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Unitary representations and theta correspondence for type I classical groups

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In remembrance of my father, Decai He, January 8, 1944–September 21, 2002

Abstract

In this paper, we discuss the positivity of the Hermitian form $(,)_\pi$ introduced by Li in Invent. Math. 27 (1989) 237–255. Let (G_1, G_2) be a type I dual pair with G_1 the smaller group. Let π be an irreducible unitary representation in the semistable range of $\theta(MG_1, MG_2)$ (see Communications in Contemporary Mathematics, Vol. 2, 2000, pp. 255–283). We prove that the invariant Hermitian form $(,)_\pi$ is positive semidefinite under certain restrictions on the size of G_2 and a mild growth condition on the matrix coefficients of π. Therefore, if $(,)_\pi$ does not vanish, $\theta(MG_1, MG_2)(\pi)$ is unitary.

Theta correspondence over $\mathbb{R}$ was established by Howe in (J. Amer. Math. Soc. 2 (1989) 535–552). Li showed that theta correspondence preserves unitarity for dual pairs in *stable range*. Our results generalize the results of Li for type I classical groups (Invent. Math. 27 (1989) 237). The main result in this paper can be used to construct irreducible unitary representations of classical groups of type I.

1. Introduction

Let (G_1, G_2) be an irreducible reductive dual pair of type I in Sp (see [7,11]). The dual pairs in this paper will be considered as ordered. For example, the pair $(O(p,q), Sp_{2n}(\mathbb{R}))$ is considered different from the pair $(Sp_{2n}(\mathbb{R}), O(p,q))$. We will in general assume that the size of $G_1(V_1)$ is less or equal to the size of $G_2(V_2)$. In other words, $dim_D(V_1) \leqslant dim_D(V_2)$. Let Mp be the unique double covering of Sp. Let $\{1, \varepsilon\}$

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be the preimage of the identity element in Sp. For a subgroup H of Sp, let MH be the preimage of H under the double covering. Whenever we use the notation MH, H is considered as a subgroup of certain Sp. Let $\omega(MG_1, MG_2)$ be a Schrödinger model of the oscillator representation of Mp. The Harish-Chandra module of $\omega(MG_1, MG_2)$ consists of polynomials multiplied by the Gaussian function.

Since the pair (G_1, G_2) is ordered, we use $\theta(MG_1, MG_2)$ to denote the theta correspondence from $\mathscr{R}(MG_1, \omega(MG_1, MG_2))$ to $\mathscr{R}(MG_2, \omega(MG_1, MG_2)$ (see [8]). In this paper, whenever we talk about "K-finite matrix coefficients" or "K-finite vectors" of a representation π of a real reductive group G, "K" is used as a generic term for a specified maximal compact subgroup of G. Throughout this paper, we will mainly work within the category of Harish-Chandra modules. A representation of a real reductive group refers to an admissible representation unless stated otherwise. Throughout this paper, a vector in an admissible representation π means that v is in the Harish-Chandra module of π which shall be evident within the context.

Let V be a vector space of finite dimension. Let W be a subspace of V. A direct complement of W in V is a subspace U such that

$$U \oplus W = V.$$

Now suppose V is equipped with a nondegenerate sesquilinear form $(,)$. The orthogonal complement of W in V consists of

$$\{v \in V \mid (v, w) = 0 \ \forall w \in W\}.$$

It is denoted by $W^{\perp}$.

Let π be an irreducible admissible representation of MG_1 such that $\pi(\varepsilon) = -1$. π is said to be in the semistable range of $\theta(MG_1, MG_2)$ if the function

$$(\omega(MG_1, MG_2)(\tilde{g}_1)\phi, \psi)(u, \pi(\tilde{g}_1)v) \quad (\forall \phi, \psi \in \omega(MG_1, MG_2); \forall u, v \in \pi)$$

is in $L^{1-\delta}(MG_1)$ for all sufficiently small nonnegative δ (i.e., $\delta \in [0, c]$ for some $c > 0$). We denote the semistable range by $\mathscr{R}_s(MG_1, \omega(MG_1, MG_2))$. Suppose from now on that π is in $\mathscr{R}_s(MG_1, \omega(MG_1, MG_2))$. For each $\phi, \psi \in \omega(MG_1, MG_2)$ and $u, v \in \pi$, we define an averaging integral

$$\int_{MG_1} (\omega(MG_1, MG_2)(\tilde{g}_1)\phi, \psi)(u, \pi(\tilde{g}_1)v)\, d\tilde{g}_1$$

and denote it by $(\phi \otimes v, \psi \otimes u)_\pi$. Thus, $(,)_\pi$ becomes a real bilinear form on $\omega(MG_1, MG_2) \otimes \pi$. Our definition of $(,)_\pi$ differs slightly from the original definition of Li in [11]. Let $\tilde{g}_2 \in MG_2$ act on $\omega(MG_1, MG_2) \otimes \pi$ by $\omega(MG_1, MG_2)(\tilde{g}_2) \otimes Id$. In [5], we show that if $(,)_\pi \neq 0$ then $(,)_\pi$ descends into a sesquilinear form on the K-finite dual representation of $\theta(MG_1, MG_2)(\pi)$. For π unitary, $(,)_\pi$ is an invariant Hermitian form on $\theta(MG_1, MG_2)(\pi)$.

For π unitary, a conjecture of Li says that $(,)_\pi$ will always be positive semidefinite (see [11]). If Li's conjecture holds and $(,)_\pi \neq 0$, then $\theta(MG_1, MG_2)(\pi)$ is unitary. In this paper, we will prove that $(,)_\pi$ is positive semidefinite under certain restrictions. This partly confirms the conjecture of Li. The nonvanishing of certain $(,)_\pi$ is proved in [4] and in [6].

We adopt the notations from [5,11,12]. Let $(G_1(V_1), G_2(V_2))$ be a dual pair of type I. Suppose $V_2 = V_2^0 \oplus V_2'$ such that

(1) $(,)_2$ restricted onto V_2^0 is nondegenerate;
(2) $V_2' = (V_2^0)^\perp$;
(3) V_2^0 is a direct sum of two isotropic subspaces:

$$V_2^0 = X_2^0 \oplus Y_2^0.$$

Obviously, V_2^0 will always be of even dimension. Let $X^0 = Hom_D(V_1, X_2^0)$. The oscillator representation $\omega(MG_1(V_1), MG_2(V_2^0))$ can be modeled on $L^2(X^0)$. The action of MG_1 on $L^2(X^0)$ is equivalent to the regular action of G_1 on $L^2(X^0)$ tensoring with a unitary character ξ of MG_1. The generic orbits of G_1 on X^0 are classified abstractly in Theorems 4.1 and 4.2.

Later in this paper, the oscillator representation $\omega(MG_1(V_1), MG_2(V_2^0))$ is denoted as $\omega(M^0G_1, M^0G_2^0)$ to indicate the fact that $MG_1(V_1)$ in $(MG_1(V_1), MG(V_2))$ might be different from $MG_1(V_1)$ in $(MG_1(V_1), MG_2(V_2^0))$. For the same reason, the oscillator representation $\omega(MG_1(V_1), MG_2(V_2'))$ is denoted by $\omega(M'G_1, M'G_2')$.

Theorem 1.1 (Main Theorem). *Let (G_1, G_2) be a dual pair. Let $\Xi(g)$ be Harish-Chandra's basic spherical function of G_1. Suppose π is an irreducible unitary representation of MG_1 in the semistable range of $\mathcal{O}(MG_1, MG_2)$. Suppose*

1. *for any $x, y \in G_1$, the function $\Xi(xgy)$ is integrable on G_{1_ϕ} for every generic $\phi \in Hom_D(V_1, X_2^0)$ (see Definition 4.1);*
2. *the tensor product $\pi_0 = \omega(M'G_1, M'G_2') \otimes \pi \otimes \bar{\xi}$, considered as a representation of G_1, is weakly contained in $L^2(G_1)$ (see [14]).*

Then $(,)_\pi$ is positive semidefinite. If $(,)_\pi$ does not vanish, then $\theta(MG_1, MG_2)(\pi)$ is unitary.

Remarks.

1. $\omega(M'G_1, M'G_2'))$, π and $\bar{\xi}$ are all projective representations of G_1. The fact that π_0 becomes a unitary representation of G_1 is explained in Part II.
2. The first condition roughly requires that

$$dim_D(X_2^0) > \frac{dim_D(V_1)}{2}.$$

The precise statement depends on the groups involved. The function $\Xi(g)|_{G_{1_\phi}}$ is in $L^1(G_{1_\phi})$ implies that $\Xi(xgy)|_{G_{1_\phi}}$ is in $L^1(G_{1_\phi})$ for any $x, y \in G_1$ and vice versa. In fact, $\Xi(g)$ is bounded by a multiple of $\Xi(xgy)$ and vice versa. Furthermore, for any compact subset Y of G_1, there exists a constant C, such that for any $x, y \in Y$,

$$\Xi(xgy) \leqslant C\Xi(g) \quad (g \in G_1). \tag{1}$$

One can prove this by studying the compact picture of the basic spherical principle series representation (see [9, Chapter VII.1]). Since this remark may have already been in the literature and a proof will incur a new set of notations, we choose not to give the proof.

