $Sp_{2n_1+2n_2}(\mathbb{R})/P_{n_1+n_2,0}$ by X .

X parametrizes Lagrangian subspaces of a symplectic space of dimension $2n_1 + 2n_2$. We are thus led to the problem of analyzing the $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ -action on X .

The existence of an open dense $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ -orbit in X can be proved using algebraic group theory. But this is not sufficient for our purpose. Not only do we need to know the existence of an open dense orbit, but also the detailed structure of this orbit. For this reason, we give two elementary lemmas and one theorem in this section. We shall use them in the next section to identify the restriction of I^α to $Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})$. The main result in this section is Theorem 4.7.

Suppose from now on $n_2 > n_1 > 0$. The case $n_1 = n_2$ has been studied in [He0], [He99]. Let me begin by recalling the following result.

Theorem 4.6 (Theorem 0.3 [He0]) *Let $L(4n)$ be the Lagrangian Grassmannian of $(\mathbb{R}^{4n}, \Omega)$. Then $Sp_{2n}(\mathbb{R}) \times Sp_{2n}(\mathbb{R})$ acts on $L(4n)$ with $2n + 1$ orbits. Furthermore, there is an open dense orbit in $L(4n)$ which can be identified with $Sp_{2n}(\mathbb{R})$.*

4.3.1 Notations

Notation 20 *Let $(\mathbb{R}^{2n_2+2n_1}, \Omega)$ be a symplectic space. Let X be the Lagrangian Grassmannian of $(\mathbb{R}^{2n_2+2n_1}, \Omega)$. Fix a standard real basis*

$$\{e_1, e_2, \dots, e_{n_2+n_1}, f_1, f_2, \dots, f_{n_2+n_1}\}$$

on $\mathbb{R}^{2n_1+2n_2}$ such that

$$\begin{aligned} \Omega(e_i, e_j) &= 0, & \Omega(f_i, f_j) &= 0 \\ \Omega(f_j, e_i) &= \delta_i^j, \end{aligned}$$

where δ_i^j is the Kronecker symbol. Write

$$\begin{aligned} \mathbb{R}^{2n_1} &= \text{span}\{e_1, \dots, e_{n_1}, f_1, \dots, f_{n_1}\} \\ \mathbb{R}^{2n_2} &= \text{span}\{e_{n_1+1}, \dots, e_{n_1+n_2}, f_{n_1+1}, \dots, f_{n_1+n_2}\}. \end{aligned}$$

Let $Sp_{2n_1}(\mathbb{R})$ be the subgroup of $Sp_{2n_1+2n_2}(\mathbb{R})$ fixing e_j, f_j for every $j > n_1$. Let $Sp_{2n_2}(\mathbb{R})$ be the subgroup of $Sp_{2n_1+2n_2}(\mathbb{R})$ fixing e_j, f_j for every $j \leq n_1$. The group $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ is then diagonally embedded into $Sp_{2n_1+2n_2}(\mathbb{R})$. Let

$$V_1 = \text{span}\{e_1 - e_{n_2+1}, e_2 - e_{n_2+2}, \dots, e_{n_1} - e_{n_2+n_1}\}$$

$$\begin{aligned}
V_2 &= \text{span}\{f_{n_2+1} + f_1, f_{n_2+2} + f_2, \dots, f_{n_1+n_2} + f_{n_1}\} \\
V_0 &= \text{span}\{e_{n_1+1}, \dots, e_{n_2}\} \\
V'_0 &= \text{span}\{f_{n_1+1}, \dots, f_{n_2}\}.
\end{aligned}$$

Put $V = V_1 \oplus V_2 \oplus V_0$.

4.3.2 $Sp_{2n_2}(\mathbb{R})$ -action and the Generic Orbit X_0

The space V is a Lagrangian subspace of $(\mathbb{R}^{2n_1+2n_2}, \Omega)$, or $V \in X$. Consider the action of $Sp_{2n_2}(\mathbb{R})$ on V . Let X_0 be the $Sp_{2n_2}(\mathbb{R})$ -orbit of V . Then X_0 is a subset of X .

Lemma 4.3.1 *Let $P_{n_2-n_1, n_1}$ be the maximal parabolic subgroup in $Sp_{2n_2}(\mathbb{R})$ stabilizing $V_0 \subset \mathbb{R}^{2n_2}$. Let $Sp_{2n_1}(\mathbb{R})GL(n_2 - n_1)$ be the Levi subgroup stabilizing both V_0 and V'_0 . Let N be the nilradical of $P_{n_2-n_1, n_1}$. Then $(Sp_{2n_2}(\mathbb{R}))_V = GL(n_2 - n_1)N$ and*

$$X_0 \cong Sp_{2n_2}(\mathbb{R})/GL(n_2 - n_1)N.$$

Furthermore, X_0 is open in X .

Proof: Let $g \in Sp_{2n_2}(\mathbb{R})$. Then g fixes e_i, f_i for every $i \leq n_1$. Suppose $gV = V$.

1. Since g stabilizes V and \mathbb{R}^{2n_2} , g stabilizes $V_0 = V \cap \mathbb{R}^{2n_2}$. So $g \in P_{n_2-n_1, n_1}$.
2. Let $j \leq n_1$. Since $ge_j = e_j$ and $g\mathbb{R}^{2n_2} = \mathbb{R}^{2n_2}$,

$$g(\mathbb{R}e_j \oplus \mathbb{R}^{2n_2}) = \mathbb{R}e_j \oplus \mathbb{R}^{2n_2}.$$

It follows that

$$g[(\mathbb{R}e_j \oplus \mathbb{R}^{2n_2}) \cap V] = [(\mathbb{R}e_j \oplus \mathbb{R}^{2n_2}) \cap V].$$

Notice that

$$(\mathbb{R}e_j \oplus \mathbb{R}^{2n_2}) \cap V = (\mathbb{R}e_j \oplus \mathbb{R}^{2n_2}) \cap (V_1 \oplus V_2 \oplus V_0) = \mathbb{R}(e_j - e_{n_2+j}) \oplus V_0.$$

So

$$g(\mathbb{R}(e_j - e_{n_2+j}) \oplus V_0) = \mathbb{R}(e_j - e_{n_2+j}) \oplus V_0.$$

Since $ge_j = e_j$,

$$g(e_j - e_{n_2+j}) \in (e_j - e_{n_2+j}) + V_0.$$

Hence for every $j \leq n_1$, $ge_{n_2+j} \in e_{n_2+j} + V_0$.

3. Similarly, since $gf_j = f_j$ for $j \leq n_1$,

$$gf_{n_2+j} \in f_{n_2+j} + V_0, \quad (j \leq n_1).$$

4. Thus, for every

$$v \in \text{span}\{e_{n_2+1}, \dots, e_{n_2+n_1}, f_{n_2+1}, \dots, f_{n_2+n_1}\} \simeq \mathbb{R}^{2n_1},$$

$g(v) \in v + V_0$. This shows that $g \in GL(n_2 - n_1)N$.

5. Conversely, if $g \in GL(n_2 - n_1)N$, then $g \in P_{n_2-n_1, n_1} \subset Sp_{2n_2}(\mathbb{R})$. So g fixes e_j, f_j for every $j \leq n_1$ and g stabilizes V_0 . Since $g \in GL(n_2 - n_1)N$, g preserves $e_{n_2+j} + V_0$ and $f_{n_2+j} + V_0$ for every $j \leq n_1$. So g preserves $e_j - e_{n_2+j} + V_0, f_{n_2+j} + f_j + V_0$ for every $j \leq n_1$. It follows that $gV = V$.

Therefore, $(Sp_{2n_2}(\mathbb{R}))_V = GL(n_2 - n_1)N$ and $X_0 \cong Sp_{2n_2}(\mathbb{R})/GL(n_2 - n_1)N$. To show that X_0 is open in X , we compute the dimensions of X_0 and X :

$$\begin{aligned}
\dim X_0 &= \dim(Sp_{2n_1}(\mathbb{R})) + \dim(Sp_{2n_2}(\mathbb{R})/Sp_{2n_1}(\mathbb{R})GL(n_2 - n_1)N) \\
&= \dim(Sp_{2n_1}(\mathbb{R})) + \frac{1}{2} \dim(Sp_{2n_2}(\mathbb{R})/Sp_{2n_1}(\mathbb{R})GL(n_2 - n_1)) \\
&= 2n_1^2 + n_1 + \frac{2n_2^2 + n_2 - 2n_1^2 - n_1 - (n_2 - n_1)^2}{2} \\
&= \frac{4n_1^2 + 2n_1 + 2n_2^2 + n_2 - 2n_1^2 - n_1 - n_1^2 + 2n_1n_2 - n_2^2}{2} \\
&= \frac{n_1^2 + n_2^2 + 2n_1n_2 + (n_1 + n_2)}{2} \\
&= \frac{2(n_1 + n_2)^2 + (n_1 + n_2) - (n_1 + n_2)^2}{2} \\
&= \frac{1}{2} \dim(Sp_{2n_1+2n_2}(\mathbb{R})/GL(n_1 + n_2, \mathbb{R})) \\
&= \dim(Sp_{2n_1+2n_2}(\mathbb{R})/P_{n_1+n_2, 0}) \\
&= \dim X.
\end{aligned} \tag{4.2}$$

□.

For $n_2 = n_1$, $P_{n_2-n_1, n_1} = Sp_{2n_2}(\mathbb{R})$. Our Lemma says that X_0 can be identified with $Sp_{2n_2}(\mathbb{R})$. This case is already treated in [He0] (see Theorem 4.6).

4.3.3 $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ -Action

Next, consider the $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ action on X . Again, our symplectic space is $\mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2}$. The basis for \mathbb{R}^{2n_1} is

$$\{e_1, e_2, \dots, e_{n_1}, f_1, f_2, \dots, f_{n_1}\}.$$

Let

$$U = \text{span}\{e_{n_2+1}, e_{n_2+2}, \dots, e_{n_2+n_1}, f_{n_2+1}, f_{n_2+2}, \dots, f_{n_2+n_1}\}.$$

Identify \mathbb{R}^{2n_1} with U by mapping

$$e_j \rightarrow e_{n_2+j}, f_j \rightarrow f_{n_2+j}, \quad (j \in [1, n_1]).$$

For each $g \in Sp_{2n_1}(\mathbb{R})$, define $i(g) \in Sp_{2n_2}(\mathbb{R})$ such that

$$i(g)|_{V_0 \oplus V'_0} = \text{identity}; \quad i(g)|_U = g.$$

In other words, $Sp_{2n_1}(\mathbb{R})$ is embedded into $Sp_{2n_2}(\mathbb{R})$ as the first factor of the Levi subgroup $Sp_{2n_1}(\mathbb{R})GL(n_2 - n_1)$. Recall for every $g \in Sp_{2n_1}(\mathbb{R})$, $\tau(g)$ is defined to be

$$\begin{pmatrix} I_{n_1} & 0 \\ 0 & -I_{n_1} \end{pmatrix} g \begin{pmatrix} I_{n_1} & 0 \\ 0 & -I_{n_1} \end{pmatrix}.$$

Lemma 4.3.2 *Let $g \in Sp_{2n_1}(\mathbb{R})$. Let $i(g)$ be an element in $Sp_{2n_2}(\mathbb{R})$ such that $i(g)|_{V_0 \oplus V'_0} = 1$. Regarding $(g, \tau(i(g)))$ as an element in $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$, we have*

$$(g, \tau(i(g)))V = V.$$

Proof: Let $g \in Sp_{2n_1}(\mathbb{R})$. We borrow some ideas from [He0]. Define an isomorphism $\phi : \mathbb{R}^{2n_1} \rightarrow U$ by

$$\begin{aligned} \phi(e_j) &= -e_{n_2+j} & (j = 1, 2, \dots, n_1) \\ \phi(f_j) &= f_{n_2+j} & (j = 1, 2, \dots, n_1). \end{aligned}$$

Then $V_1 \oplus V_2$ is the graph of ϕ and $\tau(i(g)) = \phi g \phi^{-1}$. Notice that

$$\begin{aligned} (g, \tau(i(g)))(V_1 \oplus V_2) &= \{(gu, \tau(i(g))\phi(u)) \mid u \in \mathbb{R}^{2n_1}\} \\ &= \{(u, \tau(i(g))\phi(g^{-1}u)) \mid u \in \mathbb{R}^{2n_1}\} \\ &= \{(u, \phi u) \mid u \in \mathbb{R}^{2n_1}\} \\ &= V_1 \oplus V_2. \end{aligned} \tag{4.3}$$

Thus $V_1 \oplus V_2$ is preserved by the actions of $(g, \tau(i(g)))$. Clearly, $(g, \tau(i(g)))$ also preserves vectors in V_0 . It follows that $(g, \tau(i(g)))V = V$. \square

Combining with Lemma 4.3.1, we obtain

Theorem 4.7 *Let $P_{n_2-n_1, n_1}$ be the maximal parabolic subgroup of $Sp_{2n_2}(\mathbb{R})$ which preserves V_0 . Let $Sp_{2n_1}(\mathbb{R})GL(n_2 - n_1)N$ be the Langlands decomposition of $P_{n_2-n_1, n_1}$ as in Lemma 4.3.1. Then the isotropy group*

$$(Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R}))_V = \{(\tau(g), gh) \mid g \in Sp_{2n_1}(\mathbb{R}), h \in GL(n_2 - n_1)N\}.$$

Moreover,

$$X_0 \cong (Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})) / (Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R}))_V$$

and X_0 is open and dense in X .

Here $Sp_{2n_1}(\mathbb{R})$ and $Sp_{2n_2}(\mathbb{R})$ are of the standard matrix form. The $Sp_{2n_1}(\mathbb{R})$ in the Levi factor $Sp_{2n_1}(\mathbb{R})GL(n_2 - n_1)$ of $Sp_{2n_2}(\mathbb{R})$ is identified with the standard $Sp_{2n_1}(\mathbb{R})$. We avoid using i again.

Proof: We only need to prove that X_0 is dense in X . This follows from an argument similar to the argument in [He0]. Briefly, for every Lagrangian subspace W , define

$$Ind(W) = (\dim(W \cap \mathbb{R}^{2n_1}), \dim(W \cap \mathbb{R}^{2n_2})).$$

Notice that $\dim(V \cap \mathbb{R}^{2n_1}) = 0$ and $\dim(V \cap \mathbb{R}^{2n_2}) = \dim(V_0) = n_2 - n_1$. In general, for every $W \in X$, we have

$$\dim(W \cap \mathbb{R}^{2n_1}) \geq 0, \quad \dim(W \cap \mathbb{R}^{2n_2}) \geq n_2 - n_1.$$

We say a Lagrangian subspace W is generic if $Ind(W) = (0, n_2 - n_1)$. Observe that the group action $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ preserves Ind . Therefore, every element in X_0 is generic.

Conversely, every generic Lagrangian subspace W must be an element in X_0 . Let W be generic. Since $W \cap \mathbb{R}^{2n_1} = \{0\}$, the projection

$$p_2 : \mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2} \rightarrow \mathbb{R}^{2n_2}$$

restricted to W is injective. So $\dim(p_2(W)) = n_1 + n_2$. Similarly, since $\dim(W \cap \mathbb{R}^{2n_2}) = n_2 - n_1 = \dim(W) - \dim(\mathbb{R}^{2n_1})$, the projection

$$p_1 : \mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2} \rightarrow \mathbb{R}^{2n_1}$$

restricted to W must be surjective. It can then be shown that W is uniquely determined by a symplectic map from $p_2(W)$ to \mathbb{R}^{2n_1} , namely $p_1 p_2^{-1}$. The kernel of this symplectic map is exactly $W \cap \mathbb{R}^{2n_2}$, which is an isotropic subspace. So $p_1 p_2^{-1}$ is reduced to a symplectic isomorphism from $p_2(W)/W \cap \mathbb{R}^{2n_2}$ to \mathbb{R}^{2n_1} . By basic linear group theory, the group $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R})$ acts on the set of symplectic maps from $p_2(W)$ to \mathbb{R}^{2n_1} transitively. So the orbit X_0 consists of all generic Lagrangian subspaces. Therefore X_0 is dense in X . \square

4.4 $I^\alpha(\pi)$ and $\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$

Following Notation 20, Lemma 4.3.1, and Theorem 4.7, we set

Notation 21

$$n_2 > n_1, \quad P_{n_1+n_2, 0} = Sp_{2n_1+2n_2}(\mathbb{R})_V.$$

Fix $Sp_{2n_1}(\mathbb{R}) \times Sp_{2n_2}(\mathbb{R}) \subset Sp_{2n_1+2n_2}(\mathbb{R})$ as in Notation 20. Write

$$Sp_{2n_2}(\mathbb{R})_V = GL(n_2 - n_1)N.$$

Fix a maximal compact subgroup $MU(n_1 + n_2)$ in $Mp_{2n_1+2n_2}(\mathbb{R})$ such that

$$MU(n_1 + n_2) \cap Mp_{2n_1}(\mathbb{R}), MU(n_1 + n_2) \cap Mp_{2n_2}(\mathbb{R})$$

are maximal compact subgroups for $Mp_{2n_1}(\mathbb{R})$ and $Mp_{2n_2}(\mathbb{R})$ respectively. Let

$$MU(n_1) = MU(n_1 + n_2) \cap Mp_{2n_1}(\mathbb{R}), \quad MU(n_2) = MU(n_1 + n_2) \cap Mp_{2n_2}(\mathbb{R}).$$

Let $\pi \in \Pi(Mp_{2n_1}(\mathbb{R}))$. Recall from Definition 25 that if

$$(\phi_1 \otimes u_1, \phi_2 \otimes u_2)_1 = \int_{Mp_{2n_1}(\mathbb{R})} (\pi(g_1)u_1, u_2)(I^\alpha(g_1)\phi_1, \phi_2) dg_1$$

converges for every $u_1, u_2 \in V(\pi)$, $\phi_1, \phi_2 \in V(I^\alpha)$, then $I^\alpha(\pi)$ is defined to be $V(I^\alpha) \otimes V(\pi)$ modulo the radical of $(\ , \)_1$. $I^\alpha(\pi)$ is only a $(\mathfrak{sp}_{2n_2}(\mathbb{R}), MU(n_2))$ -module. In addition, $(\ , \)_1$ descends into a nondegenerate $\mathfrak{sp}_{2n_2}(\mathbb{R})$ -invariant Hermitian form on $I^\alpha(\pi)$. The main purpose of this section is to show that $(I^\alpha(\pi), (\ , \)_1)$ is equivalent to the unitarily induced representation

$$(V(\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha), (\ , \)).$$

Here $(\ , \)$ denotes the inner product in the compact picture. We do this in three steps.

