A Morley finite element method for the displacement obstacle problem of clamped Kirchhoff plates

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1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, $f(x) \in L^2(\Omega)$, $g(x) \in H^4(\Omega)$, and $\psi_1(x), \psi_2(x) \in C^2(\Omega) \cap C(\partial \Omega)$ be obstacle functions such that

$$\psi_1 < \psi_2 \text{ in } \Omega \quad \text{and} \quad \psi_1 < g < \psi_2 \text{ on } \partial \Omega. \tag{1.1}$$

In this paper we consider the following displacement obstacle problem with general Dirichlet boundary conditions: Find $u \in K$ such that

$$u = \arg\min_{v \in K} G(v), \tag{1.2}$$

where

$$K = \{v \in H^2(\Omega) : v - g \in H^2_0(\Omega), \psi_1 \leq v \leq \psi_2 \text{ in } \Omega\}, \tag{1.3}$$

$$G(v) = \frac{1}{2} a(v, v) - (f, v), \tag{1.4}$$

$$a(w, v) = \int_{\Omega} D^2 w : D^2 v \, dx = \int_{\Omega} \sum_{i,j=1}^2 \left( \frac{\partial^2 w}{\partial x_i \partial x_j} \right) \left( \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \, dx, \tag{1.5}$$

and $(f, v) = \int_{\Omega} f v \, dx$. 

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Since $K$ is a nonempty closed convex subset of $H^2(\Omega)$ and $a(\cdot, \cdot)$ is a symmetric bounded bilinear form on $H^2(\Omega)$ that is coercive on the set $K - K \subset H^2_0(\Omega)$, the problem (1.2) has a unique solution (cf. [1–4]) which is also characterized by the variational inequality
\begin{equation}
  a(u, v - u) \geq (f, v - u) \quad \forall v \in K.
  \tag{1.6}
\end{equation}

Finite element methods for the displacement obstacle problem for second order elliptic operators were considered in [5–9], where the strong (complementary) form of the variational inequality was available for the convergence analysis because in this case the solution of the obstacle problem belongs to $H^2(\Omega)$ under appropriate conditions. However, for the obstacle problem (1.2), it was shown in [10–14] that $u \in H^{2+\alpha}(\Omega) \cap C^0(\Omega)$ under the assumptions on $\psi_i$ and $g$. Note that $\Delta^2 u = f$ in a small neighborhood of $\partial \Omega$ because of (1.1). Therefore $u \in H^{2+\alpha}(\Omega)$ where $\alpha \in (\frac{1}{2}, 1]$ is the index of elliptic regularity determined by the interior angles at the corners of $\Omega$ (cf. [15–18]). Hence the solution $u$ of (1.2) belongs to $H^{2+\alpha}(\Omega) \cap C^0(\Omega)$ in general, and the strong form of the variational inequality (1.6) is not available for the convergence analysis because of the lack of $H^4$ regularity. Consequently new techniques are required for the treatment of (1.2).

In [14] we considered (1.2) on convex domains (where $\alpha$ can be taken to be 1) with $g = 0$ and obtained $O(h)$ convergence in the energy norm for conforming, nonconforming and discontinuous Galerkin methods. This result was extended in [19] to general polygonal domains with general Dirichlet boundary conditions for a quadratic $C^0$ interior penalty method, where $O(h^s)$ convergence was obtained for both the energy norm and the $L_\infty$ norm. The convergence of the discrete coincidence sets and discrete free boundaries was also considered in [19]. The goal of this paper is to extend the results in [19] to the Morley finite element method [20], one of the simplest classical nonconforming finite element methods for clamped Kirchhoff plates.

The rest of the paper is organized as follows. We define the Morley finite element method for (1.2) in Section 2. An intermediate obstacle problem that connects the continuous and the discrete obstacle problems is introduced in Section 3. The convergence analysis is carried out in Section 4, followed by numerical results in Section 5.