3. The growth of matrix coefficients of $\omega(MG_1(V_1), MG_2(V_2'))$ can be determined easily. Thus, the second condition can be converted into a growth condition on the matrix coefficients of π (see Corollary 5.1).
4. Conditions 1 and 2 imply that π is in $\mathscr{R}_s(MG_1, \omega(MG_1, MG_2))$. Therefore, $(,)_\pi$ is an invariant Hermitian form on $\theta(MG_1, MG_2)(\pi)$. The unitarity of $\theta(MG_1, MG_2)(\pi)$ follows since $(,)_\pi$ is positive semidefinite.

This paper is organized as follows. In Part I, we prove some positivity theorems in the sense of Godement [3]. In Part II, we construct the dual pair (G_1, G_2) in terms of homomorphisms and study various subgroups and liftings concerning the tensor decomposition

$$\omega(MG_1, MG_2) \cong \omega(M^0G_1, M^0G_2^0) \otimes \omega(M'G_1, M'G_2').$$

This tensor decomposition is termed as the mixed model in [11]. The interpretation of this tensor product is not completely trivial since MG_1, M^0G_1 and $M'G_1$ may be different double coverings of G_1. In Part II, we essentially redo part of Section 4 in [11] just to be safe. In Part III, we study $(\omega(M^0G_1, M^0G_2^0), L^2(X^0))$ and classify all the generic G_1-orbits in X^0. This enables us to reduce our averaging integral $(\phi \otimes u, \phi \otimes u)_\pi$ to an integral on G_1-orbits:

$$\int_{\mathcal{O} \in G_1 \backslash X^0} \int_{G_1} \int_{x \in \mathcal{O}} \phi(g^{-1}x)\overline{\phi(x)}(u, \pi_0(g)u)\, dx\, dg\, d[\mathcal{O}].$$

We study each generic orbit integral

$$\int_{G_1} \int_{x \in \mathcal{O}} \phi(g^{-1}x)\overline{\phi(x)}(u, \pi_0(g)u)\, dx\, dg$$

in full generality and convert it into an integral on the isotropic group G_{1x}

$$\int_{G_{1x}} (\pi_0(g)u_0, u_0)\, dg.$$

Next, we apply the positivity theorem (Theorem 2.3) to show that this integral is nonnegative. Thus $(,)_\pi$ is positive semidefinite. Finally, we take the pair

$(\mathcal{O}(p,q), Sp_{2n}(\mathbb{R}))$ as an example and state our main theorem in terms of leading exponents of π.

2. Part I: positivity theorems

Let G be a real reductive Lie group. Let K be a maximal compact subgroup of G. For any unitary representation (π, H) of G and any $\sigma \in \hat{K}$, let H_σ be the K-isotypic subspace of H. Let $d(\sigma)$ be the dimension of σ. Let S be a subset of $\hat{K}$. We denote

$$\bigoplus_{\sigma \in S} H_\sigma$$

by $H(S)$.

2.1. A generic theorem

Theorem 2.1. *Let G be a real reductive Lie group. Let K be a maximal compact subgroup of G. Let $\Xi(g)$ be Harish-Chandra's basic spherical function with respect to K. Let H be a closed unimodular Lie subgroup of G. Suppose that $\Xi(g)|_H$ is in $L^1(H)$. Let ϕ be a positive definite function in $L^{2+\varepsilon}(G)(S)$ for some finite subset S of $\hat{K}$ and any $\varepsilon > 0$. Then $\int_H \phi(h)\, dh \geqslant 0$.*

Here $L^{2+\varepsilon}(G)(S)$ is defined with respect to the left regular action of G.

Proof. By the GNS construction, we construct a unitary representation $(\sigma, \mathscr{H})$ such that $\phi(g) = (\sigma(g)\eta, \eta)$ for some cyclic vector η in $\mathscr{H}(S)$. Since ϕ is a positive definite function in $L^{2+\varepsilon}(G)$ for any $\varepsilon > 0$, by Theorem 1 in [1], σ is weakly contained in $L^2(G)$. Thus, there exists a sequence of convex linear combinations of diagonal matrix coefficients of $L^2(G)(S)$,

$$A_i(g) = \sum_{l=1}^{l_i} a_i^{(l)} \left(L(g) u_i^{(l)}, u_i^{(l)} \right), \quad \sum_{l=1}^{l_i} a_i^{(l)} = 1, \quad \left(u_i^{(l)} \in L^2(G)(S), a_i^{(l)} \geqslant 0 \right)$$

such that

$$A_i(g) \to \phi(g)$$

uniformly on compacta. Let $C_c(G)(S)$ be the space of continuous and compactly supported functions in $L^2(G)(S)$. Since $C_c(G)(S)$ is dense in $L^2(G)(S)$, we choose $u_i^{(l)}$ to be in $C_c(G)(S)$. Notice that

$$A_i(e) = \sum_{l=1}^{l_i} a_i^{(l)} ||u_i^{(l)}||_{L^2}^2 \to \phi(e) = ||\eta||^2.$$

Hence $\{A_i(e)\}_{i=1}^{\infty}$ is a bounded set. Suppose $A_i(e) \leqslant C$. From Theorem 2 in [1],

$$|\left(L(g)u_i^{(l)}, u_i^{(l)}\right)| \leqslant ||u_i^{(l)}||_{L^2}^2 \left(\sum_{\sigma \in S} d(\sigma)\right)^{\frac{1}{2}} \Xi(g).$$

It follows that

$$\begin{aligned} |A_i(g)| &= \left|\sum_{l=1}^{l_i} a_i^{(l)} \left(L(g)u_i^{(l)}, u_i^{(l)}\right)\right| \\ &\leqslant \sum_{l=1}^{l_i} a_i^{(l)} ||u_i^{(l)}||_{L^2}^2 \left(\sum_{\sigma \in S} d(\sigma)\right)^{1/2} \Xi(g) \\ &\leqslant C \left(\sum_{\sigma \in S} d(\sigma)\right)^{1/2} \Xi(g). \end{aligned} \tag{2}$$

We have proved that $\phi(g)$ can be approximated by positive definite functions $A_i(g)$ such that $A_i(g)$ are uniformly bounded by a fixed multiple of $\Xi(g)$.

Now consider the restrictions of $\phi(g)$ to H. From (22.2.3) in [2], for $\left(L(g)u_i^{(l)}, u_i^{(l)}\right)$ with $u_i^{(l)}$ a compactly supported continuous function,

$$\int_H (L(h)u_i^{(l)}, u_i^{(l)})\, dh \geqslant 0.$$

Thus, $\int_H A_i(h)\, dh \geqslant 0$. But $A_i(g)|_H$ are bounded by a fixed multiple of an integrable function $\Xi(g)|_H$. By the dominated convergence theorem,

$$\int_H \phi(h)\, dh = \lim_{i \to \infty} \int_H A_i(h)\, dh \geqslant 0. \qquad \square$$

2.2. *First variation*

Theorem 2.2. *Let G be a real reductive Lie group. Let K be a maximal compact subgroup of G. Let H be a closed unimodular Lie subgroup of G. Let $\Xi(g)$ be the basic spherical function of G of Harish-Chandra. Suppose that $\Xi(g)|_H$ is in $L^1(H)$. Suppose $(\pi, \mathscr{H})$ is an irreducible unitary representation weakly contained in $L^2(G)$ (see [1]). Let*

$$v = \sum_{i=1}^{k} \int_M \phi_i(x)\pi(\gamma_i(x))u\, dx,$$

where

- *u is a K-finite vector in $\mathscr{H}$;*
- *M is a smooth manifold;*
- *ϕ_i is continuous and is supported on a compact set $X_i \subset M$;*
- *$\gamma_i : M \to G$ is smooth except a codimension 1 subset and the closure of $\gamma_i(X_i)$ is compact.*

Then

$$\int_H (\pi(h)v, v)\, dh \geqslant 0.$$

The basic idea is to control the function $(\pi(g)v, v)$ by a convergent integral of left and right translations of $\Xi(g)$.