The first step is to show that

$$I^\alpha|_{Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})} \cong L^2(Mp_{2n_2}(\mathbb{R}) \times_{MGL(n_2-n_1)N} | \det |^{\frac{n_1+n_2}{2}} \otimes \chi^\alpha, X_0)$$

where X_0 is equipped with a $MU(n_2) \times Sp_{2n_1}(\mathbb{R})$ -invariant measure. On the right hand side of the equation, $Mp_{2n_2}(\mathbb{R})$ acts on the left and $Mp_{2n_1}(\mathbb{R})$ acts on the right with an involution τ . See Theorem 4.8 for the precise statement. The main ingredient of the proof is Theorem 4.7.

From Lemma 4.3.1, we have a principal fibration

$$Mp_{2n_1}(\mathbb{R})/\{1, \epsilon\} \rightarrow X_0 \rightarrow Mp_{2n_2}(\mathbb{R})/Mp_{2n_1}(\mathbb{R})MGL(n_2-n_1)N.$$

Let $C_c^\infty(X_0, I^\alpha)$ be the set of compactly supported smooth sections of

$$Mp_{2n_2}(\mathbb{R}) \times_{MGL(n_2-n_1)N} | \det |^{\frac{n_1+n_2}{2}} \otimes \chi^\alpha \rightarrow X_0.$$

The next step is to define a map

$$\mathcal{L}^0 : C_c^\infty(X_0, I^\alpha) \otimes V(\pi) \rightarrow \text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha,$$

This map is essentially integration on the fiber $Mp_{2n_1}(\mathbb{R})$ (see Definition 29). The image of \mathcal{L}^0 are sections of

$$Mp_{2n_2}(\mathbb{R}) \times_{Mp_{2n_1}(\mathbb{R}) \times MGL(n_2-n_1)N} \pi^\tau \otimes | \det |^{\frac{n_1+n_2}{2}} \otimes \chi^\alpha \rightarrow X_0$$

in a proper sense.

Last step is to extend \mathcal{L}^0 to $V(I^\alpha) \otimes V(\pi)$ under the assumption that $(\ , \)_1$ is well-defined. We will further show that $(\ , \)_1$ on $I^\alpha(\pi)$ coincides with the Hilbert structure on

$$\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha.$$

The main result is Theorem 4.11. Let us first analyze the restriction of I^α to $Mp_{2n_2}(\mathbb{R})$.

4.4.1 I^α restricted to $Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})$

The representation I^α is a parabolically induced representation. Parabolic induction is a powerful tool in the study of representations of reductive Lie groups (see for example [KN] and [Wallach]). There are the compact picture and the noncompact picture. To study I^α , we recall the compact picture of I^α .

Notation 22 Let $P_{n_2+n_1, 0}$ be the Siegel parabolic subgroup of $Sp_{2n_1+2n_2}(\mathbb{R})$ (See Notation 21). Define a character on $P_{n_2+n_1, 0} = GL(n_2+n_1)N$ by

$$\exp(-\rho_0(gn)) = | \det g |^{-\frac{n_1+n_2+1}{2}} \quad (g \in GL(n_1+n_2), n \in N).$$

Lift $\exp(-\rho_0)$ trivially to a character of $MP_{n_2+n_1, 0}$. By abusing notation, denote this character by $\exp(-\rho_0)$.

The Hilbert space of I^α consists of L^2 sections of the linear bundle

$$Mp_{2n_1+2n_2}(\mathbb{R}) \times_{MGL(n_1+n_2)N} \mathbb{C}[\chi^\alpha \otimes \exp(\rho_0)] \rightarrow X.$$

Equivalently,

Definition 26 *The Hilbert space of I^α consists of functions on $Mp_{2n_1+2n_2}(\mathbb{R})$ such that*

$$f(gh) = \chi(h)^{-\alpha} \exp(-\rho_0(h))f(g) \quad (\forall h \in MP_{n_2+n_1,0}),$$

$$\int_{MU(n_1+n_2)} \|f(k)\|^2 dk < \infty.$$

The inner product between $f_1, f_2 \in I^\alpha$ is

$$(f_1, f_2) = \int_{MU(n_1+n_2)} f_1(k) \overline{f_2(k)} dk.$$

Notice that the restriction map

$$f \in I^\alpha \rightarrow f|_{MU(n_1+n_2)} \in \text{Ind}_{MO(n_1+n_2)}^{MU(n_1+n_2)} \chi^\alpha$$

defines an isomorphism between I^α and $\text{Ind}_{MO(n_1+n_2)}^{MU(n_1+n_2)} \chi^\alpha$. We call the restrictions of f to $MU(n_1+n_2)$ the compact picture.

Restrict the L^2 -sections to the open dense subset X_0 . Fix the base point V (see Notation 20). From Theorem 4.7, X_0 is a single $Sp_{2n_2}(\mathbb{R})$ -orbit of the form

$$Sp_{2n_2}(\mathbb{R})/GL(n_2-n_1)N.$$

We may model $I^\alpha|_{Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})}$ on the sections of

$$Mp_{2n_2}(\mathbb{R}) \times_{MGL(n_2-n_1)N} \exp(\rho_0) \otimes \chi^\alpha \rightarrow X_0.$$

Since X_0 is open and dense in X , a section f is uniquely determined by the section $f|_{X_0}$ up to a set of measure zero.

Notation 23 *Fix the base point V (see Notation 20) on X_0 . Regard $f \in I^\alpha$ as a section on*

$$Mp_{2n_2}(\mathbb{R}) \times_{MGL(n_2-n_1)N} \exp(\rho_0) \otimes \chi^\alpha \rightarrow X_0.$$

Let f^\sharp be the function on $Mp_{2n_2}(\mathbb{R})$ corresponding to this section.

The function f^\sharp satisfies

$$f^\sharp(gh) = \chi(h)^{-\alpha} \exp(-\rho_0(h))f^\sharp(g) \quad (\forall h \in MGL(n_2-n_1)N). \quad (4.4)$$

Let us consider the action of $Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})$ on I^α . Keep in mind that $Mp_{2n_1}(\mathbb{R}) \cap Mp_{2n_2}(\mathbb{R}) = \{1, \epsilon\}$ and

$$I^\alpha(\epsilon, \epsilon) = \text{Identity}.$$

Lemma 4.4.1 For $f \in I^\alpha$, $g_1 \in Mp_{2n_1}(\mathbb{R})$, $g_2 \in Mp_{2n_2}(\mathbb{R})$, (g_1, g_2) embedded in $Mp_{2n_1+2n_2}(\mathbb{R})$ diagonally, we have

$$f((g_1, g_2)) = f^\sharp(g_2\tau(g_1)^{-1}).$$

For every $h_1 \in Mp_{2n_1}(\mathbb{R})$, we have

$$[I^\alpha(h_1)f]^\sharp(g_2) = f^\sharp(g_2\tau(h_1)).$$

Proof: By Theorem 4.7,

$$f((g_1, g_2)) = f((1, g_2\tau(g_1)^{-1})(g_1, \tau(g_1))) \quad (4.5)$$

$$= f^\sharp(g_2\tau(g_1)^{-1})\chi((g_1, \tau(g_1))^{-\alpha} \exp(-\rho_0(g_1, \tau(g_1)))) \quad (4.6)$$

$$= f^\sharp(g_2\tau(g_1)^{-1}). \quad (4.7)$$

As to the action of $h_1 \in Mp_{2n_1}(\mathbb{R})$,

$$[I^\alpha(h_1)f]^\sharp(g_2) = (I^\alpha(h_1)f)((1, g_2)) = f((h_1^{-1}, g_2)) = f^\sharp(g_2\tau(h_1)).$$

□.

So for every $h_1 \in Mp_{2n_1}(\mathbb{R})$, $[I^\alpha(h_1)f]^\sharp = R(\tau(h_1))f^\sharp$. Here R stands for the right regular action. Clearly, the action of $Mp_{2n_2}(\mathbb{R})$ on f^\sharp is the left regular action. Now we shall determine the Hilbert structure on f^\sharp . More precisely, for $f_1, f_2 \in I^\alpha$, we want to express (f_1, f_2) in terms of f_1^\sharp, f_2^\sharp .

Notation 24 Let dk_2 be the normalized invariant measure for the maximal compact subgroup $MU(n_2)$. Let $d[k_2]$ be the induced measure on

$$MU(n_2)/M(O(n_2 - n_1)U(n_1)) \cong Mp_{2n_2}(\mathbb{R})/MP_{n_2-n_1, n_1}.$$

Fix an invariant measure dg_1 on $Mp_{2n_1}(\mathbb{R})$. Every element $x \in Mp_{2n_2}(\mathbb{R})/MGL(n_2 - n_1)N$ will be represented by k_2g_1 for some $k_2 \in MU(n_2)$ and some $g_1 \in Mp_{2n_1}(\mathbb{R})$. This is the case since $Sp_{2n_2}(\mathbb{R})$ has a $U(n_2)Sp_{2n_1}(\mathbb{R})GL(n_2 - n_1)N$ decomposition. Let $dg_1d[k_2]$ be the induced measure on $Mp_{2n_2}(\mathbb{R})/MGL(n_2 - n_1)N$.

Definition 27 Let

$$L^2(Mp_{2n_2}(\mathbb{R}) \times_{MGL(n_2-n_1)N} | \det |^{\frac{n_1+n_2}{2}} \otimes \chi^\alpha, dg_1d[k_2])$$

be the set of sections of

$$Mp_{2n_2}(\mathbb{R}) \times_{MGL(n_2-n_1)N} | \det |^{\frac{n_1+n_2}{2}} \otimes \chi^\alpha \rightarrow Mp_{2n_2}(\mathbb{R})/MGL(n_2 - n_1)N$$

such that

$$\|f\|^2 = \int_{Mp_{2n_2}(\mathbb{R})/MGL(n_2-n_1)N} |f(k_2g_1)|^2 d[k_2] dg_1 < \infty.$$

Define

$$(f_1, f_2)^\sharp = \int_{Mp_{2n_2}(\mathbb{R})/MGL(n_2-n_1)N} f_1(k_2g_1)\overline{f_2(k_2g_1)} d[k_2] dg_1.$$

This integral does not depend on the choices of k_2 . Let $g_2 \in Mp_{2n_2}(\mathbb{R})$ act on f by the left regular action $L(g_2)$ and $g_1 \in Mp_{2n_1}$ act on f by the right regular action $R(\tau(g_1))$.

Theorem 4.8 *Up to a proper choice of dg_1 , the map $f \rightarrow f^\sharp$ defines an isometry between I^α and*

$$L^2(Mp_{2n_2}(\mathbb{R}) \times_{MGL(n_2-n_1)N} | \det |^{\frac{n_1+n_2}{2}} \otimes \chi^\alpha, dg_1 d[k_2]).$$

In addition, this isometry induces an equivalence of $Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_2}(\mathbb{R})$ -representations:

$$I^\alpha|_{Mp_{2n_1}(\mathbb{R})Mp_{2n_2}(\mathbb{R})} \cong L^2(Mp_{2n_2}(\mathbb{R}) \times_{MGL(n_2-n_1)N} | \det |^{\frac{n_1+n_2}{2}} \otimes \chi^\alpha, dg_1 d[k_2]).$$

We call this the $Mp_{2n_2}(\mathbb{R})$ -model of I^α .

Proof: Let $f_1, f_2 \in I^\alpha$. Clearly, (f_1, f_2) can be expressed as an integral on X_0 by ignoring the measure zero set $X - X_0$. Then (f_1, f_2) can be expressed as

$$\int_{Mp_{2n_2}(\mathbb{R})/MGL(n_2-n_1)N} f_1^\sharp(k_2 g_1) \overline{f_2^\sharp(k_2 g_1)} \Delta([k_2], g_1) d[k_2] dg_1.$$

Here $\Delta([k_2], g_1)$ comes from a product of the Jacobian of a coordinate transform

$$Mp_{2n_2}(\mathbb{R})/MGL(n_2 - n_1)N \ni ([k_2], g_1) \longrightarrow [k] \in MU(n_1 + n_2)/MO(n_1 + n_2) \cong X$$

and an $\exp(-\rho_0)$ term. Since $(f_1, f_2) = (I^\alpha(k_2 g_1) f_1, I^\alpha(k_2 g_1) f_2)$ for $k_2 \in MU(n_2)$ and $g_1 \in Mp_{2n_1}(\mathbb{R})$, $\Delta([k_2], g_1)$ must be left $MU(n_2)$ invariant and right $Mp_{2n_1}(\mathbb{R})$ -invariant. Hence $\Delta([k_2], g_1)$ is a constant. The first assertion is proved.

The second assertion follows from Lemma 4.4.1. \square

$$4.4.2 \quad \mathcal{L}^0 : C_c^\infty(X_0, I^\alpha) \otimes V(\pi) \rightarrow \text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$$

Definition 28 *Let f be an element in the Hilbert space of I^α . Let f^\sharp be the restriction of f to $Mp_{2n_2}(\mathbb{R})$. We say that a vector f is compactly supported on X_0 , if f^\sharp is compactly supported on $X_0 \cong Mp_{2n_2}(\mathbb{R})/MGL(n_2 - n_1)N$. Let $C_c^\infty(X_0, I^\alpha)$ be the set of smooth vectors with compact support in X_0 . Let $C_c^\infty(X_0, I^\alpha)_{MU(n_2)}$ be the $MU(n_2)$ -finite vectors in $C_c^\infty(X_0, I^\alpha)$.*

Definition 29 *Let $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$ and $u \in V(\pi)$. For each $f \in C_c^\infty(X_0, I^\alpha)$, define $\mathcal{L}^0(f \otimes u)$ to be the function*

$$g_2 \in Mp_{2n_2}(\mathbb{R}) \rightarrow \int_{Mp_{2n_1}(\mathbb{R})} (I^\alpha(g_1) f)^\sharp(g_2) \pi(g_1) u dg_1 \in \mathcal{H}_\pi.$$

Since f^\sharp is compactly supported, for each $g_2 \in Mp_{2n_2}(\mathbb{R})$,

$$\int_{Mp_{2n_1}(\mathbb{R})} (I^\alpha(g_1) f)^\sharp(g_2) \pi(g_1) u dg_1 = \int_{Mp_{2n_1}(\mathbb{R})} f^\sharp(g_2 \tau(g_1)) \pi(g_1) u dg_1.$$

So $\mathcal{L}^0(f \otimes u)$ is well-defined as a vector valued function on $Mp_{2n_2}(\mathbb{R})$.

Theorem 4.9 *Let $f \in C_c^\infty(X_0, I^\alpha)$. Let $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$ and $u \in V(\pi)$. Then $\mathcal{L}^0(f \otimes u)$ is an element in the Hilbert space of $\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$. Furthermore,*

$$\mathcal{L}^0(C_c^\infty(X_0, I^\alpha)_{MU(2)} \otimes V(\pi)) = V(\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha)$$

and \mathcal{L}^0 preserves the $(\mathfrak{sp}_{2n_2}(\mathbb{R}), MU(n_2))$ action.

Proof: Let $f \in C_c^\infty(X_0, I^\alpha)$ and $u \in V(\pi)$.

1. Notice that for $g_2 \in Mp_{2n_2}(\mathbb{R})$ and h_1 in the $Mp_{2n_1}(\mathbb{R})$ factor of $MP_{n_2-n_1, n_1}$,

$$\begin{aligned} \mathcal{L}^0(f \otimes u)(g_2 h_1) &= \int_{Mp_{2n_1}(\mathbb{R})} (I^\alpha(g_1)f)^\sharp(g_2 h_1) \pi(g_1)u \, dg_1 \\ &= \int_{Mp_{2n_1}(\mathbb{R})} f^\sharp(g_2 h_1 \tau(g_1)) \pi(g_1)u \, dg_1 \\ &= \int_{Mp_{2n_1}(\mathbb{R})} f^\sharp(g_2 h_1 g_1) \pi(\tau(g_1))u \, dg_1 \\ &= \int_{Mp_{2n_1}(\mathbb{R})} f^\sharp(g_2 g_1) \pi(\tau(h_1^{-1} g_1))u \, dg_1 \\ &= \int_{Mp_{2n_1}(\mathbb{R})} f^\sharp(g_2 g_1) \pi(\tau(h_1)^{-1})\pi(\tau(g_1))u \, dg_1 \\ &= \pi(\tau(h_1)^{-1}) \left\{ \int_{Mp_{2n_1}(\mathbb{R})} (I^\alpha(\tau(g_1))f)^\sharp(g_2) \pi(\tau(g_1))u \, dg_1 \right\} \\ &= \pi(\tau(h_1)^{-1}) \left[\int_{Mp_{2n_1}(\mathbb{R})} (I^\alpha(g_1)f)^\sharp(g_2) \pi(g_1)u \, dg_1 \right]. \end{aligned} \tag{4.8}$$

So $\mathcal{L}^0(f \otimes u)(g_2 h_1) = \pi(\tau(h_1)^{-1})\mathcal{L}^0(f \otimes u)(g_2)$.