Proof. From the proof of Theorem 2.1, we have a sequence of K-finite compactly supported continuous positive definite functions

$$A_m(g) \to (\pi(g)u, u)$$

uniformly on any compact subset and

$$|A_m(g)| \leqslant C\Xi(g).$$

This implies that

$$|A_m(xgy)| \leqslant C\Xi(xgy).$$

By the compactness of $supp(\phi_i)$ and the unitarity of π,

$$(\pi(g)v, v) = \sum_{i,j=1}^{k} \int_{M \times M} \phi_i(x)\overline{\phi_j(y)}(\pi(g\gamma_i(x))u, \pi(\gamma_j(y))u)\, dx\, dy.$$

Since the closure of $\gamma_i(X_i)$ is compact, the closure of $\gamma_j(X_j)^{-1} g \gamma_i(X_i)$ is compact for every $g \in G$. By the inequality 1, for any m,

$$\begin{aligned} &\left| \sum_{i,j=1}^{k} \int_{M \times M} \phi_i(x)\overline{\phi_j(y)} A_m(\gamma_j(y)^{-1} g \gamma_i(x)))\, dx\, dy \right| \\ &\quad \leqslant C \sum_{i,j=1}^{k} \int_{M \times M} |\phi_i(x)||\phi_j(y)| \Xi(\gamma_j(y)^{-1} g \gamma_i(x)))\, dx\, dy \\ &\quad \leqslant C_1 \Xi(g) \end{aligned} \tag{3}$$

for some $C_1 > 0$. Furthermore,

$$\sum_{i,j=1}^{k} \int_{M\times M} \phi_i(x)\overline{\phi_j(y)}A_m(\gamma_j(y)^{-1}g\gamma_i(x))\,dx\,dy \to (\pi(g)v, v)$$

pointwisely as $m \to \infty$. By the dominated convergence theorem,

$$\int_H (\pi(h)v, v)\,dh = \lim_{m\to\infty} \int_H \sum_{i,j=1}^{k} \int_{M\times M} \phi_i(x)\overline{\phi_j(y)}A_m(\gamma_j(y)^{-1}h\gamma_i(x)))\,dx\,dy\,dh.$$

But

$$A_m(g) = \sum_{l=1}^{l_m} a_m^{(l)}\Big(L(g)u_m^{(l)}, u_m^{(l)}\Big).$$

For each l,

$$\begin{aligned}
&\int_H \sum_{i,j=1}^{k} \int_{M\times M} \phi_i(x)\overline{\phi_j(y)}\Big(L\Big(\gamma_j(y)^{-1}h\gamma_i(x)\Big)u_m^{(l)}, u_m^{(l)}\Big)\,dx\,dy\,dh \\
&\quad = \int_H \left(L(h)\left[\sum_{i=1}^{k}\int_M \phi_i(x)L(\gamma_i(x))u_m^{(l)}\,dx\right], \left[\sum_{i=1}^{k}\int_M \phi_i(x)L(\gamma_i(x))u_m^{(l)}\,dx\right]\right)dh \\
&\quad \geqslant 0 \qquad (4)
\end{aligned}$$

because $\sum_{i=1}^{k} \int_M \phi_i(x)L(\gamma_i(x))u_m^{(l)}\,dx$ is a continuous and compactly supported function on G. Hence for every m,

$$\int_H \sum_{i,j=1}^{k} \int_{M\times M} \phi_i(x)\overline{\phi_j(y)}A_m(\gamma_j(y)^{-1}h\gamma_i(x))\,dx\,dy\,dh \geqslant 0.$$

It follows that

$$\int_H (\pi(h)v, v)\,dh \geqslant 0. \qquad \square$$

2.3. Second variation

Theorem 2.3. *Let G be a real reductive Lie group. Let K be a maximal compact subgroup of G. Let H be a closed unimodular Lie subgroup of G. Let $\Xi(g)$ be the basic spherical function of G of Harish-Chandra. Suppose that $\Xi(g)|_H$ is in $L^1(H)$. Suppose $(\pi, \mathscr{H})$ is an irreducible unitary representation weakly contained in*

$L^2(G)$ (*see* [1,14]). *Let*

$$v = \sum_{i=1}^{n} \int_M \phi_i(x)\pi(\gamma_i(x))u_i\, dx,$$

where

- u_i *are* K*-finite vectors in* $\mathscr{H}$;
- M *is a smooth manifold*;
- ϕ_i *is continuous and is supported on a compact subset* $X_i \subset M$;
- $\gamma_i : M \to G$ *is smooth except a codimension* 1 *subset and the closure of* $\gamma_i(X_i)$ *is compact.*

Then

$$\int_H (\pi(h)v, v)\, dh \geqslant 0.$$

The only difference from Theorem 2.2 is

$$v = \sum_{i=1}^{n} \int_M \phi_i(x)\pi(\gamma_i(x))u_i\, dx$$

instead of

$$v = \sum_{i=1}^{n} \int_M \phi_i(x)\pi(\gamma_i(x))u\, dx.$$

Proof. Let V be the linear span of

$$\{\pi(k)u_i \mid i \in [1, n], k \in K\}.$$

Since u_i are K-finite, V is a finite-dimensional representation of K. Let u be a K-cyclic vector in V. Let $C(K)$ be the space of continuous functions on K. Consider the action of $C(K)$ on u:

$$\pi(f)u = \int_K f(k)\pi(k)u\, dk.$$

Apparently, $\pi(C(K))u = V$. Let

$$u_i = \int_K f_i(k)\pi(k)u\, dk.$$

Then

$$\begin{aligned} v &= \sum_{i=1}^{n} \int_M \phi_i(x)\pi(\gamma_i(x))u_i \, dx \\ &= \sum_{i=1}^{n} \int_M \phi_i(x)\pi(\gamma_i(x)) \int_K f_i(k)\pi(k)u \, dk \, dx \\ &= \sum_{i=1}^{n} \int_M \int_K \phi_i(x)f_i(k)\pi(\gamma_i(x)k)u \, dx \, dk. \end{aligned} \tag{5}$$

Apply Theorem 2.2 to functions $\phi_i(x)f_i(k)$ on $M \times K$ and

$$\gamma_i^* : (x,k) \in M \times K \to \gamma_i(x)k \in G.$$

The conclusion follows immediately. $\square$

Conjecture 1. *Let G be a real reductive group. Let K be a maximal compact subgroup of G. Let $\Xi(g)$ be Harish-Chandra's basic spherical function. Let H be a subgroup of G such that $\Xi(g)|_H$ is in $L^1(H)$. Let $\phi(g)$ be a positive definite continuous function bounded by $\Xi(g)$. Then $\int_H \phi(h)\, dh \geqslant 0$.*

3. Part II: dual pairs and mixed model

The basic theory on the mixed model of the oscillator representation is qcovered in [11] with reference to an unpublished note of Howe. We redo part of Section 4 of [11] with emphasis on the actions of various coverings of G_1 regarding the mixed model

$$\omega(MG_1, MG_2) \cong \omega(M^0G_1, M^0G_2^0) \otimes \omega(M'G_1, M'G_2').$$

Let V_1 be a vector space over D equipped with a sesquilinear form $(,)_1$, V_2 be a vector space over D equipped with a sesquilinear form $(,)_2$. Suppose one sesquilinear form is $\sharp$-Hermitian and the other is $\sharp$-skew Hermitian. Let G_i be the isometry group of $(,)_i$. Let $V = Hom_D(V_1, V_2)$ be the space of D-linear homomorphisms from V_1 to V_2.

3.1. Setup

Let $\phi, \psi \in V$, $v_1, u_1 \in V_1$ and $v_2 \in V_2$. We define a unique $\phi^*(v_2)$ such that

$$(\phi^*(v_2), v_1)_1 = (v_2, \phi(v_1))_2.$$

It is easy to verify that $\phi^* \in Hom_D(V_2, V_1)$. Thus, we obtain a $*$ operation from V to $V^* = Hom_D(V_2, V_1)$. Let $a \in \mathbb{R}$. Then

$$((a\phi)^*(v_2), v_1)_1 = (v_2, a\phi(v_1))_2 = a(v_2, \phi(v_1))_2 = a(\phi^*(v_2), v_1)_1 = (a\phi^*(v_2), v_1)_1.$$

Therefore, the $*$-operation is real linear.

Let $tr(*)$ be the real trace of a real linear endomorphism. Since V and V^* are real vector spaces, we can now define a real bilinear form Ω on V as follows

$$\Omega(\phi, \psi) = tr(\psi^* \phi).$$

We observe that

$$\begin{aligned}(\psi^*\phi(v_1), v_1')_1 &= (\phi(v_1), \psi(v_1'))_2 = \pm(\psi(v_1'), \phi(v_1))_2^\sharp \\ &= \pm(\phi^*\psi(v_1'), v_1)_1^\sharp = -(v_1, \phi^*\psi(v_1'))_1.\end{aligned}$$

Define a $*$-operation on $End_D(V_1)$ by

$$(A^* u_1, v_1)_1 = (u_1, A(v_1))_1 \quad (\forall\ A \in End_D(V_1)).$$

Then, $(\phi^*\psi)^* = -\psi^*\phi$. It follows that

$$\Omega(\psi, \phi) = tr(\phi^*\psi) = tr((\phi^*\psi)^*) = tr(-\psi^*\phi) = -\Omega(\phi, \psi).$$

It is easy to verify that Ω is nondegenerate. Therefore, Ω is a real symplectic form on V.

Next we define the action of G_1 on V as follows

$$(g_1\phi)(v_1) = \phi(g_1^{-1}v_1).$$

We observe that

$$\begin{aligned}&((g_1\psi)^*(g_1\phi)(u_1), v_1)_1 \\ &\quad = ((g_1\phi)(u_1), (g_1\psi)(v_1))_2 \\ &\quad = (\phi(g_1^{-1}u_1), \psi(g_1^{-1}v_1))_2 \\ &\quad = (\psi^*\phi(g_1^{-1}u_1), g_1^{-1}v_1)_1 \\ &\quad = (g_1(\psi^*\phi)g_1^{-1}u_1, v_1)_1. \end{aligned} \tag{6}$$

It follows that

$$\Omega(g_1\phi, g_1\psi) = tr((g_1\psi)^*(g_1\phi)) = tr(g_1\psi^*\phi g_1^{-1}) = tr(\psi^*\phi) = \Omega(\phi, \psi).$$

Therefore, G_1 is in $Sp(V,\Omega)$. We define the action of G_2 on V similarly by

$$(g_2\phi)(v_1) = g_2\phi(v_1).$$

One can verify that G_2 also preserves Ω. In addition, the action of G_1 commutes with the action of G_2.

3.2. Subgroups

Let V_2^0 be a D-linear subspace of V_2 such that

- $(,)_2$ restricted to V_2^0 is nondegenerate;
- There exist isotropic subspaces X_2^0 and Y_2^0 such that

$$X_2^0 \oplus Y_2^0 = V_2^0.$$

Let V_2' be the space of vectors perpendicular to V_2^0 with respect to $(,)_2$. Write

$$X^0 = Hom_D(V_1, X_2^0), \quad Y^0 = Hom_D(V_1, Y_2^0),$$

$$V' = Hom_D(V_1, V_2'), \quad V^0 = Hom_D(V_1, V_2^0).$$

For any $\phi, \psi \in X^0$,

$$(\psi^*\phi v_1, u_1)_1 = (\phi v_1, \psi u_1)_2 = 0 \quad (v_1, u_1 \in V_1).$$

Thus, $\Omega(\psi,\phi) = tr(\phi^*\psi) = 0$. X^0 is an isotropic subspace of (V,Ω). For the same reason, Y^0 is also an isotropic subspace of (V,Ω). Furthermore, we have

$$V = V' \oplus V^0, \quad V^0 = X^0 \oplus Y^0.$$

Let G_2^0 be the subgroup of G_2 such that G_2^0 restricted to V_2' is trivial. Then G_2^0 is isomorphic to $G_2(V_2^0)$. Let G_2' be the subgroup of G_2 such that G_2' restricted to V_2^0 is trivial. Then G_2' is isomorphic to $G_2(V_2')$.

Let Ω^0 be the restriction of Ω on V^0. Let Ω' be the restriction of Ω on V'. Then $Sp(V^0,\Omega^0)$ and $Sp(V',\Omega')$ can be embedded into $Sp(V,\Omega)$ diagonally. Let $GL(X^0, Y^0)$ be the subgroup of $Sp(V^0,\Omega^0)$ stabilizing X^0 and Y^0. Since G_1 and G_2^0 act on V^0, we obtain a dual pair

$$(G_1, G_2^0) \subseteq Sp(V^0, \Omega^0).$$

We denote this embedding by i^0. On the other hand, since G_1 and G_2' act on V', we obtain another dual pair

$$(G_1, G_2') \subseteq Sp(V', \Omega').$$

We denote this embedding by i'. Now the group G_1 is embedded into $Sp(V, \Omega)$ by $i^0 \times i'$. We denote this embedding by i.

3.3. Metaplectic covering and compatibility

For any symplectic group Sp, there is a unique nonsplit double covering MSp. We call this the metaplectic covering. Let ε be the nonidentity element in MSp whose image is the identity element in Sp. For any subgroup G of Sp, let MG be the preimage of G under the metaplectic covering. Then every MG contains ε.

Let $M^0Sp(V^0, \Omega^0)$, $M'Sp(V', \Omega')$ and $MSp(V, \Omega)$ be the metaplectic coverings of $Sp(V^0, \Omega^0)$, $Sp(V', \Omega')$ and $Sp(V, \Omega)$, respectively. Let M^0, M' and M be the covering maps, respectively. When we consider $Sp(V^0, \Omega^0)$ as a subgroup of $Sp(V, \Omega)$, we obtain a group $MSp(V^0, \Omega^0)$. On the other hand, $Sp(V^0, \Omega^0)$ has its own metaplectic covering, namely, $M^0Sp(V^0, \Omega^0)$.

Lemma 3.1 (compatibility). *The group $MSp(V^0, \Omega^0)$ is isomorphic to $M^0Sp(V^0, \Omega^0)$.*

Proof. It suffices to show that $MSp(V^0, \Omega^0)$ does not split. Suppose $MSp(V^0, \Omega^0)$ splits. Let K be a maximal compact subgroup of $Sp(V, \Omega)$ such that $K^0 = K \cap Sp(V^0, \Omega^0)$ is a maximal compact subgroup of $Sp(V^0, \Omega^0)$. Then MK^0 splits. On the other hand, K can be identified with a unitary group U. The metaplectic covering of U can be represented by

$$\{(\xi, g) \mid \xi^2 = det\, g, g \in U\}.$$

For the subgroup K^0, we see that MK^0 must be the nontrivial double covering of K^0. It does not split. We reach a contradiction. $\square$

This lemma basically asserts that if a smaller symplectic group is embedded in a bigger symplectic group canonically, then the metaplectic covering on the smaller group is compatible with the metaplectic covering on the bigger group. Let

$$\tilde{i^0} : (M^0G_1, M^0G_2^0) \subseteq M^0Sp(V^0, \Omega^0)$$

be the lifting of i^0. Let

$$\tilde{i'} : (M'G_1, M'G_2') \subseteq M'Sp(V', \Omega')$$

be the metaplectic lifting of i'. Let

$$\tilde{i} : (MG_1, MG_2) \subseteq MSp(V, \Omega)$$

be the lifting of i. According to the compatibility lemma, we may consider $M^0Sp(V^0, \Omega^0)$ and $M'Sp(V', \Omega')$ as subgroups of $MSp(V, \Omega)$. These two subgroups intersect. The intersection is $\{1, \varepsilon\}$.

Consider the natural multiplication map

$$j : M^0Sp(V^0, \Omega^0) \times M'Sp(V', \Omega') \to MSp(V, \Omega).$$

Its kernel is $\{(1,1), (\varepsilon, \varepsilon)\}$. If $g \in G_1$, then

$$i(g) = (i^0(g), i'(g)) \in Sp(V^0, \Omega^0) \times Sp(V', \Omega') \subseteq Sp(V, \Omega).$$

The covering group MG_1 is then isomorphic to the quotient

$$\{ j(g^0, g') \mid g^0 \in M^0G_1, g' \in M'G_1, M^0(g^0) = g = M'(g')\}/\{(1,1), (\varepsilon, \varepsilon)\}.$$

Lemma 3.2. *Each element in MG_1 can be expressed as $j(g^0, g')$ with*

$$(g^0 \in M^0G_1, g' \in M'G_1, M^0(g^0) = M'(g'))$$

up to a factor of

$$\{(1,1), (\varepsilon, \varepsilon)\}.$$

Lemma 3.3. *As a group,*

$$M^0G_1 \cong \{(g, g') \mid M(g) = M'(g'), g \in MG_1, g' \in M'G_1\}/\{(1,1), (\varepsilon, \varepsilon)\}.$$

3.4. Oscillator representation as tensor product

Theorem 3.1. *The representation*

$$\omega(M^0G_1, M^0G_2^0) \otimes \omega(M'G_1, M'G_2')$$

restricted to

$$\{ j(g^0, g') \mid g^0 \in M^0G_1, g' \in M'G_1, M^0(g^0) = g = M'(g')\}$$

descends into $\omega(MG_1, MG_2)|_{MG_1}$.

Proof. Suppose $g \in MG_1$. Then g can be written as

$$(g^0, g') \mid g^0 \in M^0G_1, g' \in M'G_1, M^0(g^0) = M'(g')$$

up to a multiplication by

$$\{(1,1), (\varepsilon, \varepsilon)\}.$$

It is easy to see that

$$\omega(MG_1, MG_2)(1,1) = id = \omega(M^0G_1, M^0G_2^0)(\varepsilon) \otimes \omega(M'G_1, M'G_2')(\varepsilon).$$

It follows that

$$\omega(MG_1, MG_2)(g) = \omega(M^0G_1, M^0G_2^0)(g^0) \otimes \omega(M'G_1, M'G_2')(g').$$

Our theorem is proved. $\square$

Let π be an irreducible unitary representation of MG_1 in the semistable range of $\theta(MG_1, MG_2)$ such that $\pi(\varepsilon) = -1$. Identify the representation $\omega(MG_1, MG_2)^c \otimes \pi$ with

$$\omega(M^0G_1, M^0G_2^0)^c \otimes (\omega(M'G_1, M'G_2')^c \otimes \pi).$$

From Lemma 3.3, $g^0 \in M^0G_1$ can be represented by a pair $(\tilde{g}, g')$ up to a multiplication of $(\varepsilon, \varepsilon)$. Since

$$\omega(M'G_1, M'G_2')^c(\varepsilon)\pi(\varepsilon) = id,$$

we can write

$$(\omega(M'G_1, M'G_2')^c \otimes \pi)(g^0) = \omega(M'G_1, M'G_2')^c(g') \otimes \pi(g).$$

The proof of Theorem 3.1 shows that

$$\omega(M'G_1, M'G_2')^c \otimes \pi$$

can be regarded as a unitary representation of M^0G_1.