2. Recall that for every $h \in MGL(n_2 - n_1)N \subset MP_{n_2-n_1, n_1}$, we have

$$f^\sharp(g_2 h) = f^\sharp(g_2)\chi(h)^{-\alpha} \exp(-\rho_0(h)).$$

It follows that

$$\int_{Mp_{2n_1}(\mathbb{R})} (I^\alpha(g_1)f)^\sharp(g_2 h) \pi(g_1)u \, dg_1 = \chi(h)^{-\alpha} \exp(-\rho_0(h)) \int_{Mp_{2n_1}(\mathbb{R})} (I^\alpha(g_1)f)^\sharp(g_2) \pi(g_1)u \, dg_1.$$

So $\mathcal{L}^0(f \otimes u)(g_2 h) = \chi(h)^{-\alpha} \exp(-\rho_0(h))\mathcal{L}^0(f \otimes u)(g_2)$.

3. Since f^\sharp is smooth and compactly supported, the restriction of

$$g_2 \rightarrow \int_{Mp_{2n_1}(\mathbb{R})} (I^\alpha(g_1)f)^\sharp(g_2) \pi(g_1)u \, dg_1$$

onto the maximal compact subgroup $MU(n_2)$ must be continuous and bounded. We see now that $\mathcal{L}^0(f \otimes u)$ is a continuous section of

$$Mp_{2n_2}(\mathbb{R}) \times_{Mp_{2n_1}(\mathbb{R})MGL(n_2-n_1)N} \pi^\tau \otimes \exp(\rho_0) \otimes \chi^\alpha \rightarrow Mp_{2n_2}(\mathbb{R})/Mp_{2n_1}(\mathbb{R})GL(n_2 - n_1)N.$$

Notice that $\exp(\rho_0) = |\det|^{\frac{n_1+n_2}{2}}$ on the GL factor is exactly the normalizing factor. Therefore $\mathcal{L}^0(f \otimes u)$ is an element in the Hilbert space of $\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$.

4. Since the right action of $Mp_{2n_1}(\mathbb{R})$ commutes with the left action of $Mp_{2n_2}(\mathbb{R})$, \mathcal{L}^0 preserves the action of $Mp_{2n_2}(\mathbb{R})$. In particular, \mathcal{L}^0 preserves the action of $MU(n_2)$. Therefore

$$\mathcal{L}^0 : C_c^\infty(X_0, I^\alpha)_{MU(n_2)} \otimes V(\pi) \rightarrow V(\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha).$$

5. Suppose that the map above is not surjective. There must exist a nonzero $MU(n_2)$ -finite vector

$$\phi : Mp_{2n_2}(\mathbb{R}) \rightarrow \mathcal{H}_{\pi^\tau}$$

such that

$$\begin{aligned} \phi(g_2 p_2) &= (\pi^\tau \otimes \chi^\alpha)(p_2^{-1})\phi(g_2) \quad (\forall p_2 \in MP_{n_2-n_1, n_1}), \\ (\phi, \mathcal{L}^0(f \otimes u)) &= 0 \quad (\forall f \in C_c^\infty(X_0, I^\alpha)_{MU(n_2)}, u \in V(\pi)). \end{aligned}$$

By an easy computation,

$$0 = (\phi, \mathcal{L}^0(f \otimes u)) = \int_{Mp_{2n_2}(\mathbb{R})/MGL(n_2-n_1)N} (\phi(k_2 g_1), u) \overline{f(k_2 g_1)} d[k_2] dg_1.$$

We see that the $MU(n_2)$ -finite function $k_2 g_1 \rightarrow (\phi(k_2 g_1), u)$ is perpendicular to all compactly supported $MU(n_2)$ -finite functions on $Mp_{2n_2}(\mathbb{R})/MGL(n_2-n_1)N$ with respect to the measure $d[k_2]dg_1$. So $(\phi(k_2 g_1), u) = 0$ for all $k_2 \in MU(n_2), g_1 \in Mp_{2n_1}(\mathbb{R}), u \in V(\pi^\tau)$. Therefore $\phi = 0$. We reach a contradiction. Hence

$$\mathcal{L}^0(C_c^\infty(X_0, I^\alpha)_{MU(2)} \otimes V(\pi)) = V(\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha).$$

□

4.4.3 The Inner Product $(\cdot, \cdot)_1$

We have shown that \mathcal{L}^0 defines a map from $C_c^\infty(X_0, I^\alpha) \otimes V(\pi)$ to $\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$. In this section, we shall extend the definition of \mathcal{L}^0 to $V(I^\alpha) \otimes V(\pi)$ under the assumption that $(\cdot, \cdot)_1$ is well-defined for $V(I^0) \otimes V(\pi)$. Let us begin with the following lemma.

Lemma 4.4.2 *Let $MU(n_1+n_2)$ be a maximal compact subgroup of $Mp_{2n_1+2n_2}(\mathbb{R})$. Let $1_{MU(n_1+n_2)}$ be the spherical vector in I^0 . Then*

1. $g \rightarrow (I^0(g)1_{MU(n_1+n_2)}, 1_{MU(n_1+n_2)})$ is positive;
2. Each $\phi \in V(I^\alpha)$, as a function on $Mp_{2n_1+2n_2}(\mathbb{R})$, is bounded by a multiple of $1_{MU(n_1+n_2)}$;
3. For every $\phi, \psi \in V(I^\alpha)$, there exists a constant $C_{\phi, \psi}$ such that

$$|(I^\alpha(g)\phi, \psi)| \leq C_{\phi, \psi} (I^0(g)1_{MU(n_1+n_2)}, 1_{MU(n_1+n_2)})$$

for every $g \in Mp_{2n_1+2n_2}(\mathbb{R})$;

4. If $(1_{MU(n_1+n_2)} \otimes u, 1_{MU(n_1+n_2)} \otimes v)_1$ converges absolutely for all $u, v \in V(\pi)$, then $(\cdot, \cdot)_1$ is well-defined for $V(I^\alpha) \otimes V(\pi)$.

Proof: In the compact picture, $1_{MU(n_1+n_2)}$ is the constant function 1 on $MU(n_1+n_2)$. $I^0(g)1_{MU(n_1+n_2)}$ is also a positive function. Therefore $(I^0(g)1_{MU(n_1+n_2)}, 1_{MU(n_1+n_2)}) > 0$. By continuity, every $MU(n_1+n_2)$ -finite function in the compact picture of I^α is bounded by a positive multiple of $1_{MU(n_1+n_2)}$. Hence every $MU(n_1+n_2)$ -finite matrix coefficient of I^α is bounded by a multiple of $(I^0(g)1_{MU(n_1+n_2)}, 1_{MU(n_1+n_2)})$, i.e.,

$$|(I^\alpha(g)u, v)| \leq C_u C_v (I^0(g)1_{MU(n_1+n_2)}, 1_{MU(n_1+n_2)}) \quad (u, v \in V(I^\alpha))$$

for some constants C_u and C_v . As a result, if $(u \otimes 1_{MU(n_1+n_2)}, v \otimes 1_{MU(n_1+n_2)})_1$ converges absolutely for all $u, v \in V(\pi)$ and $1_{MU(n_1+n_2)}$ in I^0 , then $(\cdot, \cdot)_1$ is well-defined for $V(I^\alpha) \otimes V(\pi)$. \square

Lemma 4.4.3 *Let $\phi \in V(I^\alpha)$. Then there exists a positive constant C_ϕ such that $|\phi^\sharp(k_2 g_1)| \leq C_\phi 1_{MU(n_1+n_1)}^\sharp(g_1)$ for all $k_2 \in MU(n_2)$ and $g_1 \in Mp_{2n_1}(\mathbb{R})$.*

Proof: First of all $1_{MU(n_1+n_2)}^\sharp(k_2 g_1) = \exp(-\rho_0(p(g_1)))$ where $g_1 = k(g_1)p(g_1)$ under the decomposition $MU(n_1+n_2)MP_{n_1+n_2,0}$ of $Mp_{2n_1+2n_2}(\mathbb{R})$. Here $g_1 \in Mp_{2n_1}(\mathbb{R}) \subseteq Mp_{2n_2}(\mathbb{R}) \subseteq Mp_{2n_1+2n_2}(\mathbb{R})$ (see Notation 21). Similarly, $\phi^\sharp(k_2 g_1) = \phi(k_2 k(g_1)) \exp(-\rho_0(p(g_1)))$. By Lemma 4.4.2 (2), $\phi(k_2 k(g_1))$ is bounded by a constant C_ϕ for all $k_2 \in MU(n_2), k(g_1) \in MU(n_1+n_2)$. So

$$|\phi^\sharp(k_2 g_1)| \leq C_\phi 1_{MU(n_1+n_1)}^\sharp(g_1)$$

for all $k_2 \in MU(n_2)$ and $g_1 \in Mp_{2n_1}(\mathbb{R})$. \square

Lemma 4.4.4 *Let $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$. The following are equivalent.*

1. $I^0(\pi)$ is well-defined.
2. $(1_{MU(n_1+n_2)} \otimes u, 1_{MU(n_1+n_2)} \otimes v)_1$ converges absolutely for all $u, v \in V(\pi)$.
3. For every $u, v \in V(\pi)$, the following integral

$$\int_{Mp_{2n_1}(\mathbb{R})} (R(\tau(h_1))1_{MU(n_1+n_2)}^\sharp(g_1), 1_{MU(n_1+n_2)}^\sharp(g_1))_{Mp_{2n_1}(\mathbb{R})}(\pi(h_1)u, v) dh_1$$

converges absolutely. Here $(\cdot, \cdot)_{Mp_{2n_1}(\mathbb{R})}$ is the inner product for $L^2(Mp_{2n_1}(\mathbb{R}))$.

4. For every $u, v \in V(\pi)$, the function

$$(g_1, h_1) \mapsto 1_{MU(n_1+n_2)}^\sharp(g_1 \tau(h_1)) 1_{MU(n_1+n_2)}^\sharp(g_1) |(\pi(h_1)u, v)|$$

is in $L^1(Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_1}(\mathbb{R}), dg_1 dh_1)$.

5. $I^\alpha(\pi)$ is well-defined for every α .

Proof: By Definition 27 and Theorem 4.8, we have

$$(L(h_1)1_{MU(n_1+n_2)}, 1_{MU(n_1+n_2)}) = (R(\tau(h_1))1_{MU(n_1+n_2)}^\sharp, 1_{MU(n_1+n_2)}^\sharp)^\sharp.$$

The right hand side is equal to

$$C(R(\tau(h_1))1_{MU(n_1+n_2)}^\sharp(g_1), 1_{MU(n_1+n_2)}^\sharp(g_1))_{Mp_{2n_1}(\mathbb{R})}$$

where C is a constant related to the measure of $Mp_{2n_2}(\mathbb{R})/MP_{n_2-n_1, n_1}$. We obtain that

$$\begin{aligned} & (1_{MU(n_1+n_2)} \otimes u, 1_{MU(n_1+n_2)} \otimes v)_1 \\ &= C \int_{Mp_{2n_1}(\mathbb{R})} (R(\tau(h_1))1_{MU(n_1+n_2)}^\sharp(g_1), 1_{MU(n_1+n_2)}^\sharp(g_1))_{Mp_{2n_1}(\mathbb{R})}(\pi(h_1)u, v) dh_1. \end{aligned} \quad (4.9)$$

So (2) \leftrightarrow (3). (3) \leftrightarrow (4) follows from Fubini's theorem for nonnegative functions.

Suppose $I^0(\pi)$ is well-defined. Then $(1_{MU(n_1+n_2)} \otimes u, 1_{MU(n_1+n_2)} \otimes v)_1$ converges absolutely for all $u, v \in V(\pi)$. So (1) \rightarrow (2). By Lemma 4.4.2 (4), $I^\alpha(\pi)$ is well-defined for every α . So (2) \rightarrow (5). (5) \rightarrow (1) is obvious. We have proved (1) \rightarrow (2) \rightarrow (5) \rightarrow (1). \square

Suppose that $I^0(\pi)$ is well-defined. Let $\phi, \psi \in V(I^\alpha)$. By Lemma 4.4.3 and Lemma 4.4.4 (4),

$$\int \int_{Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_1}(\mathbb{R})} |\phi^\sharp(k_2 g_1) \overline{\phi^\sharp(k_2 h_1)}(\pi(\tau(g_1))u, \pi(\tau(h_1))u)| dh_1 dg_1 < \infty$$

for all $k_2 \in MU(n_2)$.

Lemma 4.4.5 *Let \mathcal{H} be a separable Hilbert space and Φ be a \mathcal{H} -valued continuous function on $Mp_{2n_1}(\mathbb{R})$. If*

$$\int \int_{x, y \in Mp_{2n_1}(\mathbb{R})} |(\Phi(x), \Phi(y))| dx dy < \infty,$$

then $\int \Phi(x) dx \in \mathcal{H}$ is well-defined, as the unique limit of $\int_K \Phi(x) dx$ as $K \rightarrow Mp_{2n_1}(\mathbb{R})$. Furthermore,

$$\left(\int_{x \in Mp_{2n_1}(\mathbb{R})} \Phi(x) dx, \int_{y \in Mp_{2n_1}(\mathbb{R})} \Phi(y) dy \right) = \int_{x, y \in Mp_{2n_1}(\mathbb{R})} (\Phi(x), \Phi(y)) dx dy$$

The proof of this will be given in the Appendix. See Theorem 7.1.

Definition 30 *Suppose that $I^0(\pi)$ is well-defined for a $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$. Let $\phi \in V(I^\alpha)$ and $u \in V(\pi)$. Consider the \mathcal{H}_π -valued function*

$$g_1 \in Mp_{2n_1}(\mathbb{R}) \rightarrow \phi^\sharp(k_2 g_1) \pi(\tau(g_1))u \quad (k_2 \in MU(n_2))$$

Define

$$\mathcal{L}^0(\phi \otimes u) : MU(n_2) \ni k_2 \mapsto \int_{g_1} \phi^\sharp(k_2 g_1) \pi(\tau(g_1))u dg_1 \in \mathcal{H}_\pi.$$

By Lemma 4.4.5, $\mathcal{L}^0(\phi \otimes u)$ is a \mathcal{H}_π -valued function on $MU(n_2)$. Now \mathcal{L}^0 is defined on $C_c^\infty(X_0, I^\alpha) \otimes V(\pi) + V(I^\alpha) \otimes V(\pi)$.

Theorem 4.10 *Let $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$. Suppose that $(\ , \)_1$ is well-defined for $V(I^0) \otimes V(\pi)$. Then for every $u, v \in V(\pi)$ and $\phi, \psi \in V(I^\alpha)$, we have*

1. $\mathcal{L}^0(\phi \otimes u), \mathcal{L}^0(\psi \otimes v) \in \text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$;
2. $2(\phi \otimes u, \psi \otimes v)_1 = (\mathcal{L}^0(\phi \otimes u), \mathcal{L}^0(\psi \otimes v))_{\text{Ind } \pi^\tau \otimes \chi^\alpha}$.

In particular, the Hermitian form $(\cdot, \cdot)_1$ is positive semidefinite and $\mathcal{L}^0(V(I^\alpha) \otimes V(\pi)) \cong I^\alpha(\pi)$.

Proof: To prove (1), we will construct a sequence $\phi_m \in C_c^\infty(X_0, I^\alpha) \rightarrow \phi$ such that $\mathcal{L}^0(\phi_m \otimes u)(k)$ is a sequence approaching $\mathcal{L}^0(\phi \otimes u)(k)$ in \mathcal{H}_π uniformly for all $k \in MU(n_2)$. We start with a sequence of functions $\mu_m \in C_c^\infty(Mp_{2n_1}(\mathbb{R}))$ such that

1. $\mu_m(k_1 g k'_1) = \mu_m(g), \forall k_1, k'_1 \in MU(n_1)$;
2. $\mu_m(\exp H) = 1$ for all $\|H\| \leq m$ where $H \in \mathfrak{a}$ and $\|H\|^2 = |(H, H)|$ (see 1.1).
3. $0 \leq \mu_m(g) \leq 1$ for all g .

Define $\phi_m \in C_c^\infty(X_0, I^\alpha)$ such that $\phi_m^\sharp(k_2 g_1) = \phi^\sharp(k_2 g_1) \mu_m(g_1)$. By Lemma 4.4.3 and the proof of Lemma 4.4.5 for the function $g_1 \rightarrow 1_{MU(n_1+n_2)}^\sharp(g_1) \pi(\tau(g_1))u$, we have

$$\begin{aligned}
& \|\mathcal{L}^0(\phi_m \otimes u)(k_2) - \mathcal{L}^0(\phi \otimes u)(k_2)\|^2 \\
&= \int \int_{Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_1}(\mathbb{R})} [\phi_m^\sharp(k_2 g_1) - \phi^\sharp(k_2 g_1)] \overline{[\phi_m^\sharp(k_2 h_1) - \phi^\sharp(k_2 h_1)]} (\pi(\tau(g_1))u, \pi(\tau(h_1))u) dg_1 dh_1 \\
&\leq 4C_\phi^2 \int \int_{\|g_1\| \geq m \text{ or } \|h_1\| \geq m} |1_{MU(n_1+n_2)}^\sharp(g_1) \pi(\tau(g_1))u, 1_{MU(n_1+n_2)}^\sharp(h_1) \pi(\tau(h_1))u| dg_1 dh_1 \\
&\rightarrow 0
\end{aligned} \tag{4.10}$$

where $\|g_1\| = \|H\|$ for $g_1 = k_1 \exp H k'_1$. So

$$\mathcal{L}^0(\phi_m \otimes u)(k_2) \rightarrow \mathcal{L}^0(\phi \otimes u)(k_2)$$

uniformly for all k_2 . By the proof of Theorem 4.9, each $\mathcal{L}^0(\phi_m \otimes u)$ is continuous in

$$\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha.$$

Therefore $\mathcal{L}^0(\phi \otimes u)$ is also continuous in $\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$.

To prove (2), we compute

$$\begin{aligned}
& 2(\phi \otimes u, \psi \otimes v)_1 \\
&= 2 \int_{g_1 \in Mp_{2n_1}(\mathbb{R})} (\pi(g_1)u, v) ((I^\alpha(g_1)\phi)(x), \psi(x)) dg_1 \\
&= \int_{g_1 \in Mp_{2n_1}(\mathbb{R})} (\pi(g_1)u, v) \int_{Mp_{2n_2}(\mathbb{R})/MGL(n_2-n_1)N} \phi^\sharp(k_2 h_1 \tau(g_1)) \overline{\psi^\sharp(k_2 h_1)} d[k_2] dh_1 dg_1 \\
&= \int_{g_1 \in Mp_{2n_1}(\mathbb{R})} (\pi(\tau(g_1))u, v) \int_{[k_2] \in Mp_{2n_2}(\mathbb{R})/MP_{n_2-n_1, n_1}} \int_{h_1 \in Mp_{2n_1}(\mathbb{R})} \phi^\sharp(k_2 h_1 g_1) \overline{\psi^\sharp(k_2 h_1)} dh_1 d[k_2] dg_1
\end{aligned} \tag{4.11}$$

By Lemma 4.4.2 (1), the triple integral (4.11) converges absolutely for $\phi, \psi = 1_{MU(n_1+n_2)} \in I^0$. By Lemma 4.4.2 (2)(3), the triple integral (4.11) converges absolutely for all $\phi, \psi \in V(I^\alpha)$. So by

Fubini's Theorem, changing the order of integrations in Integral (4.11), we obtain

$$\begin{aligned}
& 2(\phi \otimes u, \psi \otimes v)_1 \\
&= \int_{[k_2] \in Mp_{2n_2}(\mathbb{R})/MP_{n_2-n_1}} \int \int_{Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_1}(\mathbb{R})} \phi^\sharp(k_2 h_1 g_1) \overline{\psi^\sharp(k_2 h_1)} (\pi(\tau(g_1))u, v) dh_1 dg_1 d[k_2]. \\
&= \int_{[k_2] \in Mp_{2n_2}(\mathbb{R})/MP_{n_2-n_1}} \int \int_{Mp_{2n_1}(\mathbb{R}) \times Mp_{2n_1}(\mathbb{R})} \phi^\sharp(k_2 g_1) \overline{\psi^\sharp(k_2 h_1)} (\pi(\tau(g_1))u, \pi(\tau(h_1))v) dh_1 dg_1 d[k_2] \\
&= \int_{[k_2] \in Mp_{2n_2}(\mathbb{R})/MP_{n_2-n_1}} (\mathcal{L}^0(\phi \otimes u)(k_2), \mathcal{L}^0(\psi \otimes v)(k_2)) d[k_2] \\
&= (\mathcal{L}^0(\phi \otimes u), \mathcal{L}^0(\psi \otimes v))_{\text{Ind } \pi^\tau \otimes \chi^\alpha}.
\end{aligned} \tag{4.12}$$

It follows that $\mathcal{L}^0(V(I^\alpha) \otimes V(\pi)) \cong I^\alpha(\pi)$. \square

Corollary 4.1 *Let $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$. Then for every $u, v \in V(\pi)$ and $\phi, \psi \in C_c^\infty(X_0, I^\alpha)$,*

$$2(\phi \otimes u, \psi \otimes v)_1 = (\mathcal{L}^0(\phi \otimes u), \mathcal{L}^0(\psi \otimes v))_{\text{Ind } \pi^\tau \otimes \chi^\alpha}.$$

Suppose that $I^0(\pi)$ is well-defined. Then for every $\phi, \psi \in C_c^\infty(X_0, I^\alpha) \oplus V(I^\alpha)$,

$$2(\phi \otimes u, \psi \otimes v)_1 = (\mathcal{L}^0(\phi \otimes u), \mathcal{L}^0(\psi \otimes v))_{\text{Ind } \pi^\tau \otimes \chi^\alpha}.$$

By Theorem 4.10, $I^\alpha(\pi)$ can be identified with $\mathcal{L}^0(V(I^\alpha) \otimes V(\pi))$, which is a subspace of

$$\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha.$$

It is also K -finite with respect to the maximal compact subgroup $MU(n_2)$.

4.4.4 A Density Theorem

In this section, we will show that

$$\mathcal{L}^0(V(I^\alpha) \otimes V(\pi)) = V(\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha).$$

Theorem 4.11 *Let τ and π^τ be as in Definition 8. Let $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$. Suppose that $I^0(\pi)$ is well-defined. Suppose $n_1 < n_2$ and $\pi(\epsilon)\chi^\alpha(\epsilon) = 1$. Let $P_{n_2-n_1, n_1} = Sp_{2n_1}(\mathbb{R})GL(n_2 - n_1, \mathbb{R})N$ be as in Lemma 4.3.1. Realize $MP_{n_2-n_1, n_1}$ as the quotient group*

$$Mp_{2n_1}(\mathbb{R}) \times MGL(n_2 - n_1)N / \{(1, 1), (\epsilon, \epsilon)\}.$$

Then

$$I^\alpha(\pi) \cong \mathcal{L}^0(V(I^\alpha) \otimes V(\pi)) = V(\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha).$$

Furthermore, $\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$ is equivalent to the completion of $I^\alpha(\pi)$ with respect to the inner product $(\ , \)_1$.

By Theorems 4.9 and 4.10, $I^\alpha(\pi) \cong \mathcal{L}^0(V(I^\alpha) \otimes V(\pi))$ is a subspace of $V(\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha)$ and $\mathcal{L}^0(C_c^\infty(X_0, I^\alpha) \otimes V(\pi))$ is a dense subspace of $\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$. Since every element in $C_c^\infty(X_0, I^\alpha) \otimes V(\pi)$ can be approximated by vectors in $V(I^\alpha) \otimes V(\pi)$, it is reasonable to believe that $\mathcal{L}^0(V(I^\alpha) \otimes V(\pi))$ is dense in $\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$. This may be a direct consequence of some theorem in functional analysis about the averaging operator \mathcal{L}^0 . Unfortunately, we are not aware of any such theorem. We are forced to go back and dig into the compact picture in which the Harish-Chandra module is visible.

Proof of Theorem 4.11: It suffices to prove that $\mathcal{L}^0(V(I^\alpha) \otimes V(\pi))$ is dense in

$$\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha.$$

We shall prove that every element in $\mathcal{L}^0(C_c^\infty(X_0, I^\alpha) \otimes V(\pi))$ can be approximated by a sequence of elements in $\mathcal{L}^0(V(I^\alpha) \otimes V(\pi))$.

Consider $\psi \otimes u \in C_c^\infty(X_0, I^\alpha) \otimes V(\pi)$. Regard ψ as a smooth function on $MU(n_1 + n_2)$. By the Stone-Weierstrass Theorem, there exists a sequence of functions $\phi_i \in V(I^\alpha)$, such that $\phi_i \rightarrow \psi$ under the sup norm. In other words, for i sufficiently large,

$$|\phi_i(k) - \psi(k)| \leq \delta 1_{MU(n_1+n_2)}(k) \quad (\forall k \in K).$$

By Theorem 4.10 and Corollary 4.1,

$$(\mathcal{L}^0((\phi_i - \psi) \otimes u), \mathcal{L}^0((\phi_i - \psi) \otimes u))_{\text{Ind} \pi^\tau \otimes \chi^\alpha} = 2((\phi_i - \psi) \otimes u, (\phi_i - \psi) \otimes u)_1$$

The latter is no greater than $2\delta^2(1_{MU(n_1+n_2)} \otimes u, 1_{MU(n_1+n_2)} \otimes u)_1$ with respect to $I^0 \otimes \pi$. Therefore, $\mathcal{L}^0(\phi_i \otimes u) \rightarrow \mathcal{L}^0(\psi \otimes u)$ under the Hilbert norm. By Theorem 4.9, $\mathcal{L}^0(C_c^\infty(X_0, I^\alpha) \otimes V(I^\alpha))$ is already dense in $\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$. So $\mathcal{L}^0(V(I^\alpha) \otimes V(\pi))$ is dense in $\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha$.

By an easy K -finiteness argument with respect to $MU(n_2)$,

$$\mathcal{L}^0(V(I^\alpha) \otimes V(\pi)) = V(\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha).$$

We have proved that

$$I^\alpha(\pi) \cong \text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha.$$

□

Combining this with Theorem 4.5, we obtain

Theorem 4.12 *Assume that $\alpha \equiv n_1 + n_2 + 1 \pmod{2}$. If $\pi \in \Pi_u(Mp_{2n_1}(\mathbb{R}))$ and $I^0(\pi)$ is well-defined, then*

$$\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha \cong \bigoplus_{p+q=n_1+n_2+1, p-q \equiv \alpha \pmod{4}} \mathcal{Q}(2n_1; p, q; 2n_2)(\pi)$$

as unitary representations.

4.5 Wave Front Set and the Nonvanishing Theorem

Theorem 4.13 ([BV85] [VO01]) *Let $P = MAN$ be a parabolic subgroup of a reductive Lie group G . Let σ be an irreducible representation of $L = MA$. Then*

$$WF(\text{Ind}_P^G \sigma) = cl(\text{Ind}_1^{\mathfrak{g}} WF(\sigma)).$$

Notice that $\text{Ind}_1^{\mathfrak{g}}(WF(\sigma))$ may contain several irreducible components of the same dimension. Applying Theorem 4.13 to I^α for $Mp_{2n}(\mathbb{R})$, we obtain

Corollary 4.2 *The wave front set $WF(I^\alpha) = \text{Ind}_{\mathfrak{gl}(n, \mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})} \mathcal{O}_{[1^n]}$.*

Again assume $p + q = n + 1$. By the theorem of Kudla-Rallis-Lee-Zhu, we have

$$I^\alpha = \bigoplus_{p+q=n+1, p-q \equiv \alpha \pmod{4}} \theta(p, q; 2n)(\text{trivial}).$$

Consider the wave front sets. From Corollary 2.3, $\text{Ind}_{\mathfrak{gl}(n, \mathbb{R})}^{\mathfrak{sp}_{2n}(\mathbb{R})} \mathcal{O}_{[1^n]}$ consists of $n + 1$ components $\mathbf{D}^{(j)}$ and is equal to the closure of $\mathcal{O}_{[2^n]} \cap \mathfrak{sp}_{2n}(\mathbb{R})$. We have

$$cl\left(\bigcup D^{(j)}\right) = \bigcup_{p+q=n+1, p-q \equiv \alpha \pmod{4}} WF(\theta(p, q; 2n)(\text{trivial})).$$

This implies that these $WF(\theta(p, q; 2n)(\text{trivial}))$ must be in the closure of $\mathcal{O}_{[2^n]} \cap \mathfrak{sp}_{2n}(\mathbb{R})$ ([BV80]). In fact, they are distinct.

Theorem 4.14 *Assume $p + q = n + 1$. Let $[2^n]^{(i)}$ be the signed Young diagram with i rows starting with $+$. Then*

$$WF(\theta(p, q; 2n)(\text{trivial})) = cl(\mathcal{O}_{[2^n]^{(p)}} \cup \mathcal{O}_{[2^n]^{(p-1)}}).$$

Here we take $\mathcal{O}_{[2^n]^{(-1)}}$ and $\mathcal{O}_{[2^n]^{(n+1)}}$ to be the empty set. In particular,

$$WF(I^\alpha) = \bigcup_{p+q=n+1, p-q \equiv \alpha \pmod{4}} WF(\theta(p, q; 2n)(\text{trivial}))$$

Proof: From Lemma 3.5.3, only $cl(\mathcal{O}_{[2^n]^{(p)}} \cup \mathcal{O}_{[2^n]^{(p-1)}})$ can occur in the image of the moment map m_2 for $(O(p, q), Sp_{2n}(\mathbb{R}))$. Thus by Theorem 3.11 of Przebinda,

$$WF(\theta(p, q; 2n)(\text{trivial})) \subseteq cl(\mathcal{O}_{[2^n]^{(p)}} \cup \mathcal{O}_{[2^n]^{(p-1)}}).$$

But by the decomposition Theorem 4.4,

$$\bigcup_{j=0}^n cl(\mathcal{O}_{[2^n]^{(j)}}) = WF(I^\alpha) = \bigcup_{p+q=n+1, p-q \equiv \alpha \pmod{4}} WF(\theta(p, q; 2n)(\text{trivial})) \subseteq \bigcup_{j=0}^n cl(\mathcal{O}_{[2^n]^{(j)}})$$

We must have $WF(\theta(p, q; 2n)(\text{trivial})) = cl(\mathcal{O}_{[2^n]^{(p)}} \cup \mathcal{O}_{[2^n]^{(p-1)}})$. \square

This theorem partly explains the decomposition theorem of Kudla-Rallis-Lee-Zhu in terms of the

orbit philosophy. Generalizing this idea to $I^\alpha(\pi)$, we give a proof of the Generic Nonvanishing Theorem.

Proof of Theorem 4.2: By Lemma 4.4.2 (3), every $MU(n_1 + n_2)$ -finite matrix coefficient of I^α is bounded by a multiple of

$$(I^0(g)1_{MU(n_1+n_2)}, 1_{MU(n_1+n_2)}),$$

which is a spherical function. Notice that the infinitesimal character of I^0 is equal to

$$\mu = \left(\overbrace{\left(\frac{n_1 + n_2 - 1}{2}, \frac{n_1 + n_2 - 1}{2}, \frac{n_1 + n_2 - 3}{2}, \frac{n_1 + n_2 - 3}{2}, \dots, \frac{n_1 + n_2 - 1}{2} - \left[\frac{n_1 + n_2 - 1}{2} \right] \right)}^{n_1+n_2} \right),$$

and

$$\mu - \rho(Mp_{2n_1+2n_2}(\mathbb{R})) \prec \left(\overbrace{\left(\frac{-n_1 - n_2}{2}, \frac{-n_1 - n_2}{2} + 1, \dots, \frac{-n_1 - n_2}{2} + \left[\frac{n_1 + n_2}{2} \right], 0, \dots, 0 \right)}^{n_1+n_2} \right).$$

By Theorem 1.1 or by Harish-Chandra's spherical expansion, $(I^0(g)1_{MU(n_1+n_2)}, 1_{MU(n_1+n_2)})$ is bounded by a multiple of

$$a(g)^{\left(\frac{-n_1 - n_2}{2}, \frac{-n_1 - n_2}{2} + 1, \dots, \frac{-n_1 - n_2}{2} + \left[\frac{n_1 + n_2}{2} \right], 0, \dots, 0 \right)}.$$

In particular, $(I^0(g_1)1_{MU(n_1+n_2)}, 1_{MU(n_1+n_2)})$ is bounded by a multiple of

$$a(g_1)^{\left(\frac{-n_1 - n_2}{2}, \frac{-n_1 - n_2}{2} + 1, \dots, \frac{n_1 - n_2}{2} + 1 \right)} \quad (g_1 \in Mp_{2n_1}(\mathbb{R})).$$

Since $\pi \in R_{ss}(Mp_{2n_1}(\mathbb{R}), \omega(n_1 + n_2 + 1, 0; 2n_1))$ (Def. 18), for every leading exponent v of π ,

$$\Re(v) \preceq \frac{\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{1}}{2} - \mathbf{n}_1 - \mathbf{1} - \rho(Sp_{2n_1}(\mathbb{R})).$$

We obtain

$$\begin{aligned} & \Re(v) - \left(\frac{n_1 + n_2}{2}, \frac{n_1 + n_2}{2} - 1, \dots, \frac{n_2 - n_1}{2} + 1 \right) + 2\rho(Mp_{2n_1}(\mathbb{R})) \\ & \preceq \frac{\mathbf{n}_2 - \mathbf{n}_1 - \mathbf{1}}{2} - \rho(Mp_{2n_1}(\mathbb{R})) - \left(\frac{n_1 + n_2}{2}, \frac{n_1 + n_2}{2} - 1, \dots, \frac{n_2 - n_1}{2} + 1 \right) + 2\rho(Mp_{2n_1}(\mathbb{R})) \quad (4.13) \\ & \preceq \frac{\mathbf{n}_2 - \mathbf{n}_1 - \mathbf{1}}{2} - \left(\frac{n_1 + n_2}{2}, \frac{n_1 + n_2}{2} - 1, \dots, \frac{n_2 - n_1}{2} + 1 \right) + (n_1, n_1 - 1, \dots, 1) \\ & \prec 0. \end{aligned}$$

Inequality 4.13 and Theorem 1.1 imply that $(\cdot, \cdot)_1$ is well-defined for $V(I^0 \otimes V(\pi))$. So $I^\alpha(\pi)$ is well-defined for any α . Suppose that $\alpha \equiv n_1 + n_2 + 1 \pmod{2}$. Theorem 4.12 applies. We have

$$\text{Ind}_{MP_{n_2 - n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha = \bigoplus_{p+q=n_1+n_2+1, p-q \equiv \alpha \pmod{4}} \mathcal{Q}(2n_1; p, q; 2n_2)(\pi).$$

By Theorem 3.8,

$$\text{Ind}_{MP_{n_2 - n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \pi^\tau \otimes \chi^\alpha = \bigoplus_{p+q=n_1+n_2+1, p-q \equiv \alpha \pmod{4}} \theta_s(p, q; 2n_2) \theta_s(2n_1; p, q)(\pi).$$

It follows that

$$\text{Ind}_{\mathfrak{sp}_{n_1}(\mathbb{R})\mathfrak{gl}(n_2-n_1)}^{\mathfrak{sp}_{2n_2}(\mathbb{R})} WF(\pi^\tau) = \bigcup_{p+q=n_1+n_2+1, p-q \equiv \alpha \pmod{4}} WF(\theta_s(p, q; 2n_2)\theta_s(2n_1; p, q)(\pi)).$$

Let $\mathcal{O}_{\mathbf{D}}$ be a nilpotent orbit of maximal dimension in

$$\text{Ind}_{\mathfrak{sp}_{2n_1}(\mathbb{R})\mathfrak{gl}(n_2-n_1)}^{\mathfrak{sp}_{2n_2}(\mathbb{R})} WF(\pi^\tau).$$

Define $S \subseteq \{(p, q) \mid p+q = n_1 + n_2 + 1, p \text{ fixed parity}\}$ to be the set of those (p, q) such that $\mathcal{O}_{\mathbf{D}}$ occurs in the image of the moment map m_2 associated with $(O(p, q), Sp_{2n_2}(\mathbb{R}))$ (see Definition 23). By Theorem 3.11, $\mathcal{O}_{\mathbf{D}}$ can only appear in $WF(\theta_s(p, q; 2n_1)\theta_s(2n_1; p, q)(\pi))$ with $(p, q) \in S$. There must exist at least a $(p_0, q_0) \in S$ such that

$$\mathcal{O}_{\mathbf{D}} \subseteq WF(\theta_s(p_0, q_0; 2n_2)\theta_s(2n_1; p_0, q_0)(\pi)).$$

Hence $\mathcal{Q}(2n_1; p_0, q_0; 2n_2)(\pi) \neq 0$ and $\theta_s(2n_1; p_0, q_0)(\pi) \neq 0$. \square

Under certain favorable circumstances, $\mathcal{O}_{\mathbf{D}}$ only occurs in the image of m_2 for a unique (p, q) with $p+q = n_1 + n_2 + 1$. In this case, we are able to determine for which (p, q) , $\theta_s(2n_1; p, q)(\pi) \neq 0$.