3.5. Schrödinger model of $\omega(M^0G_1, M^0G_2^0)$

Recall $V^0 = X^0 \oplus Y^0$ and both X^0, Y^0 are Lagrangian in (V^0, Ω^0). Let $GL(X^0, Y^0)$ be the subgroup of $Sp(V^0, \Omega^0)$ stabilizing X^0 and Y^0. Then

$$GL(X^0, Y^0) \cong GL(X^0) \cong GL(Y^0).$$

Let $L^2(X^0)$ be a Schrödinger model of $\omega(M^0G_1, M^0G_2^0)$ (see [5,13]). The group $M^0GL(X^0, Y^0)$ acts on $L^2(X^0)$ naturally. Since G_1 is a subgroup of $GL(X^0, Y^0)$, an element in the group M^0G_1 can be written as

$$(\xi, g) \mid g \in G_1, \xi \in \mathbb{C}$$

such that the operator

$$(\omega(M^0G_1, M^0G_2^0)(\xi, g)\phi)(x) = \xi\phi(g^{-1}x) \quad (x \in X^0, \phi \in L^2(X^0))$$

is unitary.

Consider

$$\int_{M^0 G_1} (\omega(M^0 G_1, M^0 G_2^0)(\xi, g)\phi, \psi)(u, (\omega(M' G_1, M' G_2')^{c} \otimes \pi)(\xi, g)v)\, dg\, d\xi \quad (7)$$

with $u, v \in \omega(M' G_1, M' G_2') \otimes \pi$. Since the group action of G_1 on $L^2(X^0)$ is already unitary, ξ is a unitary character of $M^0 G_1$. Thus, $\bar{\xi} \otimes \omega(M^0 G_1, M^0 G_2^0)$ can be viewed as a unitary representation of G_1. Moreover,

$$\bar{\xi}\omega(M^0 G_1, M^0 G_2^0)(g, \xi)\phi(x) = \phi(g^{-1}x).$$

Define

$$\pi_0 = \bar{\xi} \otimes (\omega(M' G_1, M' G_2')^{c} \otimes \pi).$$

Viewing $(\omega(M' G_1, M' G_2')^{c} \otimes \pi)$ as a representation of $M^0 G_1$, π_0 descends into a unitary representation of G_1.

Tensor products with $\bar{\xi}$ here do not change the ambient spaces. However, the group actions differ by a unitary character. Now, the integral (7) becomes a multiple of

$$\int_{G_1} \int_{X^0} \phi(g^{-1}x)\overline{\psi(x)}\, dx (u, \pi_0(g)v)\, dg. \quad (8)$$

This integral can be expressed as orbital integral

$$\int_{G_1} \int_{\mathcal{O} \in G_1 \backslash X^0} \int_{x \in \mathcal{O}}.$$

In Part III, we will classify the generic G_1-orbits in X^0 and study each generic orbital integral

$$\int_{G_1} \int_{x \in \mathcal{O}} \phi(g^{-1}x)\overline{\psi(x)}(u, \pi_0(g)v)\, dx\, dg.$$

4. Part III: orbital integrals

Recall that $X^0 = Hom_D(V_1, X_2^0)$. We need to classify the orbital structure of the G_1-action on X^0. Let $m = dim_D V_1$ and $dim_D X_2^0 = p$. If $m \leqslant p$, (G_1, G_2) is said to be in the stable range. The action of G_1 on X^0 is almost free. This case is already treated in [11]. For (G_1, G_2) in the stable range, our approach can be simplified and indeed coincides with Li's approach in [11]. From now on, assume $m \geqslant p$. The set of nonsurjective homomorphisms from V_1 to X_2^0 is of measure zero. Hence, we will focus on surjective homomorphisms in X^0. We denote the set of surjective homomorphisms by X_0^0. Let $\phi \in X_0^0$.

4.1. The isotropic subgroup G_{1_ϕ}

Let $e_1, e_2, \ldots, e_m$ be a D-linear basis for V_1, and $f_1, f_2, \ldots, f_p$ be a D-linear basis for X_2^0. Then ϕ is uniquely determined by

$$\phi(e_1), \phi(e_2), \ldots, \phi(e_m).$$

We will determine the "generic" isotropic subgroups of the G_1-action on X_0^0. Suppose $g \in G_1$ stabilizes ϕ. In other words,

$$\phi(u) = (g\phi)(u) = \phi(g^{-1}u) \quad (\forall u).$$

This implies that $ker(\phi)$ is stabilized by g. Therefore, $ker(\phi)^{\perp}$ is also stabilized by g.

Lemma 4.1. *Let $g \in G_1$ and $\phi \in X_0^0$. Then ϕ is fixed by g if and only if any vector in $ker(\phi)^{\perp}$ is fixed by g.*

Proof. Suppose ϕ is fixed by g. Let $(v, ker\,\phi)_1 = 0$. We choose an arbitrary $u \in V_1$. Since $\phi(g^{-1}u) = \phi(u)$, $g^{-1}u - u \in ker\,\phi$. This implies that $(v, g^{-1}u - u)_1 = 0$. Thus, $(gv, u)_1 = (v, u)_1$ for every $u \in V_1$. It follows that $gv = v$. g fixes every vector in $v \in ker\,\phi^{\perp}$.

Conversely, suppose $gv = v$ for any $(v, ker\,\phi) = 0$. We choose an arbitrary $u \in V_1$. Then $(gv - v, u)_1 = 0$. Hence, $(v, g^{-1}u - u)_1 = 0$ for every $v \in ker\,\phi^{\perp}$. From the nondegeneracy of $(,)_1$,

$$g^{-1}u - u \in (ker\,\phi^{\perp})^{\perp} = ker\,\phi$$

Therefore, $\phi(g^{-1}u - u) = 0$ for every $u \in V_1$. It follows that $g\phi = \phi$. $\square$

Theorem 4.1. *Let ϕ be a surjective homomorphism from V_1 to X_2^0. Then the isotropic subgroup G_{1_ϕ} is the subgroup that fixes all vectors in $ker(\phi)^{\perp}$.*

The restriction of $(,)_1$ onto $ker\,\phi^{\perp}$ contains a null space, namely,

$$W = ker\,\phi \cap ker\,\phi^{\perp}. \tag{9}$$

W is an isotropic subspace of V_1 and it may or may not be trivial. Let U be a direct complement of W in $ker\,\phi^{\perp}$, i.e.,

$$U \oplus W = ker\,\phi^{\perp}. \tag{10}$$

Then $(,)_1$ restricted to U is nondegenerate. Thus, $(,)_1$ restricted onto $U^{\perp}$ is a nondegenerate sesquilinear form. Since the group G_{1_ϕ} fixes all vectors in $ker\,\phi^{\perp}$ and

$U \subseteq ker\,\phi^{\perp}$, G_{1_ϕ} can be identified with the subgroup of $G_1(U^{\perp})$ that fixes all vectors in W.

From Eqs. (9) and (10), $ker\,\phi$ is the orthogonal complement of W in $U^{\perp}$. From Eqs. (28) and (29) in [11], G_{1_ϕ} is a twisted product of $G_1(ker\,\phi/W)$ with a at most two-step nilpotent group N.

Theorem 4.2. *For orthogonal groups, we take $G_1 = SO(p,q)$. The isotropic subgroup G_{1_ϕ} is a twisted product of a classical group of the same type with a at most two-step nilpotent group N. It is always unimodular.*

Proof. To show that G_{1_ϕ} is unimodular, one must show that the adjoint action of $G_1(ker\,\phi/W)$ on the Lie algebra $\mathfrak{n}$ has determinant 1. This is obvious since $\mathfrak{n}$ as a G_1 $(ker\,\phi/W)$ module decomposes into direct sum of trivial representations and the standard representations. $\square$

4.2. Generic element

The homomorphism ϕ induces an isomorphism

$$[\phi] : V_1/ker\,\phi \to X_2^0.$$

Notice that $ker\,\phi$ can be regarded as a point in the Grassmannian $\mathscr{G}(m, m-p)$. We obtain a fibration

$$GL_p(D) \to X_0^0 \to \mathscr{G}(m, m-p).$$

The projection maps ϕ to $ker\,\phi$. The fiber contains all isomorphisms from $V_1/ker\,\phi$ to X_2^0. Thus, the fiber can be identified with $GL_p(D)$.

Definition 4.1. Generic elements in X^0 are those surjective ϕ such that

1. either $(,)_1$ restricted on $ker(\phi)$ is nondegenerate;
2. or if the above case is not possible,

$$dim_D(ker(\phi) \cap ker(\phi)^{\perp}) = 1.$$

Let X_{00}^0 be the subset of generic elements. The subspaces $ker(\phi)$ for generic ϕ are called generic $(m-p)$-subspaces. The set of generic $(m-p)$-subspaces is denoted by $\mathscr{G}_0(m, m-p)$.

Consider the following fibration,

$$GL_p(D) \to X_{00}^0 \to \mathscr{G}_0(m, m-p).$$

Since the set $\mathscr{G}_0(m, m-p)$ is open and dense in $\mathscr{G}(m, m-p)$, the set X^0_{00} is open and dense in X^0_0. Therefore, X^0_{00} is open and dense in X^0.

First, suppose $(,)_1$ restricted to $ker(\phi)$ is nondegenerate. We must have

$$ker(\phi) \oplus ker(\phi)^{\perp} = V_1.$$

The isotropic subgroup G_{1_ϕ} can be identified with $G_1(ker(\phi))$ by restriction according to Theorem 4.1. It is a smaller group of type G_1. The group G_{1_ϕ} is automatically unimodular.

Secondly, suppose

$$dim_D(ker(\phi) \cap ker(\phi)^{\perp}) = 1.$$

Notice that this case does not occur for $O(p,q)$. From Theorem 4.2, G_{1_ϕ} is a unimodular group. We obtain

Corollary 4.1. *For any generic element $\phi \in X^0_{00}$, the isotropy subgroup G_{1_ϕ} is always unimodular.*

4.3. Averaging integral revisited

Let π be an irreducible unitary representation in the semistable range of $\theta(MG_1, MG_2)$. Recall that

$$\pi_0 = \omega(M'G_1, M'G'_2)^{c} \otimes \pi \otimes \xi$$

is a unitary representation of G_1. Consider the integral

$$\int_{G_1} \int_{X^0} \phi(g^{-1}x)\overline{\psi(x)}\, dx(u, \pi_0(g)v)\, dg, \tag{11}$$

where ϕ, ψ are K-finite vectors in $L^2(X^0)$ and $u, v \in \pi_0$.

Theorem 4.3. *Let π be an irreducible unitary representation in the semistable range of $\theta(MG_1, MG_2)$. Let ϕ, ψ be in the Harish-Chandra module of $\omega(M^0G_1, M^0G^0_2)$. Let $u, v \in \pi_0$. Then the function $\phi(g^{-1}x)\overline{\psi(x)}(u, \pi_0(g)v)$ is continuous and absolutely integrable on $G_1 \times X^0$. Therefore, we have*

$$\int_{G_1} \int_{X^0} \phi(g^{-1}x)\overline{\psi(x)}\, dx(u, \pi_0(g)v)\, dg = \int_{X^0} \int_{G_1} (\phi(g^{-1}x)\overline{\psi(x)}(u, \pi_0(g)v)\, dg\, dx.$$

From our discussion in Part II, the integral (11) is a form of the averaging integral under the mixed model

$$\omega(MG_1, MG_2) \cong \omega(M^0G_1, M^0G^0_2) \otimes \omega(M'G_1, M'G'_2).$$

The absolute integrability of $\phi(g^{-1}x)\overline{\psi(x)}(u, \pi_0(g)v)$ is guaranteed by the semistable condition (see [5]). We skip the proof.

4.4. Orbital integral in general

First, let me quote a simplified version of Theorem 8.36 from [10].

Theorem 4.4. *Let G be a unimodular group and H be a closed unimodular subgroup of G. Let dg and dh be their Haar measures, respectively. Then up to a scalar, there exists a unique G-invariant measure $d[gH]$ on G/H. Furthermore, this measure can be normalized such that for any L^1 function on G,*

$$\int_G f(g)\, dg = \int_{G/H} \int_H f(gh)\, dh\, d[gH].$$

Suppose τ is a unitary representation of G, u and v are K finite vectors in τ.

Theorem 4.5. *Let G be a real reductive group, and M be a G-homogeneous space.*

- *Let x_0 be a fixed base point and G_0 be the isotropy group of x_0. Suppose that G_0 is unimodular. Then M is isomorphic to G/G_0 and possesses a G-invariant measure.*
- *Let $\gamma : M \to G$ be a smooth section of the principle bundle*

  $$B : G_0 \to G \to M$$

  except for a subset of at least codimension 1. Assume $\phi(y)$ is an absolutely integrable function on M. Then

  $$v_0 = \int_M \overline{\phi(y)}\tau(\gamma(y)^{-1})v\, dy$$

  is well-defined.
- *Assume $\phi(g^{-1}x_0)(u, \tau(g)v)$ is integrable as a function on G. Then we have*

  $$\int_G \phi(gx_0)(\tau(g)u, v)\, dg = \int_{G_0} (\tau(g_0)u, v_0)\, dg_0.$$

Proof. (1) follows directly from Theorem 4.4 by identifying M with G/G_0. Since τ is unitary and $\phi(y)$ is integrable, v_0 is well-defined, (2) is proved. Notice that $\gamma(y)G_0x_0 = y$. We compute

$$\begin{aligned}&\int_G \phi(gx_0)(\tau(g)u, v)\, dg \\ &\quad= \int_{[g]\in G/G_0} \int_{G_0} \phi(gg_0x_0)(\tau(gg_0)u, v)\, dg_0\, d[gG_0]\end{aligned}$$

$$
\begin{aligned}
&= \int_{y \in M} \phi(y) \int_{G_0} (\tau(\gamma(y)g_0)u, v)\, dg_0\, dy \\
&= \int_M \phi(y) \int_{G_0} (\tau(g_0)u, \tau(\gamma(y)^{-1})v)\, dg_0\, dy \\
&= \int_{G_0} \left(\tau(g_0)u, \int_M \overline{\phi(y)}\tau(\gamma(y)^{-1})v\, dy\right) dg_0 \\
&= \int_{G_0} (\tau(g_0)u, v_0)\, dg_0. \qquad \Box
\end{aligned} \tag{12}
$$

We can further utilize the right invariance of the Haar measure on G by changing x_0 into an arbitrary $x \in M$.

Theorem 4.6. *Under the same assumptions from Theorem* 4.5, *suppose* $\psi(x)$ *is an absolutely integrable function on* M. *Let*

$$u_0 = \int_M \overline{\psi(x)}\tau(\gamma(x)^{-1})u\, dx.$$

Suppose the function

$$\phi(g^{-1}x)\overline{\psi(x)}(u, \tau(g)v)$$

is in $L^1(G \times M)$. *Then we have*

$$\int_M \int_G \phi(g^{-1}x)\overline{\psi(x)}(u, \tau(g)v)\, dg\, dx = \int_{G_0} (\tau(g_0)u_0, v_0)\, dg_0.$$

Proof. First of all, since τ is unitary and $\psi(x)$ is integrable, u_0 is well-defined. According to Fubini's theorem, we can interchange the order of integrations. We obtain

$$
\begin{aligned}
&\int_M \int_G \phi(g^{-1}x)\overline{\psi(x)}(u, \tau(g)v)\, dg\, dx \\
&\quad = \int_M \int_G \phi(gx)\overline{\psi(x)}(\tau(g)u, v)\, dg\, dx \\
&\quad = \int_M \overline{\psi(x)} \int_G \phi(gx)(\tau(g)u, v)\, dg\, dx \\
&\quad = \int_M \overline{\psi(x)}\left(\int_G \phi(g\gamma(x)x_0)(\tau(g)u, v)\, dg\right) dx
\end{aligned}
$$

$$= \int_M \overline{\psi(x)} \left(\int_G \phi(gx_0)(\tau(g\gamma(x)^{-1})u, v)\, dg \right) dx \quad \text{by the right invariance of } dg$$

$$= \int_M \overline{\psi(x)} \left(\int_G \phi(gx_0)(\tau(g)\tau(\gamma(x)^{-1})u, v)\, dg \right) dx$$

$$= \int_M \overline{\psi(x)} \int_{G_0} (\tau(g_0)\tau(\gamma(x)^{-1})u, v_0)\, dg_0\, dx \quad \text{by Theorem 4.5}$$

$$= \int_{G_0} \left(\tau(g_0) \left(\int_M \overline{\psi(x)} \tau(\gamma(x)^{-1})u\, dx \right), v_0 \right) dg_0$$

$$= \int_{G_0} (\tau(g_0)u_0, v_0)\, dg_0. \qquad \square \tag{13}$$

4.5. *Orbital integral $I(\phi, u, \mathcal{O}_x)$*

Let $\mathcal{O}_x$ be a generic G_1-orbit in X^0_{00}. Then $\mathcal{O}_x$ possesses an G_1-invariant measure. Let π be a unitary representation in the semistable range of $\theta(MG_1, MG_2)$. Let us recall some notations and facts from Part II.

1. ξ is a central unitary character of M^0G_1 and any element g^0 in M^0G_1 can be expressed as a pair (ξ, g) with g in G_1.
2. $\pi_0 = \omega(M'G_1, M'G'_2)^c \otimes \pi \otimes \bar{\xi}$ is a representation of G_1.