Corollary 4.3 *Consider the group $Mp_{2n_1+2n_2}(\mathbb{R})$ with $n_1 < n_2$. Let π be a unitary representation in*

$$R_{ss}(Mp_{2n_1}(\mathbb{R}), \omega(n_1 + n_2 + 1, 0; 2n_1)),$$

(see Def. 18). Let $\mathcal{O}_{\mathbf{D}}$ be a nilpotent orbit of maximal dimension in

$$\text{Ind}_{\mathfrak{sp}_{2n_1}(\mathbb{R}) \oplus \mathfrak{gl}(n_2-n_1, \mathbb{R})}^{\mathfrak{sp}_{2n_2}(\mathbb{R})} WF(\pi^\tau).$$

Fix a parity of p . Suppose that $\mathcal{O}_{\mathbf{D}}$ is not contained the image of the moment map m_2 associated with $(O(p, n_1 + n_2 + 1 - p), Sp_{2n_2}(\mathbb{R}))$ for $p \neq p_0$. Then $\mathcal{Q}(2n_1; p_0, n_1 + n_2 + 1 - p_0; 2n_2)(\pi) \neq 0$ and $\theta_s(2n_1; p_0, n_1 + n_2 + 1 - p_0)(\pi) \neq 0$.

Chapter 5

Construction of Unipotent Representations, Unitarity and Infinitesimal Character

Start with a real nilpotent orbit $\mathcal{O} = \mathcal{O}_{\mathbf{D}}$ in $\mathcal{U}(O(p, q))$ or $\mathcal{U}(Mp_{2n}(\mathbb{R}))$ (see Definition 12). First, construct an alternating sequence of nilpotent orbits of metaplectic groups and nilpotent orbits of orthogonal groups:

$$\mathcal{O}(d_1) = \mathcal{O}_{\mathbf{D}}, \mathcal{O}(d_1 - 1) = \mathcal{O}_{\mathbf{D}-1}, \dots, \mathcal{O}(1) = \mathcal{O}_{\mathbf{D}-d_1+1}.$$

Write

$$G(1) = G(\mathcal{O}(1)), G(2) = G(\mathcal{O}(2)), \dots, G(d_1) = G(\mathcal{O}(d_1)).$$

Clearly, $\mathcal{O}(k) \in \mathcal{U}$ for every k (see Cor. 2.2). Let \mathcal{O}' be the negative of \mathcal{O}

$$\mathcal{O}' = \{-x \mid x \in \mathcal{O}\}.$$

\mathcal{O}' may or may not equal to \mathcal{O} .

Definition 31 *We define $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ inductively.*

1. For $\mathcal{O}(1)$, let $\mathcal{N}(\mathcal{O}(1))$ be the set of all one-dimensional unitary characters of $G(1)$.
2. Suppose $\mathcal{N}(\mathcal{O}(k))$ is defined.

(a) For $G(k+1)$ orthogonal, define $\mathcal{N}_0(\mathcal{O}(k+1))$ to be

$$\{\pi(k+1) \otimes \eta \mid \pi(k+1) = \theta_s(G(k), G(k+1))(\pi), \pi \in \mathcal{N}(\mathcal{O}(k)), \eta \text{ a character of } O(p, q)\}.$$

$$\text{Let } \mathcal{N}(\mathcal{O}(k+1)) = \mathcal{N}_0(\mathcal{O}(k+1)) \cup \{\pi^* \mid \pi \in \mathcal{N}_0(\mathcal{O}(k+1)')\}.$$

(b) For $G(k+1)$ metaplectic, define $\mathcal{N}_0(\mathcal{O}(k+1))$ to be

$$\{\pi(k+1) \mid \pi(k+1) = \theta_s(G(k), G(k+1))(\pi), \pi \in \mathcal{N}(\mathcal{O}(k))\}.$$

$$\text{Let } \mathcal{N}_1(\mathcal{O}(k+1)) = \mathcal{N}_0(\mathcal{O}(k+1)) \cup \{\pi^\tau \mid \pi \in \mathcal{N}_0(\tau(\mathcal{O}(k+1)))\} \text{ and}$$

$$\mathcal{N}(\mathcal{O}(k+1)) = \mathcal{N}_1(\mathcal{O}(k+1)) \cup \{\pi^* \mid \pi \in \mathcal{N}_1(\mathcal{O}(k+1)')\}.$$

Any of the operations in the definition of \mathcal{N} , tensoring with a character, involution by τ or $*$, do not alter the infinitesimal character, the real part of the leading exponent, the unitarity, or the associated variety of the annihilator. In most cases, we will ignore these operations. The reader should be aware of the fact that $*$ and τ operations alter the wave front set by the involution \prime and τ . The author would like to thank the referee for suggesting this definition.

Theorem 5.1 *For $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}(G)$, the set $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ is not empty.*

For technical reasons, we will postpone the proof till the next chapter. In this chapter, we work under the assumption that $\mathcal{N}(\mathcal{O}_{\mathbf{D}}) \neq \emptyset$. We will show that

Theorem 5.2 *Let $\mathcal{O}_{\mathbf{D}}$ be in $\mathcal{U}(O(p, q))$ or $\mathcal{U}(Sp_{2n}(\mathbb{R}))$. Then any $\pi(k) \in \mathcal{N}(\mathcal{O}(k))$ is in $R_{ss}(G(k), \omega)$ (Def. 18) for the pair $(G(k), G(k+1))$. The set $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ is well-defined. The representations in $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ are all unitary.*

Thus $\mathcal{N}(\mathcal{O}_{\mathbf{D}}) \subset \Pi_u(G(\mathcal{O}_{\mathbf{D}}))$. We will further determine the infinitesimal character of $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$. It turns out that $\mathcal{I}(\pi)$ only depends on \mathbf{d} and consists of ρ -like segments.

5.1 Unitarity of $\mathcal{N}(\mathcal{O})$

Let $\mathcal{O}_{\mathbf{D}}$ be in $\mathcal{U}(O(p, q))$ or $\mathcal{U}(Sp_{2n}(\mathbb{R}))$. The group $G(\mathcal{O}_{\mathbf{D}})$ can be read off from D^+ , the number of positive boxes in \mathbf{D} , and from D^- , the number of negative boxes in \mathbf{D} . Construct an alternating sequence of symplectic signed Young diagrams and orthogonal signed Young diagrams:

$$\mathbf{D}(d_1) = \mathbf{D}, \mathbf{D}(d_1 - 1) = \mathbf{D}(d_1) - \mathbf{1}, \dots, \mathbf{D}(2) = \mathbf{D}(3) - \mathbf{1}, \mathbf{D}(1) = \mathbf{D}(2) - \mathbf{1}.$$

Let $\mathbf{D}(0) = 0$. Recall that $\|\mathbf{d}\|$ stands for the number of boxes in \mathbf{d} . We have the following.

Lemma 5.1.1 *Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$. Then*

1. *For every $k \in [1, d_1 - 1]$,*

$$D(k+1)^+ - D(k)^+ \geq D(k)^- - D(k-1)^-, \quad D(k+1)^- - D(k)^- \geq D(k)^+ - D(k-1)^+.$$

Thus,

$$D(k+1)^+ + D(k-1)^- \geq \|\mathbf{d}(k)\|, \quad D(k+1)^- + D(k-1)^+ \geq \|\mathbf{d}(k)\|.$$

2. *For every $k \in [1, d_1 - 1]$,*

$$D(k+1)^+ \geq D(k)^-, \quad D(k+1)^- \geq D(k)^+.$$

3. *For every $k \in [1, d_1 - 1]$,*

$$\|\mathbf{d}(k+1)\| + \|\mathbf{d}(k-1)\| \geq 2\|\mathbf{d}(k)\| + 2$$

if $\mathbf{D}(k)$ is symplectic;

4. for every $k \in [1, d_1 - 1]$,

$$\|\mathbf{d}(k+1)\| + \|\mathbf{d}(k-1)\| \geq 2\|\mathbf{d}(k)\|$$

if $\mathbf{D}(k)$ is orthogonal.

5. $\|\mathbf{d}(k+2)\| \equiv \|\mathbf{d}(k)\| \pmod{2}$ for all $k > 0$.

Proof: All the statements can be read off from \mathbf{D} . The number of $+$ boxes on the $k+1$ -th column must be greater than or equal to the number of $-$ boxes on the k -th column. We obtain (1). Adding all columns together, we obtain (2). (3) and (4) are consequences of (2.a) and (3.a) from Definition 12. (5) follows from (2.c) and (3.c) from Definition 12. \square

Proof of Theorem 5.2: Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$. Consider the sequences $\mathbf{D}(k)$ and $\mathcal{O}(k)$. We shall prove that $\pi(k)$ is unitary and lies in $R_{ss}(G(k), \omega)$ for the pair $(G(k), G(k+1))$. We will use induction.

1. For $k = 1$, by definition, every $\pi(1)$ in $\mathcal{N}(\mathcal{O}(1))$ is unitary. If $G(1)$ is metaplectic, then we have

- $D(1)^+ = D(1)^-$;
- $D(2)^+ \geq 2D(1)^-, D(2)^- \geq 2D(1)^+$, by Lemma 5.1.1 (1);
- $D(2)^+ + D(2)^- \geq 4D(1)^- + 2$, by Lemma 5.1.1 (3).

if $G(1)$ is orthogonal, then we have $D(2)^+ = D(2)^- \geq D(1)^- + D(1)^+$. It follows from the Definition 18, that every unitary representation is in $R_{ss}(G(1), \omega)$ with respect to the dual pair $(G(1), G(2))$.

2. Suppose that $\mathcal{N}(\mathcal{O}(k-1))$ is well-defined and $\mathcal{N}(\mathcal{O}(k-1)) \subseteq \Pi_u(G(k-1))$. Suppose that $\mathcal{N}(\mathcal{O}(k-1)) \subseteq R_{ss}(G(k-1), \omega)$ with respect to the pair $(G(k-1), G(k))$. This is our induction hypothesis.

If $G(k)$ is metaplectic, let

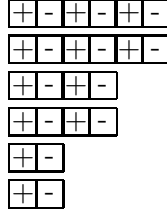
$$p' = D(k+1)^+, q' = D(k+1)^-, n = D(k)^+ = D(k)^-, p = D(k-1)^+, q = D(k-1)^-.$$

Then by Lemma 5.1.1, $p+q \equiv p'+q' \pmod{2}$, $p'+q'-2n \geq 2n-p-q+2$, $p' \geq n$ and $q' \geq n$. Clearly $\max(p', q') > n$. If $n \neq \min(p', q')$, then Theorem 3.6 holds and $\pi(k) = \theta_s(\pi(k-1))$ is well-defined and unitary. Furthermore, $\pi(k) \in R_{ss}(G(k), \omega)$ with respect to $(G(k), G(k+1))$. If $n = \min(p', q')$, without loss of generality, assume, $n = p'$. We shall prove that this case does not occur if $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$.

Since $D(k)^- = n = p' = D(k+1)^+$, the diagram $\mathbf{D}(k+1)$ must have its first column marked with $-$. By the structure of signed Young diagram, $\mathbf{D}(k)$ must have its first column marked with $+$. $\mathbf{D}(k)$ must have its second column marked with $-$. Furthermore, the first column of $\mathbf{D}(k)$ and the second column of $\mathbf{D}(k)$ must have the same length. Otherwise, there are at least two rows of length one in $\mathbf{D}(k)$. At least one \square would occur in the first column of $\mathbf{D}(k)$ according to our definition of $\mathcal{YD}_-(n, n)$. This contradicts the fact that the first column of $\mathbf{D}(k)$ is all marked with $+$. Thus,

$$\|\mathbf{d}(k)\| - \|\mathbf{d}(k-1)\| = \|\mathbf{d}(k-1)\| - \|\mathbf{d}(k-2)\|.$$

By the same argument, $\mathbf{D}(k)$ must be of the following shape and sign pattern:



In other words, the j -th column of $\mathbf{D}(k)$ must be marked by $(-1)^j$ and the $2j + 1$ -th column of $\mathbf{D}(k)$ must be of the same length as the $2j + 2$ -th column. This kind of $\mathbf{D}(k)$ is excluded from \mathcal{U} . So the case $n = \min(p', q')$ does not occur.

If $G(k)$ is orthogonal, let

$$n' = D(k+1)^+ = D(k+1)^-, \quad p = D(k)^+, q = D(k)^-, \quad n = D(k-1)^+ = D(k-1)^-.$$

Then by Lemma 5.1.1, we have

$$2n' - p - q \geq p + q - 2n > p + q - 2n - 2, \quad p \geq n, \quad q \geq n.$$

By the same argument as for metaplectic $G(k)$, $n \neq \min(p, q)$. By Theorem 3.7, $\pi(k) = \theta_s(\pi(k-1))$ is well-defined, unitary and $\pi(k) \in R_{ss}(G(k), \omega)$ with respect to the pair $(G(k), G(k+1))$.

3. It follows that $\mathcal{N}(\mathcal{O}(k)) \subseteq R_{ss}(G(k), \omega)$ for the pair $(G(k), G(k+1))$ and every $\pi(k) \in \mathcal{N}(\mathcal{O}(k))$ is unitary.

□

5.2 Infinitesimal Character

Theorem 5.3 (Thm 1.19, [PR96]) *Let $\mathcal{I}(\pi)$ be the infinitesimal character of π . Suppose that π occurs in the theta correspondence with respect to $(Sp_{2n}(\mathbb{R}), O(p, q))$.*

1. *Suppose $p + q < 2n$. Then $\mathcal{I}(\theta(p, q; 2n)(\pi))$ can be obtained by augmenting $\mathcal{I}(\pi)$ by*

$$\left(n - \frac{p+q}{2}, n - \frac{p+q}{2} - 1, \dots, 1 + \left[\frac{p+q}{2}\right] - \frac{p+q}{2}\right)$$

2. *Suppose $2n + 1 < p + q$. Then $\mathcal{I}(\theta(2n; p, q)(\pi))$ can be obtained by augmenting $\mathcal{I}(\pi)$ by*

$$\left(\frac{p+q}{2} - n - 1, \frac{p+q}{2} - n - 2, \dots, \frac{p+q}{2} - \left[\frac{p+q}{2}\right]\right)$$

3. *Suppose $p + q = 2n$ or $p + q = 2n + 1$. Then $\mathcal{I}(\theta(p, q; 2n)(\pi))$ is just $\mathcal{I}(\pi)$.*

Notation 25 *We define the orthogonal segment*

$$\mathcal{I}_+(m) = \overbrace{\left(\frac{m}{2} - 1, \frac{m}{2} - 2, \dots, \frac{m}{2} - \left[\frac{m}{2}\right]\right)}^{\left[\frac{m}{2}\right]}$$

and the symplectic segment

$$\mathcal{I}_-(m) = \overbrace{\left(\frac{m}{2}, \frac{m}{2} - 2, \dots, \frac{m}{2} + 1 - \left[\frac{m+1}{2}\right]\right)}^{\left[\frac{m+1}{2}\right]}.$$

The orthogonal segment $\mathcal{I}_+(m)$ is just $\rho(\mathfrak{o}(m, \mathbb{C}))$. For m even, $\mathcal{I}_-(m)$ is $\rho(\mathfrak{sp}_m(\mathbb{C}))$. For m odd, $\mathcal{I}_-(m)$ is the infinitesimal character of the oscillator representation of $Mp_{m+1}(\mathbb{R})$.

Theorem 5.4 *Let $\mathcal{O}_{\mathbf{D}}$ be in $\mathcal{U}(O(p, q))$ or $\mathcal{U}(Sp_{2n}(\mathbb{R}))$. Let $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$ and*

$$\mathbf{d}^t = (m_1 \geq m_2 \geq \dots \geq m_{d_1}).$$

If $G(\mathbf{D})$ is an orthogonal group, then

$$\mathcal{I}(\pi) = (\mathcal{I}_+(m_1), \mathcal{I}_-(m_2), \mathcal{I}_+(m_3), \mathcal{I}_-(m_4), \dots).$$

If $G(\mathbf{D})$ is a symplectic group, then

$$\mathcal{I}(\pi) = (\mathcal{I}_-(m_1), \mathcal{I}_+(m_2), \mathcal{I}_-(m_3), \mathcal{I}_+(m_4), \dots).$$

Proof: Let $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$. We prove this theorem by induction on d_1 . If $d_1 = 1$, by definition of $\mathcal{N}(\mathcal{O}(1))$, π restricted to the identity component of $G(\mathcal{O}_{\mathbf{D}})$ must contain a trivial constituent. Thus $\mathcal{I}(\pi) = \mathcal{I}_{\pm}(m_1)$. Suppose our assertion holds for any \mathbf{D} with $d_1 \leq k$. Let $d_1 = k + 1$.

If $G(\mathbf{D})$ is $O(p, q)$, then according to our definition of π , $\pi|_{SO_0(p, q)}$ must be equivalent to

$$\theta(G(\mathcal{O}_{\mathbf{D}-1}), G(\mathcal{O}_{\mathbf{D}}))(\sigma)|_{SO_0(p, q)}$$

or its contragredient for some $\sigma \in \mathcal{N}(\mathcal{O}_{\mathbf{D}-1})$. $\mathcal{I}(\pi)$ can be obtained by augmenting $\mathcal{I}(\sigma)$ with

$$\overbrace{\left(\frac{m_1}{2} - 1, \frac{m_1}{2} - 2, \dots, \frac{m_1}{2} - \left[\frac{m_1}{2}\right]\right)}^{\left[\frac{m_1}{2}\right]} = \mathcal{I}_+(m_1).$$

If $G(\mathbf{D})$ is $Mp_{2n}(\mathbb{R})$, observe that $\mathcal{I}(\pi^\tau) = \mathcal{I}(\pi)$. Then

$$\mathcal{I}(\pi) = \mathcal{I}(\theta(G(\mathcal{O}_{\mathbf{D}-1}), G(\mathcal{O}_{\mathbf{D}}))(\sigma))$$

for some $\sigma \in \mathcal{N}(\mathcal{O}_{\mathbf{D}-1})$. $\mathcal{I}(\pi)$ can be obtained by augmenting $\mathcal{I}(\sigma)$ with

$$\overbrace{\left(\frac{m_1}{2}, \frac{m_1}{2} - 1, \dots, \frac{m_1}{2} - \left[\frac{m_1 - 1}{2}\right]\right)}^{\left[\frac{m_1 + 1}{2}\right]} = \mathcal{I}_-(m_1).$$

□

Notation 26 *Let*

$$\mathbf{d}^t = (m_1 \geq m_2 \geq \dots \geq m_{d_1}).$$

For each orthogonal Young diagram \mathbf{d} , we define $\mathcal{I}_+(\mathbf{d})$ to be

$$(\mathcal{I}_+(m_1), \mathcal{I}_-(m_2), \mathcal{I}_+(m_3), \mathcal{I}_-(m_4), \dots).$$

For each symplectic Young diagram \mathbf{d} , we define $\mathcal{I}_-(\mathbf{d})$ to be

$$(\mathcal{I}_-(m_1), \mathcal{I}_+(m_2), \mathcal{I}_-(m_3), \mathcal{I}_+(m_4), \dots).$$

Theorem 5.4 states that $\mathcal{I}(\pi) = \mathcal{I}(\mathbf{d})$ for $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$. Thus $\mathcal{I}(\pi)$ only depends on \mathbf{d} , not on the signs of \mathbf{D} .

5.3 $\mathcal{I}_{\pm}(\mathbf{d})$ and \mathbf{d} : An algorithm

In this section, we will seek a direct way of obtaining $\mathcal{I}_{\pm}(\mathbf{d})$ from the Young diagram \mathbf{d} . We assume \mathbf{d}^t is very odd or very even. The Young diagram \mathbf{d} does not have to be pre-rigid or in \mathcal{U} . The algorithm can be described as follows.

1. First, cover the Young diagram \mathbf{d} by horizontal and vertical dominoes



as follows. We cover each column of the Young diagram \mathbf{d} by consecutive vertical dominoes, starting from the bottom row. If \mathbf{d}^t is very even, vertical dominoes covers \mathbf{d} completely. If \mathbf{d}^t is very odd, we have the first row left uncovered. We cover the first row of the Young diagram \mathbf{d} by consecutive horizontal dominoes, starting from the *right*. We may have the leftmost block uncovered. In that case, cover it with a horizontal domino anyway. We call this domino an open domino.

2. For each vertical domino DO , we can enumerate the number of dominoes above it. A horizontal domino will be counted as $\frac{1}{2}$ domino and a vertical domino will be counted as a full domino. Thus we obtain a number $n(DO)$.
3. If \mathbf{d}^t is odd, fill the open domino with no number and fill the other horizontal dominoes with $\frac{1}{2}$. For $\mathcal{I}_+(\mathbf{d})$, if a vertical domino DO is in the k -th column, we fill in DO with the number $n(DO) + \frac{1+(-1)^k}{2}$. For $\mathcal{I}_-(\mathbf{d})$, if a domino DO is in the k -th column, we fill in DO with the number $n(DO) + \frac{1-(-1)^k}{2}$. Extracting the numbers in all the dominoes, we obtain $\mathcal{I}(\mathbf{d})$.

Notation 27 *Let* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$. We define $\bar{\lambda}$ to be the reordering of λ such that

$$\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \dots \geq \bar{\lambda}_n.$$

5.4 Orderings and Reversal Phenomena

Recall from [CM] that complex nilpotent orbits of classical groups have a partial ordering \preceq .

Notation 28 Write $\mathbf{d}_1 \preceq \mathbf{d}_2$ if $\mathcal{O}_{\mathbf{d}_1}$ is contained in the closure of $\mathcal{O}_{\mathbf{d}_2}$.

The ordering \preceq for Young diagrams is different from the ordering \preceq for numerical sequences (see Notation 2). If one regards \mathbf{d}_1^t and \mathbf{d}_2^t as numerical sequences arranged in descending orders, then $\mathbf{d}_1 \preceq \mathbf{d}_2$ if and only if $\mathbf{d}_1^t \succeq \mathbf{d}_2^t$.

Theorem 5.5 (Reversal Phenomena) *Let $\mathcal{O}_{\mathbf{D}_1}$ and $\mathcal{O}_{\mathbf{D}_2}$ be nilpotent orbits of a fixed orthogonal group or symplectic group. Suppose that \mathbf{d}_1^t and \mathbf{d}_2^t are either both very even or both very odd. If $\mathbf{d}_1 \preceq \mathbf{d}_2$ then $\overline{\mathcal{I}_{\pm}(\mathbf{d}_1)} \succeq \overline{\mathcal{I}_{\pm}(\mathbf{d}_2)}$ (see Notation 27).*

Later we will show that $\mathcal{V}(\text{Ann } \pi) = \mathcal{O}_{\mathbf{d}}$ for every $\pi \in \mathcal{N}(\mathbf{D})$. This theorem suggests that, among the representations \mathcal{N} of a fixed group G , the smaller the infinitesimal character, the bigger is the associated variety and conversely. For example, the trivial representation has the smallest associated variety, namely $\{0\}$. Its infinitesimal character is given by $\rho(G)$, the greatest among all $\mathcal{I}(\pi)$ with $\pi \in \mathcal{N}$. Theorem 5.5 will be used in the Appendix to establish the estimates of $\mathcal{I}_{\pm}(\mathbf{d})$ given in Theorem 6.1 and Theorem 6.2.

We prove the reversal phenomena through a number of lemmas. **The conditions in the theorem are always assumed in these lemmas.**

Lemma 5.4.1 *If the Young diagram \mathbf{d}_1 can be obtained from the Young diagram \mathbf{d}_2 by moving one domino to its lower left without changing other dominoes, then $\mathbf{d}_1 \preceq \mathbf{d}_2$.*

We call this procedure a *move*.

Lemma 5.4.2 *If the Young diagram \mathbf{d}_1 can be obtained from \mathbf{d}_2 by a finite number of moves, then $\mathbf{d}_1 \preceq \mathbf{d}_2$ and vice versa.*

This lemma holds only under the assumption of the theorem. If \mathbf{d}_1^t is very even and \mathbf{d}_2^t is very odd, this lemma is no longer valid.

Lemma 5.4.3 *If Young diagram \mathbf{d}_1 can be obtained from the Young diagram \mathbf{d}_2 by a move, then $\overline{\mathcal{I}_{\pm}(\mathbf{d}_1)} \succeq \overline{\mathcal{I}_{\pm}(\mathbf{d}_2)}$.*

This Lemma is an easy consequence of our algorithm from \mathbf{d} to $\mathcal{I}_{\pm}(\mathbf{d})$.

By the above three lemmas, Theorem 5.5 is proved. \square

Corollary 5.1 *Let $\mathcal{O}_{\mathbf{D}_1}$ and $\mathcal{O}_{\mathbf{D}_2}$ be pre-rigid nilpotent orbits of a fixed orthogonal group or symplectic group (see Definition 11). Suppose that \mathbf{d}_1^t and \mathbf{d}_2^t are both very even or both very odd. Then $\overline{\mathcal{I}_{\pm}(\mathbf{d}_1)} \succeq \overline{\mathcal{I}_{\pm}(\mathbf{d}_2)}$ if and only if $\mathbf{d}_1 \preceq \mathbf{d}_2$.*

Proof: For pre-rigid orbits, one can reconstruct \mathbf{d}^t from $\mathcal{I}(\mathbf{d})$ in a unique way and the partial orderings are reversed in this reconstruction. \square

Chapter 6

Associated Varieties and Existence of $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$

Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}(G)$ and $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$. In this chapter, we study the matrix coefficients and the associated variety of π . We first bound $\overline{\mathcal{I}(\pi)}$ by a multiple of $\rho(G)$. By Theorem 1.2, we obtain a bound on the matrix coefficients of π . Then we apply Przebinda's theorem to show that

$$\mathcal{V}(\text{Ann } \pi) = cl(\mathcal{O}_{\mathbf{d}}).$$

With $\mathcal{V}(\text{Ann } \pi)$ in hand, we gain some control of $WF(\pi)$. Our next task is to prove Theorem 5.1. Let me briefly describe the proof of Theorem 5.1. Suppose $\pi(k) \in \mathcal{N}(\mathcal{O}(k))$. Then $\mathcal{I}(\pi(i))$ and $\mathcal{V}(\text{Ann } \pi(i))$ are all known for every $i \leq k$ based on our computation. If $G(\mathcal{O}(k))$ is orthogonal, then $\mathcal{N}(\mathcal{O}(k+1)) \neq \emptyset$ follows from Theorem 3.3. If $G(\mathcal{O}(k))$ is the metaplectic group, $\mathcal{N}(\mathcal{O}(k+1)) \neq \emptyset$ follows from Theorem 4.2 and Cor. 4.3. The details of the proof of 5.1 are given at the end of this chapter.

6.1 Estimates on Infinitesimal Characters: I

Let $G = Mp_{2n}(\mathbb{R})$ and $\rho(G) = (n, n-1, \dots, 1)$. Let $\mathcal{O}_{\mathbf{D}}$ be in $\mathcal{U}(G)$. Let

$$\mathbf{d}^t = (m_1 \geq m_2 \geq \dots \geq m_{d_1})$$

Then we have $2n = \|\mathbf{d}\|$.

Theorem 6.1 *Suppose $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}(Mp_{2n}(\mathbb{R}))$. Let*

$$\mathbf{d}^t = (m_1, m_2, \dots, m_{d_1}).$$

Then

$$\overline{\mathcal{I}_-(\mathbf{d})} \prec \frac{m_1 + 2}{2n} \rho(Mp_{2n}(\mathbb{R})).$$

A similar statement holds for $G = O(p, q)$.

Theorem 6.2 *Suppose $\mathcal{O}_{\mathbf{d}} \in \mathcal{U}(O(p, q))$. Let*

$$\mathbf{d}^t = (m_1, m_2, \dots, m_{d_1}).$$

Then

$$\overline{\mathcal{I}_+(\mathbf{d})} \prec \frac{m_1 + 2}{p + q - 2} \left(\frac{p + q}{2} - 1, \frac{p + q}{2} - 2, \dots, \frac{p + q}{2} - \left\lfloor \frac{p + q}{2} \right\rfloor \right).$$

In fact, we will have that

$$\begin{aligned} \overline{\mathcal{I}_-(\mathbf{d})} &\preceq \frac{m_1}{2n} \rho(Mp_{2n}(\mathbb{R})) \\ \overline{\mathcal{I}_+(\mathbf{d})} &\preceq \frac{m_1}{p + q - 2} \left(\frac{p + q}{2} - 1, \frac{p + q}{2} - 2, \dots, \frac{p + q}{2} - \left\lfloor \frac{p + q}{2} \right\rfloor \right). \end{aligned} \tag{6.1}$$

for $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$. These two statements are slightly stronger than those of Theorems 6.1 and 6.2. In the Appendix, we will give a proof for the first statement and Theorem 6.1. We skip the proof for Theorem 6.2.

6.2 Associated Varieties of $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$

Theorem 6.3 (Estimates on Leading Exponents) *Let $\mathcal{O}_{\mathbf{D}}$ be in \mathcal{U} . Let $\mathbf{d}^t = (m_1, m_2, \dots, m_{d_1})$. Let $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$.*

1. *If $G(\mathcal{O}_{\mathbf{D}})$ is $Mp_{2n}(\mathbb{R})$, then every leading exponent v of π satisfies*

$$\Re(v) \prec \left(\frac{m_1 + 2}{2n} - 1 \right) \rho(Mp_{2n}(\mathbb{R})).$$

2. *If $G(\mathcal{O}_{\mathbf{D}})$ is $O(p, q)$, then every leading exponent v of π satisfies*

$$\Re(v) \prec \left(\frac{m_1 + 2}{p + q - 2} - 1 \right) \rho(O(p, q)).$$

Proof: Part (1) is a direct consequence of Theorem 6.1 and Theorem 1.1. Part (2) is a direct consequence of Theorem 6.2 and Theorem 1.1. \square

Theorem 6.3 can be established independently in the framework of [Heq] without resorting to the infinitesimal character estimate in Theorems 6.1 and 6.2. However, the argument in [Heq] relies heavily on the estimates of twisted integral. It is more general but less intuitive. The estimate of the the leading exponents using infinitesimal character only works for the unitarily small representation, namely unitary representations with infinitesimal character sitting inside the convex hull of $\{w\rho \mid w \in W(\mathfrak{g}, \mathfrak{h})\}$. This method can be used to establish strongly semistable condition for unitarily small representations.

Theorem 6.4 *Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}(G)$. Let $\pi \in \mathcal{N}(\mathcal{O}_{\mathbf{D}})$. Then $\mathcal{V}(\text{Ann } \pi) = \text{cl}(\mathcal{O}_{\mathbf{d}})$.*

Proof: Notice that none of the operations in the definition of $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ changes $\mathcal{I}(\pi)$ or $\mathcal{V}(\text{Ann } \pi)$. Without loss of generality, we assume $\pi(k) = \theta_s(G(k-1), G(k))(\pi(k-1))$ and $\pi(d_1) = \pi$. Recall Theorem 3.9 and Theorem 3.10.

1. Condition (1) is automatic by the definition of $(G(k-1), G(k))$.
2. Conditions (2) from Theorem 3.9 and 3.10 are satisfied due to Theorem 6.3 and the definition of $(G(k-1), G(k))$.

3. $\pi(k-1)$ is not a discrete series representation. If $\pi(k-1)$ is in the discrete series and $G(k-1)$ is of rank r , then the infinitesimal character $\overline{\mathcal{I}(\pi(k-1))} \succeq (r-1, r-2, \dots, 1, 0)$ (for example, see Theorem 9.20 [KN]). By Theorem 6.1 and Inequalities (6.1) this is only possible for several large $\mathcal{I}(\pi(k-1))$'s. For these large $\mathcal{I}(\pi(k-1))$'s, the representation $\pi(k-1)$ can not be in the discrete series.
4. Condition (4) is assumed.
5. Condition (5), which basically say that $\theta_s(\pi(k-1))$ is unitary, is readily verified by Theorem 5.2.
6. Conditions (6) are checked in Lemma 3.6.1 and Lemma 3.6.2.

Thus Theorem 3.9 and Theorem 3.10 hold. Therefore, by Lemma 3.5.2 of DKP,

$$\mathcal{V}(\text{Ann } \pi(k)) = \Theta(G(k-1), G(k))(\mathcal{V}(\text{Ann } \pi(k-1))) = cl(\mathcal{O}(k)).$$

By induction we have $\mathcal{V}(\text{Ann } \pi) = cl(\mathcal{O}_{\mathbf{d}})$. \square

Corollary 6.1 *Fix an $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}(Mp_{2n}(\mathbb{R}))$. Let $O(p_0, q_0) = G(\mathcal{O}_{\mathbf{D}-1})$. Let $\pi \in \mathcal{N}_0(\mathcal{O}_{\mathbf{D}})$. Then there exists a nilpotent orbit $\mathcal{O}_{\mathbf{D}_0}$ in $WF(\pi)$ such that*

1. $\mathbf{d}_0 = \mathbf{d}$.
2. $G(\mathcal{O}_{\mathbf{D}_0-1}) = O(p_0, q_0)$.

Proof: Suppose that $\pi = \theta_s(p_0, q_0; 2n)(\sigma)$ with $\sigma \in \mathcal{N}(\mathcal{O}_{\mathbf{D}-1})$. Then $\mathcal{V}(\text{Ann } \sigma) = \mathcal{O}_{\mathbf{d}-1}$ and $\mathcal{V}(\text{Ann } \pi) = \mathcal{O}_{\mathbf{d}}$. By Theorem 3.11, $WF(\pi)$ must be in the image of m_2 with respect to $(O(p_0, q_0), Sp_{2n}(\mathbb{R}))$. Then Lemma 3.5.3 says that every orbit $\mathcal{O}_{\mathbf{S}}$ in $WF(\pi)$ satisfies $G(\mathcal{O}_{\mathbf{S}-1}) = O(s, t)$ with $s \leq p_0$ and $t \leq q_0$. There must be at least one $\mathcal{O}_{\mathbf{D}_0}$ in $WF(\pi)$ such that $G(\mathcal{O}_{\mathbf{D}_0-1}) = O(p_0, q_0)$. Otherwise, $\mathcal{V}(\text{Ann } \pi)$, which is the complexification of $WF(\pi)$, will be strictly smaller than $cl(\mathcal{O}_{\mathbf{d}})$ and will not be equal to $cl(\mathcal{O}_{\mathbf{D}})$. \square

6.3 Nonvanishing of $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$

Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$. Let $\pi \in \mathcal{N}_0(\mathcal{O}_{\mathbf{D}})$. We have proved that π must be unitary and $\mathcal{V}(\text{Ann } \pi) = \mathcal{O}_{\mathbf{d}}$. In other words, any representation in $\mathcal{N}(\mathcal{O})$ must be unitary and $\mathcal{V}(\text{Ann } \pi)$ must be equal to $cl(\mathcal{O}_{\mathbf{d}})$. In this section, we will prove that $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ is nonempty. In fact, we will prove that $\mathcal{N}_0(\mathcal{O}_{\mathbf{D}})$ is nonempty.