We fix a K-finite vector u in $\pi \otimes \bar{\xi}$. Let

$$\phi = \sum_{i=1}^{s} \phi_i^0 \otimes \phi'_i$$

with $\phi_i^0 \in \omega(M^0G_1, M^0G_2^0)$ and $\phi'_i \in \omega(M'G_1, M'G'_2)$. Then we have

$$\begin{aligned}
&(\phi \otimes u, \phi \otimes u)_\pi \\
&\quad = \int_{MG_1} (\omega(MG_1, MG_2)(\tilde{g})\phi, \phi)(u, \pi(\tilde{g})u)\, d\tilde{g} \\
&\quad = \sum_{i,j} \int_{M^0G_1} (\omega(M^0G_1, M^0G_2^0)(g^0)\phi_i^0, \phi_j^0)(\phi'_j \otimes u, (\omega(M'G_1, M'G'_2)^c \otimes \pi) \\
&\quad \quad \times (g^0)(\phi'_i \otimes u))\, dg^0
\end{aligned}$$

$$= \sum_{i,j} \int_{M^0 G_1} ((\omega(M^0 G_1, M^0 G_2^0) \otimes \bar{\xi})(g^0)\phi_i^0, \phi_j^0)(\phi_j' \otimes u,$$
$$(\omega(M' G_1, M' G_2')^c \otimes \pi \otimes \bar{\xi})(g^0)(\phi_i' \otimes u))\, dg^0$$
$$= 2\sum_{i,j} \int_{G_1} \int_{X^0} \phi_i^0(g^{-1}x)\overline{\phi_j^0(x)}\, dx(\phi_j' \otimes u, \pi_0(g)(\phi_i' \otimes u))\, dg$$
$$= 2\sum_{i,j} \int_{X^0} \int_{G_1} \phi_i^0(g^{-1}x)\overline{\phi_j^0(x)}(\phi_j' \otimes u, \pi_0(g)(\phi_i' \otimes u))\, dg\, dx. \tag{14}$$

First of all, due to Theorem 4.3, the above integral converges absolutely. Since X_{00}^0 is open and dense in X^0,

$$2\int_{X_{00}^0} \int_{G_1} \sum_{i,j} \phi_i^0(g^{-1}x)\overline{\phi_j^0(x)}(\phi_j' \otimes u, \pi_0(g)(\phi_i' \otimes u))\, dg\, dx$$

converges absolutely. Due to Fubini's Theorem, for almost all the orbits $\mathcal{O}_x$ in X_{00}^0, the function

$$\phi_i^0(g^{-1}x)\overline{\phi_j^0(x)}(\phi_j' \otimes u, \pi_0(g)(\phi_i' \otimes u)) \quad \forall i,j \in [1,s]$$

is absolutely integrable on $\mathcal{O}_x \times G_1$. Secondly, since $\{\phi_j^0\}_{j=1}^s$ are rapidly decaying functions in the Schrödinger model of $\omega(M^0 G_1, M^0 G_2^0)$, $\{\phi_j^0\}_{j=1}^s$ are absolutely integrable on X_{00}^0. Hence, $\{\phi_j^0\}_{j=1}^s$ are absolutely integrable on almost every G_1 orbit $\mathcal{O}_x$.

Take M to be an G_1-orbit $\mathcal{O}_x$ such that

1. ϕ_j^0 is absolutely integrable on $\mathcal{O}_x$ for every j;
2. The function

$$\phi_i^0(g^{-1}x)\overline{\phi_j^0(x)}(\phi_j' \otimes u, \pi_0(g)(\phi_i' \otimes u))$$

   is absolutely integrable on $\mathcal{O}_x \times G_1$ for every $i,j \in [1,s]$.

Denote the orbital integral

$$\sum_{i,j} \int_{\mathcal{O}_x} \int_{G_1} \phi_i^0(g^{-1}x)\overline{\phi_j^0(x)}(\phi_j' \otimes u, \pi_0(g)(\phi_i' \otimes u))\, dg\, dx$$

by $I(\phi, u, \mathcal{O}_x)$. Take τ to be π_0. Since M can be identified with G_1/G_{1x}, G_1 forms a fiber bundle over M. By local triviality, we choose a smooth section $\gamma: M \to G_1$ over an open dense subset of M. Then Theorem 4.6 implies

$$\begin{aligned}
& I(\phi, u, \mathcal{O}_x) \\
&= \sum_{i,j} \int_{\mathcal{O}_x} \int_{G_1} \phi_i^0(g^{-1}x)\overline{\phi_j^0(x)}(\phi_j' \otimes u, \pi_0(g)(\phi_i' \otimes u))\, dg\, dx \\
&= \sum_{i,j} \int_{G_{1x}} \left(\pi_0(g_0) \int_{\mathcal{O}_x} \overline{\phi_j^0(y)}\pi_0(\gamma(y)^{-1})(\phi_j' \otimes u)\, dy, \right. \\
& \quad \left. \int_{\mathcal{O}_x} \overline{\phi_i^0(y)}\pi_0(\gamma(y)^{-1})(\phi_i' \otimes u)\, dy\right) dg_0 \\
&= \int_{G_{1x}} (\pi_0(g_0)u_0, u_0)\, dg_0.
\end{aligned} \tag{15}$$

Here

$$u_0 = \int_{\mathcal{O}_x} \sum_i \overline{\phi_i^0(y)}\pi_0(\gamma(y)^{-1})(\phi_i' \otimes u)\, dy.$$

4.6. Compactly supported continuous functions

The theorems we have so far proved hold for compactly supported continuous (not necessarily smooth) functions ϕ_i^0, ψ_i^0 as well. In fact, any compactly supported continuous function on X^0 can be dominated by a multiple of the Gaussian function $\mu(x)$ on X^0. Therefore, the function

$$\left|\phi_i^0(g^{-1}x)\overline{\psi_j^0(x)}(\psi_j' \otimes u, \pi_0(g)(\phi_i' \otimes v))\right|$$

is always in $L^1(G_1 \times X^0)$. The rest of the argument from Part III goes through. Again, we obtain

Theorem 4.7. *Let π be a unitary representation in the semistable range of $\theta(MG_1, MG_2)$. Let u be a K-finite vector in $\pi \otimes \bar{\xi}$. Let ϕ_i^0 be compactly supported continuous functions on X^0 and $\phi_i' \in \omega(M'G_1, M'G_2')$. Write*

$$\phi = \sum_{i=1}^{s} \phi_i^0 \otimes \phi_i'.$$

Then the integral

$$(\phi \otimes u, \phi \otimes u)_\pi = 2 \sum_{i,j} \int_{X_{00}^0} \int_{G_1} \phi_i^0(g^{-1}x)\overline{\phi_j^0(x)}(\phi_j' \otimes u, \pi_0(g)(\phi_i' \otimes u))\, dg\, dx$$

is absolutely convergent. For almost every G_1-orbit $\mathcal{O}$ (except a subset of measure zero), $I(\phi, u, \mathcal{O})$ converges absolutely. Fix such an orbit $\mathcal{O}_x$ and a base point x. Choose any smooth section $\gamma : \mathcal{O}_x \to G_1$ over an open dense subset of $\mathcal{O}_x$. Let

$$u_0 = \int_{\mathcal{O}_x} \sum_i \overline{\phi_i^0(y)} \pi_0(\gamma(y)^{-1})(\phi_i' \otimes u)\, dy.$$

Then

$$I(\phi, u, \mathcal{O}_x) = \int_{G_{1x}} (\pi_0(g)u_0, u_0)\, dg.$$

5. Part IV: positivity and unitarity

Lemma 5.1. *Suppose π is a unitary representation in $\mathscr{R}(MG_1, \omega(MG_1, MG_2))$. Suppose for every $\phi \in \omega(MG_1, MG_2)$ and a fixed nonzero $u \in \pi$*

$$(\phi \otimes u, \phi \otimes u)_\pi \geqslant 0.$$

Then $(,)_\pi$ is positive semidefinite. If $(,)_\pi$ does not vanish, Then $\theta(MG_1, MG_2)(\pi)$ is unitary.

A similar statement can be found in [12].

Proof. If $(,)_\pi$ vanishes, the lemma holds automatically. Suppose $(,)_\pi$ does not vanish. Let $\mathscr{R}_\pi$ be the radical of $(,)_\pi$. The linear space

$$(\mathscr{P} \otimes u)/(\mathscr{R}_\pi \cap (\mathscr{P} \otimes u))$$

must be nontrivial. Otherwise $\mathscr{P} \otimes u \subseteq \mathscr{R}_\pi$. Since $\mathscr{R}_\pi$ is a $(\mathfrak{g}_1, MK_1)$-module, by the $(\mathfrak{g}_1, MK_1)$-action,

$$\mathscr{P} \otimes \pi^c \subseteq \mathscr{R}_\pi.$$

This contradicts the nonvanishing of $(,)_\pi$.

Observe that

$$(\mathscr{P} \otimes u)/(\mathscr{R}_\pi \cap (\mathscr{P} \otimes u))$$

is an admissible Harish-Chandra module of MG_2.