Theorem 6.5 *Let $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$. Let $O(p_0, q_0) = G(k-1)$, $Mp_{2n_1}(\mathbb{R}) = G(k)$ and $O(p, q) = G(k+1)$. Suppose*

$$\pi(k-1) \in \mathcal{N}(\mathcal{O}(k-1)), \text{ and } \pi(k) = \theta_s(p_0, q_0; 2n_1)(\pi(k-1)) \neq 0.$$

Then $\pi(k+1) = \theta_s(2n_1; p, q)(\pi(k)) \neq 0$.

Proof: Since $\mathcal{O}_{\mathbf{D}} \in \mathcal{U}$, write

$$\mathbf{d}(k+1)^t = (m_1 > m_2 \geq m_3 > m_4 \geq \dots).$$

Then

$$\mathbf{d}(k)^t = (m_2 \geq m_3 > m_4 \geq \dots), \quad \mathbf{d}(k-1)^t = (m_3 > m_4 \geq \dots),$$

$$p+q = 2n_1 + m_1, \quad 2n_1 = p_0 + q_0 + m_2.$$

By Theorem 6.4, $\mathcal{V}(\text{Ann } \pi(k)) = \mathcal{O}_{\mathbf{d}(k)}$ and $\mathcal{V}(\text{Ann } \pi(k-1)) = \mathcal{O}_{\mathbf{d}(k-1)}$. Let $n_2 = p + q - n_1 - 1$. Clearly $n_2 - n_1 = p + q - 2n_1 - 1 = m_1 - 1 > 0$ and $n_1 + n_2 + 1 \geq p + q$. By Theorem 4.1 and Theorem 5.2, it suffices to show that

$$\mathcal{Q}(2n_1; p, q; 2n_2)(\pi(k)) \neq 0.$$

1. By Corollary 6.1, there exists a nilpotent orbit $\mathcal{O}_{\mathbf{D}_0}$ of $Sp_{2n_1}(\mathbb{R})$ such that

- (a) $\mathbf{d}_0 = \mathbf{d}(k)$;
- (b) $\mathbf{D}_0 - 1$ contains p_0 $\boxed{+}$'s and q_0 $\boxed{-}$'s;
- (c) $\mathcal{O}_{\mathbf{D}_0} \subseteq WF(\pi(k))$.

2. By Theorem 4.13,

$$\text{Ind}_{\mathfrak{sp}_{2n_1}(\mathbb{R}) \oplus \mathfrak{gl}(n_2-n_1, \mathbb{R})}^{\mathfrak{sp}_{2n_2}(\mathbb{R})} \tau(\mathcal{O}_{\mathbf{D}_0}) \subseteq WF(\text{Ind}_{MP_{n_2-n_1, n_1}}^{Mp_{2n_2}(\mathbb{R})} \chi^\alpha \otimes \pi^\tau).$$

Observe that $n_2 - n_1 = m_1 - 1$, and $m_1 - 1 \geq m_2$. m_2 is the first entry of $\mathbf{d}_0^t = \mathbf{d}(k)^t$. By Theorem 2.8,

$$\text{Ind}_{\mathfrak{sp}_{2n_1}(\mathbb{R}) \oplus \mathfrak{gl}(n_2-n_1, \mathbb{R})}^{\mathfrak{sp}_{2n_2}(\mathbb{R})} \tau(\mathcal{O}_{\mathbf{D}_0})$$

contains $m_1 - m_2$ irreducible components

$$\cup_{j=0}^{m_1-m_2-1} \mathcal{O}_{\mathbf{S}^{(j)}}.$$

And $\mathbf{S}^{(j)}$ can be obtained by

- merging 2 columns of length $m_1 - 1$ to \mathbf{D}_0 from left;
- extending the signs of \mathbf{D}_0 for the first m_2 rows;
- assigning j $\boxed{-+}$'s and $m_1 - m_2 - 1 - j$ $\boxed{+-}$'s to rows of length 2.

3. Notice that for every j ,

$$(\mathbf{s}^{(j)})^t = \mathbf{s}^t = (m_1 - 1, m_1 - 1, m_2, m_3, \dots);$$

$$(\mathbf{s}^{(j)} - \mathbf{1})^t = (\mathbf{s} - \mathbf{1})^t = (m_1 - 1, m_2, m_3, \dots).$$

Consider the signature of $\mathbf{S}^{(j)} - \mathbf{1}$. For the first m_2 rows, there are $p_0 + m_2$ $\boxed{+}$'s and $q_0 + m_2$ $\boxed{-}$'s. For the last $m_1 - m_2 - 1$ rows of length 1, there are j $\boxed{+}$'s and $m_1 - m_2 - 1 - j$ $\boxed{-}$'s. Thus the signature of $\mathbf{S}^{(j)} - \mathbf{1}$ is

$$(p_0 + m_2 + j, q_0 + m_2 + m_1 - m_2 - 1 - j) = (p_0 + m_2 + j, q_0 + m_1 - 1 - j)$$

4. Fix the parity of p . Let $j = p - p_0 - m_2$. Here $p - p_0$ is equal to the number of \boxplus in the first two columns of $\mathbf{D}(k+1)$. It is greater or equal to the number of boxes in the second column of $\mathbf{D}(k+1)$. It follows that $j = p - p_0 - m_2 \geq 0$. Since $p + q = m_1 + m_2 + p_0 + q_0$, the signature of $\mathbf{S}^{(j)} - \mathbf{1}$ equals

$$(p_0 + m_2 + j, q_0 + m_1 - 1 - j) = (p, q_0 + m_1 - 1 + p_0 + m_2 - p) = (p, q - 1).$$

From Lemma 3.5.3, $\mathcal{O}_{\mathbf{S}^{(j)}}$ occurs in the image of m_2 associated with $(O(p, q), Sp_{2n_2}(\mathbb{R}))$. Also from Lemma 3.5.3, $\mathcal{O}_{\mathbf{S}^{(j)}}$ does not occurs in the image of m_2 associated with

$$(O(p - 2i, q + 2i), Sp_{2n_2}(\mathbb{R})), (O(p + 2i, q - 2i), Sp_{2n_2}(\mathbb{R}))$$

for any $i \neq 0$.

5. By Theorem 5.2, $\pi(k) \in R_{ss}(Mp_{2n_1}(\mathbb{R}), \omega(n_1 + n_2 + 1, 0; 2n_1))$. So the generic nonvanishing theorem and consequently Cor. 4.3 hold. Therefore,

$$\mathcal{Q}(2n_1; p, q; 2n_2)(\pi(k)) \neq 0.$$

and

$$\pi(k+1) = \theta_s(2n_1; p, q)(\pi(k)) \neq 0.$$

□

Proof of Theorem 5.1: It suffices to prove that $\mathcal{N}_0(\mathcal{O})$ is nonempty. We prove this by induction. The existence of $\pi(1)$ is trivial. The existence of $\mathcal{N}_0(\mathcal{O}(2))$ is established by Jian-Shu Li. Suppose that $k \geq 2$ and $\pi(k) \in \mathcal{N}(\mathcal{O}(k))$. If $G(k)$ orthogonal, from Theorem 3.3, $\mathcal{N}(\mathcal{O}(k+1))$ is nonempty. If $G(k)$ is metaplectic, from Theorem 6.5, $\pi(k+1)$ is not zero. Thus $\mathcal{N}_0(\mathcal{O}(k+1))$ is not empty. □

6.4 Perspectives

In this paper, we only treat $O(p, q)$ and $Mp_{2n}(\mathbb{R})$. Quantum induction for other classical group of type I can be defined similarly. Excluding the pair $(O(m, \mathbb{C}), Sp_{2n}(\mathbb{C}))$, the groups are connected and the metaplectic lift on these groups split. Thus the discussion on these groups should be considerably easier. We do not intend to work out the details here. Rather, we shall list some problems concerning quantum induction and unipotent representations for classical groups of type I. We will also point out problems in connection with representation theory and the theory of automorphic forms. The main question is whether quantum induction can supplement parabolic induction (including complementary series) and cohomological induction ([K-V]) to produce a complete classification of Π_u , Π_{auto} and Π_{rama} . In any case, this article should be viewed as a starting point for new development in that direction.

6.4.1 Wave Front Sets of $\mathcal{N}(\mathcal{O})$

Wave front sets under Howe's local theta correspondence are discussed in detail in [PR00]. There are still open questions which need to be answered. For the unipotent representations in $\mathcal{N}(\mathcal{O})$, we have showed that their associated varieties are the complexification of $d(\mathcal{O})$. Now one may

speculate that $cl(\mathcal{O})$ is the wave front set of $\pi \in \mathcal{N}(\mathcal{O})$. This is far from the case. In fact, there are irreducible representations in $\mathcal{N}(\mathcal{O})$ such that $WF(\pi) \neq cl(\mathcal{O})$ as demonstrated by Theorem 4.14. What should be true is that $\mathcal{O} \subseteq WF(\pi)$. We fall short of proving it.

Conjecture 3 *For every $\pi \in \mathcal{N}(\mathcal{O})$, $\mathcal{O} \subseteq WF(\pi)$.*

Let me give some hint on how one may proceed. Przebinda started a program attempting to obtain the Harish-Chandra character of $\theta_s(\pi)$ from the Harish-Chandra character of π . If his program works in our situation, we will have a confirmation of this conjecture. Let me point out a critical step.

Conjecture 4 *Let $\pi \in R_s(MG_1, MG_2)$. Then*

$$V(\theta_s(MG_1, MG_2)(\pi)) \cong [\text{Hom}_{\mathfrak{g}_1, \text{MK}_1}(\omega^c, \pi)]_{\text{MK}_2}.$$

Roughly speaking, this is saying that every MK_2 -finite π^c -valued MG_1 -equivariant distribution on ω can be obtained by integration over MG_1 as in [He00].

Notation 29 *Let $\mathcal{O}_{\mathbf{D}}$ be a nilpotent orbit of G . Let $\Pi_{\mathcal{O}_{\mathbf{D}}}(G)$ be the set of $\pi \in \Pi(G)$ such that $cl(\mathcal{O}_{\mathbf{D}})$ is of maximal dimension in $WF(\pi)$. Let $\Pi_{u, \mathcal{O}_{\mathbf{D}}}(G) = \Pi_{\mathcal{O}_{\mathbf{D}}}(G) \cap \Pi_u(G)$.*

According to a conjecture of Barbasch-Vogan ([BV85]),

$$\{\pi \in \Pi_{\mathcal{O}_{\mathbf{D}}}(G) \mid \|\mathcal{I}(\pi)\| = \min\{\|\mathcal{I}(\sigma)\| \mid \sigma \in \Pi_{\mathcal{O}_{\mathbf{D}}}(G)\} \subseteq \Pi_u(G).$$

Assume that Conjecture 3 is true. Then $\mathcal{N}(\mathcal{O}_{\mathbf{D}}) \subseteq \Pi_{u, \mathcal{O}_{\mathbf{D}}}(G)$. In fact, we conjecture that the representations in $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ are the ones that occur in the Barbasch-Vogan conjecture.

Conjecture 5 *Let G be either an orthogonal group or a metaplectic group. Let $\pi \in \Pi_{\mathcal{O}_{\mathbf{D}}}(G)$ with $\mathcal{O}_{\mathbf{d}}$ rigid. Let λ be the shortest infinitesimal character among all $\mathcal{I}(\pi)$ with $\pi \in \Pi_{\mathcal{O}_{\mathbf{D}}}(G)$. Then $\lambda = \mathcal{I}(\mathbf{d})$.*

This conjecture is false if $\mathcal{O}_{\mathbf{d}}$ fails to be rigid. Now assuming Conjecture 3, we formulate the following conjecture which implies Conjecture 5.

Conjecture 6 *Let \mathcal{O} be a rigid orbit in \mathcal{U} . Suppose that $G(1)$ is not $O(l)$ ($l \geq 2$). Then $\mathcal{N}(\mathcal{O}_{\mathbf{D}})$ exhausts all $\Pi_{u, \mathcal{O}_{\mathbf{D}}}(G)$ for $Mp_{2n}(\mathbb{R})$ and $O(p, q)$.*

Notice that the rigid orbits in \mathcal{U} include all special rigid orbits. For nonspecial rigid orbits of $O(p, q)$, by the works of Brylinski-Kostant ([BK]) and Huang-Li ([HL]), there might be unitary representations of the nontrivial covering of $SO_0(p, q)$ attached to them. Our construction does not cover their cases. However, for rigid nilpotent orbits in Conjecture 6, our construction should be exhaustive.

6.4.2 Wave Front Sets and Induction Functors

A critical observation in this paper is that $\mathbf{D} - 2$ remains to be a symplectic (orthogonal) Young diagram if \mathbf{D} is a symplectic (orthogonal) Young diagram. On the philosophical level, this paper

raises the question whether there always exists a functor

$$F(\mathcal{O}_{\mathbf{D}-2}, \mathcal{O}_{\mathbf{D}}) : \Pi_{u, \mathcal{O}_{\mathbf{D}-2}}(G) \rightarrow \Pi_{u, \mathcal{O}_{\mathbf{D}}}(G).$$

This question does not always have a positive answer. For example, there are $\mathcal{O}_{\mathbf{D}}$ with $\Pi_{u, \mathcal{O}_{\mathbf{D}}}(G) = \emptyset$.

Problem 1 *For all real forms of symplectic and orthogonal groups, construct functors*

$$F(\mathcal{O}_{\mathbf{D}-2}, \mathcal{O}_{\mathbf{D}}) : \Pi_{u, \mathcal{O}_{\mathbf{D}-2}}(G^0) \rightarrow \Pi_{u, \mathcal{O}_{\mathbf{D}}}(G)$$

whenever $\Pi_{u, \mathcal{O}_{\mathbf{D}}}(G) \neq \emptyset$. Here G^0 is a group attached to $\mathcal{O}_{\mathbf{D}-2}$ that is of the same type as G .

We call each one of these functors, an *induction functor*.

The second question is whether one can construct enough number of induction functors $\mathcal{F}(\mathcal{O}_{\mathbf{D}-2}, \mathcal{O}_{\mathbf{D}})$ such that all $\pi \in \Pi_{u, \mathcal{O}_{\mathbf{D}}}(G)$ can be obtained from $\Pi_{u, \mathcal{O}_{\mathbf{D}-2}}(G^0)$ through one of the induction functors.

Indeed, if $\mathcal{O}_{\mathbf{d}}$ is induced from $\mathcal{O}_{\mathbf{d}-2}$, then unitary parabolic induction functor is the right candidate. Under different assumptions, complementary parabolic induction and cohomological induction can also be applied ([B01] [K-V]). This situation has been studied intensively in the past.

The case that $\mathcal{O}_{\mathbf{D}}$ is not induced from $\mathcal{O}_{\mathbf{D}-2}$ is the main focus of this paper. It seems that the quantum induction \mathcal{Q} is the right choice. For example, one can study the following problem.

Problem 2 *Consider $Mp_{2n}(\mathbb{R})$. Let $\mathcal{O}_{\mathbf{D}}$ be a symplectic nilpotent orbit. Suppose that $\mathcal{O}_{\mathbf{D}}$ is not induced from $\mathcal{O}_{\mathbf{D}-2}$. Let $2n_2 = \|\mathbf{d}\|$, $(p, q) = ((\mathbf{D} - 1)^+, (\mathbf{D} - 1)^-)$ and $2n_1 = \|\mathbf{d} - 2\|$. Then is*

$$\mathcal{Q}(2n_1; p, q; 2n_2) : \Pi_{u, \mathcal{O}_{\mathbf{D}-2}}(G) \rightarrow \Pi_{u, \mathcal{O}_{\mathbf{D}}}(G)?$$

Notice that $\Pi_{u, \mathcal{O}_{\mathbf{D}}}(G)$ may be empty for some \mathbf{D} . Even in those cases, this problem seems to make sense, that is, $\mathcal{Q}(2n_1; p, q; 2n_2)$ will conjecturally map $\Pi_{u, \mathcal{O}_{\mathbf{D}-2}}(G)$ to zero. We shall come back to this problem in a subsequent paper.

6.4.3 Unitary Dual, Automorphic Dual and Ramanujan Dual

Let G be a classical group of type I. Let $\Pi_{herm}(G)$ be the set of irreducible Harish-Chandra modules with an invariant Hermitian structure. Let $\Pi_{temp}(G)$ be the set of irreducible Harish-Chandra modules with almost L^2 matrix coefficients. Recall that

$$\Pi(G) \supset \Pi_{herm}(G) \supset \Pi_u(G) \supset \Pi_{temp}(G).$$

The classification of the admissible dual $\Pi(G)$ is due to Langlands. Langlands proved that every $\pi \in \Pi(G)$ occurs as the unique quotient of certain induced representation from a tempered representation. The quotients are often known as Langlands quotients. The classification of Π_{temp} is due to Knapp-Zuckerman. The classification of the Hermitian dual is also known. See for example Theorem 16.6 and its remark [KN].