From Theorem 7.8 [5], it must be irreducible and equivalent to $\mathscr{P} \otimes \pi^c / \mathscr{R}_\pi$. Since

$$\int_{MG_1} (\phi, \omega(g)\phi)(\pi(g)u, u)\, dg \geqslant 0$$

for a fixed $u \in \pi$ and any K-finite ϕ, $(,)_\pi|_{\mathscr{P}\otimes u}$ induces an invariant positive definite form on $\theta(MG_1, MG_2)(\pi)$. Thus $\theta(MG_1, MG_2)(\pi)$ must be unitary. Consequently, $(,)_\pi$ must be positive semidefinite. $\square$

5.1. Proof of the main theorem

Theorem 5.1. *Let $\Xi(g)$ be Harish-Chandra's basic spherical function of G_1. Suppose*

1. *π is a unitary representation in the semistable range of $\theta(MG_1, MG_2)$.*
2. *For any $x, y \in G_1$, the function $\Xi(xgy)$ is integrable on G_{1_ϕ} for every generic $\phi \in Hom_D(V_1, X_2^0)$ (see Definition 4.1).*
3. *π_0 is weakly contained in $L^2(G_1)$.*

Then $(,)_\pi$ is positive semidefinite. If $(,)_\pi$ does not vanish, then $\theta(MG_1, MG_2)(\pi)$ is unitary.

Roughly speaking, the second condition requires G_{1_ϕ} be half the "size" of G_1. The first condition is redundant assuming the second and the third conditions are true. The third conditions can be converted into a growth condition on the matrix coefficients of π.

Proof. Let u be a fixed K-finite vector in $\pi \otimes \bar{\xi}$. Write

$$\mathscr{S} = \left\{ \phi = \sum_{i=1}^{s} \phi_i^0 \otimes \phi_i' \mid \phi_i^0 \in C_c^\infty(X^0), \phi_i' \in \omega(M'G_1, M'G_2') \right\}.$$

Let $\phi \in \mathscr{S}$. Choose an arbitrary G_1-orbit $\mathcal{O}_x$ in X_{00}^0 such that $I(\phi, u, \mathcal{O}_x)$ converges absolutely. There is a canonical fiber bundle

$$G_{1_x} \to G_1 \to \mathcal{O}_x.$$

Fix a smooth section $\gamma : \mathcal{O}_x \to G_1$ over an open dense subset of $\mathcal{O}_x$ such that the closure of $\gamma(supp(\phi_i^0))$ is compact for every i. Let

$$u_0 = \int_{\mathcal{O}_x} \sum_i \overline{\phi_i^0(y)} \pi_0(\gamma(y)^{-1})(\phi_i' \otimes u)\, dy.$$

From Theorem 2.3, we have

$$\int_{G_{1x}} (\pi_0(g)u_0, u_0)\, dg \geqslant 0.$$

Combined with Theorem 4.7, we obtain

$$I(\phi, u, \mathcal{O}_x) \geqslant 0,$$

$$(\phi \otimes u, \phi \otimes u)_\pi = \int_{\mathcal{O} \in G_1 \backslash X_{00}^0} I(\phi, u, \mathcal{O})\, d[\mathcal{O}] \geqslant 0.$$

We have thus proved that the Hermitian form $(,)_\pi$ restricted to $\mathcal{S} \otimes u$ is positive semidefinite, i.e.,

$$\int_{MG_1} (\omega(MG_1, MG_2)(\tilde{g})\phi, \phi)(u, \pi(\tilde{g})u)\, d\tilde{g} \geqslant 0$$

for every $\phi \in \mathcal{S}$.

For an arbitrary K-finite vector f in $\omega(MG_1, MG_2)$, write

$$f = \sum_{k=1}^{s} f_k^0(x) \otimes f_k' \quad (f_k^0 \in \omega(M^0 G_1, M^0 G_2^0), f_k' \in \omega(M' G_1, M' G_2')).$$

For each k, choose a sequence $\psi_k^{(j)}(x) \in C_c^\infty(X^0)$ such that

$$\left|\psi_k^{(j)}(x)\right| \leqslant |f(x)|,$$

$$\psi_k^{(j)}(x) \to f(x).$$

Let $\psi^{(j)} = \sum_{k=1}^{s} \psi_k^{(j)} \otimes f_k'$. Apparently, $\psi^{(j)} \in \mathcal{S}$ and

$$\left(\omega(MG_1, MG_2)(\tilde{g})\psi^{(j)}, \psi^{(j)}\right)(u, \pi(\tilde{g})u) \to (\omega(MG_1, MG_2)(\tilde{g})f, f)(u, \pi(\tilde{g})u)$$

pointwise. Furthermore,

$$|(\omega(MG_1, MG_2)(\tilde{g})\psi_j, \psi_j)(u, \pi(\tilde{g})u)| \leqslant \sum_{k,i=1}^{s} |(\omega(MG_1, MG_2)(\tilde{g})|f_k^0| \otimes f_k', |f_i^0| \otimes f_i')(u, \pi(\tilde{g})u)|.$$

By the definition of semistable range, the function

$$|((\omega(MG_1, MG_2)(\tilde{g})|f_k^0| \otimes f_k', |f_i^0| \otimes f_i')(u, \pi(\tilde{g})u)|$$

is absolutely integrable on MG_1 (see [5]). Hence, by dominated convergence theorem,

$$(f \otimes u, f \otimes u)_\pi = \lim_{j \to \infty} \left(\psi^{(j)} \otimes u, \psi^{(j)} \otimes u \right)_\pi \geqslant 0.$$

Therefore, the form $(,)_\pi$ is positive semidefinite. If $(,)_\pi$ does not vanish, then $(,)_\pi$ considered as a form on

$$\theta(MG_1, MG_2)(\pi)$$

is positive definite (see [5]). We conclude that $\theta(MG_1, MG_2)(\pi)$ is unitary. $\square$

For (G_1, G_2) in the stable range, the generic isotropic group G_{1_ϕ} will be trivial. In this case, if π is an irreducible unitary representation of MG_1, then $(,)_\pi$ is positive semidefinite and nonvanishing. This result is due to Li [11].

5.2. $G_1 = Sp_{2n}(\mathbb{R})$

Take $G_1 = Sp_{2n}(\mathbb{R})$ as an example. We can make our theorem more precise. First let me define a partial order $\preceq$ in $\mathbb{R}^n$. We say that $a \preceq b$ if and only if

$$\sum_{j=1}^{k} a_j \leqslant \sum_{j=1}^{k} b_j$$

for all k.

Corollary 5.1. *Suppose $n < p \leqslant q$. Let π be an irreducible unitary representation of $MSp_{2n}(\mathbb{R})$. Suppose for every leading exponent (see [9, Chapter 8.8]) v of π we have*

$$\Re(v) - \left(\frac{p+q}{2} - n - 1\right) \preceq -\rho(Sp_{2n}(\mathbb{R})).$$

Then $(,)_\pi$ is positive semidefinite. In addition, if $(,)_\pi$ is nonvanishing, then

$$\theta(MG_1, MG_2)(\pi)$$

is unitary.

Proof. Take $V_1 = \mathbb{R}^{2n}$ and $X_2^0 = \mathbb{R}^{n+1}$. Then V_2' is a linear space equipped with a nondegenerate symmetric form of signature $(p-n-1, q-n-1)$. We verify the conditions in Theorem 5.1.

- For $x \in Hom(V_1, X_2^0)$, the generic isotropic group G_{1_x} is just $Sp_{n-1}(\mathbb{R})$ for n odd. For n even, the generic G_{1_x} can be identified with $Sp_{n-2}(\mathbb{R}) \times N$ where $N \cong \mathbb{R}^n$. One can easily check that $\Xi(g)$ for $Sp_{2n}(\mathbb{R})$ is integrable on G_{1_x}.

- Since

$$\Re(v) - \left(\frac{p+q}{2} - n - 1\right) \preceq -\rho(Sp_{2n}(\mathbb{R})),$$

$\pi_0 = \omega(M'G_1, M'G_2')^c \otimes \pi \otimes \bar{\xi}$ has almost square integrable matrix coefficients. According to Theorem 1 of [1], π_0 is weakly contained in $L^2(G_1)$.
- By Theorem 3.2 [11], matrix coefficients of $\omega(MO(n+1, n+1), MSp_{2n}(\mathbb{R}))$ are in $L^{2-\delta}(MSp_{2n}(\mathbb{R}))$ for small $\delta > 0$. Since π_0 is almost square integrable, the matrix coefficients of $\omega(MO(p,q), MSp_{2n}(\mathbb{R})) \otimes \pi$ are in $L^{1-\delta_0}(MG_2)$ for small $\delta_0 > 0$. Thus, π must be in the semistable range of $\theta(MG_1, MG_2)$.

We conclude that $(,)_\pi$ is positive semidefinite. $\square$

5.3. $G_1 = O(p,q)$

Similarly, we obtain

Corollary 5.2. *Suppose $p + q \leqslant 2n+1$. Let π be an irreducible unitary representation of $MO(p,q)$. Suppose for every leading exponent v of π we have*

$$\Re(v) - \left(n - \frac{p+q}{2}\right) \preceq -\rho(O(p,q)).$$

Then $(,)_\pi$ is positive semidefinite. In addition, if $(,)_\pi$ is nonvanishing, then

$$\theta(MG_1, MG_2)(\pi)$$

is unitary.

For $p+q$ odd, the growth condition concerning the leading exponent v can be strengthened to allow

$$\Re(v) - \left(n - \frac{p+q-1}{2}\right) \preceq -\rho(O(p,q)).$$

The proof is omitted.

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