With these classifications in hand, classification of unitary dual can be translated as the determination of unitarity of the Langlands quotients. But this approach is of great mathematical complexity. It often involves the positivity of certain analytically defined intertwining operator. In [VO86] and [VO87], Vogan envisioned a more geometric approach based on Kirillov-Kostant's orbit philosophy. We believe that quantum induction should be sufficient to supplement the existing techniques to produce a classification of the unitary dual for classical groups of type I.

Problem 3 *Arrange a process to construct irreducible unitary representations of a classical group G using unitary parabolic induction, cohomological induction ($[K-V]$) and complementary series construction, combined with quantum induction. Can this process be arranged to produce all irreducible unitary representations?*

Recently, Salamanca-Riba and Vogan conjectured that all irreducible unitary representation can be constructed by cohomological method from an irreducible unitary representation of a smaller group except unitarily small representations [SV]. Notice that all the representations in $\mathcal{N}(\mathcal{O})$ are unitarily small and quantum inductions seem to keep the infinitesimal characters small. So investigations on Problem 3 may give some hope to a classification of the unitary dual of classical groups of type I, modulo the conjecture of Salamanca-Riba and Vogan. For coverings of classical groups, the situation is a little more delicate. But the same idea should apply.

Alternatively, one can follow the route of Problem 1 and construct the unitary dual by applying different induction functors based on the wave front sets.

Theta correspondences as formulated by Howe ([Ho79]) originated from the theory of theta series ([We65], [Si]). The history of theta series went back to the theta functions defined by Jacobi. Special cases of theta series were studied by many people in number theory. The upshot is that theta correspondences should produce automorphic representations of a bigger group from automorphic representations of a smaller group. This line of ideas was further explored by Rallis, Li in some quite general settings ([Ra87], [Li94]) and by many other people in special cases. We are tempted to make the following conjecture.

Conjecture 7 *All representations in $\mathcal{N}(\mathcal{O})$ are automorphic.*

In fact, we can even say more about $\mathcal{N}(\mathcal{O})$ and quantum induction. Recall that

$$\Pi_u(G) \supset \Pi_{auto}(G) \supset \Pi_{rama}(G).$$

We adopt the definitions from [BLS]. In principle, according to the conjectures of Ramanujan and Selberg ([Se], [Sa]), complementary series should not occur in $\Pi_{auto}(G)$ or $\Pi_{rama}(G)$. Assuming that,

Problem 4 *Can $\pi \in \Pi_{auto}(G)$ be constructed by unitary quantum induction, unitary cohomological induction and unitary parabolic induction?*

The argument that parabolic inductions should send Π_{auto} of a Levi subgroup to $\Pi_{auto}(G)$ was given in [BLS] with their H taking to be the parabolic subgroup rather than the Levi subgroup. Finally, for split classical groups, Barbasch showed that the irreducible spherical unitary representations can be constructed as a parabolically induced representation from a complementarily

induced representation tensored with a special unipotent representation ([B01]). The Ramanujan dual is simply the intersection of the automorphic dual with the spherical dual. Thus one may ask

Problem 5 *For G a split classical group of type I, can one construct all $\pi \in \Pi_{\text{rama}}(G)$ by unitary quantum induction and unitary parabolic induction?*

This exhaustion question is perhaps too difficult to answer. For $SL(2, \mathbb{R})$ where no quantum induction is involved, this is equivalent to the Ramanujan-Selberg conjecture.

Chapter 7

Appendix

7.1 Infinitesimal Characters for Mp

Recall that $\mathcal{I}_{\pm}(\mathbf{d})$ consists of an arithmetic sequence with multiplicities. For \mathbf{d}^t odd, the sequence ends with $\frac{1}{2}$; for \mathbf{d}^t even, the sequence ends with *zero* if $\mathbf{d} \neq [1]^{2n}$. In this chapter, we shall give a proof for Theorem 6.1. We start with a lemma.

Lemma 7.1.1 (Triviality)

1. If $\bar{\lambda} \prec \bar{\mu}$ and $\bar{\lambda}' \preceq \bar{\mu}'$, then $\bar{\lambda} + \bar{\lambda}' \prec \bar{\mu} + \bar{\mu}'$.
2. If $\bar{\lambda} \prec \bar{\mu}$ and $\bar{\lambda}' \prec \bar{\mu}'$, then $\overline{(\lambda, \lambda')} \prec \overline{(\mu, \mu')}$.
3. If $\lambda_l \leq \mu_l$ for all l , then $\bar{\lambda} \preceq \bar{\mu}$.
4. If $\lambda_l < \mu_l$ for all l , then $\bar{\lambda} \prec \bar{\mu}$.

Let $G = Sp_{2n}(\mathbb{R})$. Then

$$\rho(G) = (n, n-1, \dots, 1) = \mathcal{I}_-(2n).$$

Lemma 7.1.2 *Suppose that $m \equiv r \pmod{2}$. If $r \leq m$, then*

$$\overline{(\mathcal{I}_-(m), \mathcal{I}_+(r))} \preceq \frac{m}{m+r} \mathcal{I}_-(m+r).$$

Proof: Write

$$B = \overline{(\mathcal{I}_-(m), \mathcal{I}_+(r))}, \quad A = \frac{m}{m+r} \mathcal{I}_-(m+r).$$

We prove our lemma by induction on $\frac{m-r}{2}$. If $\frac{m-r}{2} = 0$, $m = r$. Then

$$A = \left(\frac{m}{2}, \frac{m-1}{2}, \frac{m-2}{2}, \frac{m-3}{2}, \dots, \frac{2}{2}, \frac{1}{2} \right),$$

$$B = \left(\frac{m}{2}, \frac{m-2}{2}, \frac{m-2}{2}, \frac{m-4}{2}, \frac{m-4}{2}, \dots \right).$$

B ends with $(1, 0)$ if m is even and $(\frac{1}{2}, \frac{1}{2})$ if m is odd. Obviously, $A_k \geq B_k$. So $B \preceq A$. Assume that $m - 2 \geq r$ and

$$\overline{(\mathcal{I}_-(m-2), \mathcal{I}_+(r))} \preceq \frac{m-2}{m-2+r} \mathcal{I}_-(m+r-2).$$

Notice that $\frac{m-2}{m-2+r} \leq \frac{m}{m+r}$. We have

$$\begin{aligned} B = \overline{(\mathcal{I}_-(m), \mathcal{I}_+(r))} &= \overline{\left(\frac{m}{2}, \mathcal{I}_-(m-2), \mathcal{I}_+(r)\right)} \\ &\preceq \overline{\left(\frac{m}{2}, \frac{m-2}{m-2+r} \mathcal{I}_-(m+r-2)\right)} \\ &\preceq \overline{\left(\frac{m}{m+r} \frac{m+r}{2}, \frac{m}{m+r} \mathcal{I}_-(m+r-2)\right)} \\ &= \frac{m}{m+r} \overline{\mathcal{I}_-(m+r)} = A. \end{aligned} \tag{7.1}$$

□

Lemma 7.1.3 *Consider a partition of $2n$*

$$\mathbf{d}^t = \overbrace{(m, m, m-2, m-2, \dots, m-2j+2, m-2j+2, m_0, r)}^{2j}$$

with $m \equiv m_0 \equiv r \pmod{2}$ and $m - 2j \geq m_0 \geq r \geq 0$. Then

$$\overline{\mathcal{I}_-(\mathbf{d})} \preceq \frac{m}{2n} \mathcal{I}_-(2n).$$

Proof: Clearly, $2n = m_0 + r + (2m - 2j + 2)j$. If $j = 0$, $\mathbf{d}^t = (m_0, r)$. We have $\mathcal{I}_-(\mathbf{d}) \preceq \frac{m_0}{m_0+r} \mathcal{I}_-(m_0+r)$ by Lemma 7.1.2. Assume $j \geq 1$.

Use induction on m . When $m = 2$, $B(2, 0, 0) = \mathcal{I}_-(2) \preceq I_-(2)$. So our lemma holds for $m = 2$. Assume our lemma holds for $m - 1$.

Suppose that $r \geq 1$. The case $r = 0$ can be treated similarly. Obviously, $2n < 2m(j + 1)$. So

$$m(2n - 2j - 2) = 2mn - m(2j + 2) \leq 2nm - 2n = 2n(m - 1).$$

It follows that $\frac{m}{2n} \leq \frac{m-1}{2n-2j-2}$. Consider

$$\mathbf{e}^t = \overbrace{(m-1, m-1, m-3, m-3, \dots, m-2j+1, m-2j+1, m_0-1, r-1)}^{2j}.$$

Suppose that m is even. Then $\overline{\mathcal{I}_-(\mathbf{d})} = \overline{(\mathcal{I}_-(\mathbf{e}) + \frac{1}{2}, \overbrace{0, 0, \dots, 0}^{j+1})}$. By induction hypothesis,

$$\overline{\mathcal{I}_-(\mathbf{e})} \preceq \frac{m-1}{2n-2j-2} \mathcal{I}_-(2n-2j-2).$$

Then we have

$$\begin{aligned}
\overline{\mathcal{I}_-(\mathbf{e})} + \frac{1}{2} &\leq \frac{m-1}{2n-2j-2} \mathcal{I}_-(2n-2j-2) + \frac{1}{2} \\
&= \frac{\mathbf{m}-1}{2} - \frac{m-1}{2n-2j-2} (0, 1, \dots, n-j-2) + \frac{1}{2} \\
&\leq \frac{\mathbf{m}}{2} - \frac{m}{2n} (0, 1, \dots, n-j-2) \\
&= \frac{m}{2n} \mathbf{n} - \frac{m}{2n} (0, 1, \dots, n-j-2) \\
&= \frac{m}{2n} \overbrace{(n, n-1, \dots, j+2)}^{n-j-1}.
\end{aligned} \tag{7.2}$$

We obtain $\overline{\mathcal{I}_-(\mathbf{d})} \leq \frac{m}{2n} \mathcal{I}_-(2n)$.

Suppose m is odd. Then $\overline{\mathcal{I}_-(\mathbf{d})} = \overline{\mathcal{I}_-(\mathbf{e})} + \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}$. We will again have

$$\overline{\mathcal{I}_-(\mathbf{e})} + \frac{1}{2} \leq \frac{m}{2n} \overbrace{(n, n-1, \dots, j+2)}^{n-j-1}.$$

As l changes from $n-j$ to n , $\frac{m}{2n} \mathcal{I}_-(2n)_l - \frac{1}{2}$ decreases from positive to negative. So

$$\frac{m}{2n} \sum_{i=1}^l \mathcal{I}_-(2n)_i - \sum_{i=1}^l \overline{\mathcal{I}_-(\mathbf{d})}_i$$

increases and then decreases. It suffices to show that

$$\frac{m}{2n} \sum_{i=1}^n \mathcal{I}_-(2n)_i - \sum_{i=1}^n \overline{\mathcal{I}_-(\mathbf{d})}_i \geq 0$$

Notice that

$$\sum_i (\mathcal{I}_-(m-2s))_i = \frac{(m-2s+1)^2}{8}, \quad \sum_i (\mathcal{I}_+(m-2s))_i = \frac{(m-2s-1)^2}{8}$$

We obtain

$$\begin{aligned}
& \frac{m}{2n} \sum_{i=1}^n \mathcal{I}_-(2n)_i - \sum_{i=1}^n \overline{\mathcal{I}_-(\mathbf{d})}_i \\
&= \frac{m}{2n} \frac{(n+1)n}{2} - \left[\sum_{s=0}^{j-1} \frac{(m-2s+1)^2}{8} + \frac{(m-2s-1)^2}{8} \right] - \frac{(m_0+1)^2}{8} - \frac{(r-1)^2}{8} \\
&= \frac{1}{8} [m(2n+2) - \left(\sum_{s=0}^{j-1} 2(m-2s)^2 + 2 \right) - (m_0+1)^2 - (r-1)^2] \\
&= \frac{1}{8} \left\{ m(m_0+1+r-1+2 + \sum_{s=0}^{j-1} 2(m-2s)) - (m_0+1)^2 - (r-1)^2 - 2j - \sum_{s=0}^{j-1} 2(m-2s)^2 \right\} \\
&= \frac{1}{8} \left\{ (m_0+1)(m-m_0-1) + (r-1)(m-r+1) + 2(m-j) + 2 \sum_{s=0}^{j-1} (m-2s)(m-m+2s) \right\} \\
&\geq 0
\end{aligned} \tag{7.3}$$

Therefore $\overline{\mathcal{I}_-(\mathbf{d})} \preceq \frac{m}{2n} \mathcal{I}_-(2n)$. By induction, $\overline{\mathcal{I}_-(\mathbf{d})} \preceq \frac{m}{2n} \mathcal{I}_-(2n)$ for all m . \square

Proof of Theorem 6.1: Fix n first. Let $\mathcal{O}_{\mathbf{d}} \in \mathcal{U}(Mp_{2n}(\mathbb{R}))$. Fix the number of row $m_1 = m$. Consider

$$\mathbf{d}_0^t = (m, m, m-2, m-2, \dots, m-2j+2, m-2j+2, m_0, r) \quad (j \geq 0) \tag{7.4}$$

with

$$0 \leq r \leq m_0 \leq m-2j-2, \quad m \equiv m_0 \equiv r \pmod{2}. \tag{7.5}$$

Here $j=0$ means that $\mathbf{d}_0^t = (m_0, r)$. By Lemma 7.1.3,

$$\overline{\mathcal{I}_-(\mathbf{d}_0)} \preceq \frac{m}{2n} \mathcal{I}_-(2n).$$

Notice that $\mathcal{O}_{\mathbf{d}_0}$ is the minimal orbit in $\mathcal{U}(Mp_{2n}(\mathbb{R}))$ with a fixed m . Therefore, by Theorem 5.5, for any \mathbf{d} with m rows, $\overline{\mathcal{I}_-(\mathbf{d})} \preceq \overline{\mathcal{I}_-(\mathbf{d}_0)}$. We obtain for any $\mathcal{O}_{\mathbf{d}} \in \mathcal{U}(Mp_{2n}(\mathbb{R}))$,

$$\overline{\mathcal{I}_-(\mathbf{d})} \preceq \frac{m_1}{2n} \rho(G) \prec \frac{m_1+2}{2n} \rho(G).$$

\square

7.2 Integration of Functions with Values in Hilbert Spaces

In this section, we shall give a proof of Lemma 4.4.5. By changing coordinates, it suffices to prove the following Lemma.

Lemma 7.2.1 *Let \mathcal{H} be a Hilbert space and Φ be a \mathcal{H} -valued continuous function on \mathbb{R}^n . If*

$$\int \int_{x,y \in \mathbb{R}^n} |(\Phi(x), \Phi(y))| dx dy < \infty,$$

then $\int_x \Phi(x) dx \in \mathcal{H}$ is well-defined as the unique limit of $u_{K_i} = \int_{K_i} \Phi(x) dx$ for any increasing sequence of compact sets $K_i \rightarrow \mathbb{R}^n$. Furthermore,

$$\left(\int_{x \in \mathbb{R}^n} \Phi(x) dx, \int_{y \in \mathbb{R}^n} \Phi(y) dy \right) = \int_{x, y \in \mathbb{R}^n} (\Phi(x), \Phi(y)) dx dy$$

Proof: Let $\Phi(x)$ be a continuous function from \mathbb{R}^n to \mathcal{H} such that

$$\int_{x, y \in \mathbb{R}^n} |(\Phi(x), \Phi(y))| dx dy < \infty.$$

Let K_i be an increasing sequence of compact sets $K_i \rightarrow \mathbb{R}^n$. Then as $i \rightarrow \infty$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n - K_i \times K_i} |(\Phi(x), \Phi(y))| dx dy \rightarrow 0.$$

Define $u_{K_i} = \int_{K_i} \Phi(x) dx$. Then for any $m > i$, we have

$$\begin{aligned} (u_{K_m} - u_{K_i}, u_{K_m} - u_{K_i}) &= \left(\int_{K_m - K_i} \Phi(x) dx, \int_{K_m - K_i} \Phi(x) dx \right) \\ &= \int_{(K_m - K_i) \times (K_m - K_i)} (\Phi(x), \Phi(y)) dx dy \\ &\leq \int_{(\mathbb{R}^n - K_i) \times (\mathbb{R}^n - K_i)} |(\Phi(x), \Phi(y))| dx dy \\ &\leq \int_{\mathbb{R}^n \times \mathbb{R}^n - K_i \times K_i} |(\Phi(x), \Phi(y))| dx dy \rightarrow 0. \end{aligned} \tag{7.6}$$

Therefore $\{u_{K_i}\}$ is a Cauchy sequence and $\lim u_{K_i}$ exists. So $u = \int \Phi(x) dx \in \mathcal{H}$ is well-defined. Furthermore, $\|u - u_{K_i}\| \rightarrow 0$. By a similar argument, we have

$$(u, u) = \int (\Phi(x), \Phi(y)) dx dy.$$

We can generalize Lemma 7.2.1 as follows.

Theorem 7.1 *Let H be a Hilbert space and X a locally compact space. Let μ be a σ -finite regular Borel measure. Let Φ be a \mathcal{H} -valued continuous function on X . If*

$$\int \int_{x, y \in X} |(\Phi(x), \Phi(y))| d\mu(x) d\mu(y) < \infty,$$

then $\int_x \Phi(x) d\mu(x) \in \mathcal{H}$ is well-defined as the unique limit of $\int_{K_i} \Phi(x) d\mu(x)$ for any increasing sequence K_i with $\mu(X - \cup K_i) = 0$. Furthermore,

$$\left(\int_{x \in X} \Phi(x) d\mu(x), \int_{y \in X} \Phi(y) d\mu(y) \right) = \int_{x, y \in X} (\Phi(x), \Phi(y)) d\mu(x) d\mu(y)$$

The convergence of $\int_x \Phi(x) d\mu(x)$ is neither a strong convergence nor a weak convergence. We refer to it as *convergence on compact sets*.

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