2. A Morley finite element method

Let $T_h$ be a regular triangulation of $\Omega$ with mesh size $h$. The Morley finite element space associated with $T_h$ (cf. [20]) is defined by
\begin{equation}
  V_h = \{ v \in L_2(\Omega) : v|_T \in P_2(T), v \text{ is continuous at the vertices and the normal derivative of } v \text{ is continuous at the midpoints of the edges} \},
\end{equation}
and $\tilde{V}_h$ is the subspace of $V_h$ defined by
\begin{equation}
  \tilde{V}_h = \{ v \in V_h : \text{ the degrees of freedom of } v \text{ vanish on } \partial \Omega \},
\end{equation}
i.e., $v \in \tilde{V}_h$ belongs to $\tilde{V}_h$ if and only if $v$ vanishes at the vertices of the edges along $\partial \Omega$ and $\partial v / \partial n$ vanishes at the midpoints of the edges along $\partial \Omega$.

Let $\Pi_h : H^2(\Omega) \rightarrow V_h$ be the interpolation operator defined by the following conditions:
\begin{align}
  (\Pi_h \xi)(p) &= \xi(p) \quad \forall p \in V_h, \\
  \int_e \frac{\partial (\Pi_h \xi)}{\partial n} \, ds &= \int_e \frac{\partial \xi}{\partial n} \, ds \quad \forall e \in E_h,
\end{align}
where $E_h$ is the set of all the edges of $T_h$. Then we have (cf. [21])
\begin{equation}
  \| \xi - \Pi_h \xi \|_h \leq Ch^\alpha |\xi|_{H^{2+\alpha}(\Omega)} \quad \forall \xi \in H^{2+\alpha}(\Omega),
\end{equation}
where $\| \cdot \|_h = \sqrt{a_h(\cdot, \cdot)}$ is the energy norm.

Remark 2.1. From here on, we will use $C$ (with or without a subscript) to denote a generic positive constant independent of $h$ that can take different values at different places.

Let $V_h$ be the set of the vertices of $T_h$. We consider the following discrete obstacle problem: Find $u_h \in K_h$ such that
\begin{equation}
  u_h = \arg\min_{v \in K_h} G_h(v),
\end{equation}
where
\begin{equation}
  K_h = \{ v \in V_h : v - \Pi_h g \in \tilde{V}_h, \psi_1(p) \leq v(p) \leq \psi_2(p) \forall v \in \tilde{V}_h \},
\end{equation}
and
\begin{equation}
  G_h(v) = \frac{1}{2} a_h(v, v) - (f, v), \quad a_h(w, v) = \sum_{T \in T_h} \int_T D^2 w : D^2 v \, dx.
\end{equation}
Since $a_h(\cdot, \cdot)$ is symmetric positive definite on the set $K_h - K_h \subseteq \tilde{V}_h$, the discrete problem (2.3) is well-posed and its unique solution is also characterized by the discrete variational inequality

\[ a_h(u_h, v - u_h) \geq (f, v - u_h) \quad \forall v \in K_h. \]  

(2.5)

**Remark 2.2.** It follows from (1.3), (2.1) and (2.4) that $I_h$ maps $K$ into $K_h$.

3. Connection between continuous and discrete problems

In this section we build a connection between the continuous problem and the discrete problem that will be useful in the convergence analysis.

3.1. An intermediate obstacle problem

We consider the following intermediate obstacle problem: Find $\tilde{u}_h \in \tilde{K}_h$ such that

\[ \tilde{u}_h = \arg\min_{v \in \tilde{K}_h} G(v), \]  

(3.1)

where

\[ \tilde{K}_h = \{ v \in H^2(\Omega) : v - g \in H_0^2(\Omega), \psi_1(p) \leq v(p) \leq \psi_2(p) \quad \forall p \in \mathcal{V}_h \}. \]  

(3.2)

It is clear that $K \subset \tilde{K}_h$ and (3.1) has a unique solution $\tilde{u}_h$ that is also determined by the following intermediate variational inequality:

\[ a(\tilde{u}_h, v - \tilde{u}_h) \geq (f, v - \tilde{u}_h) \quad \forall v \in \tilde{K}_h. \]  

(3.3)

In view of (1.2) and (3.1), we can consider $u$ as an internal approximation of $\tilde{u}_h$ and therefore the distance between $u$ and $\tilde{u}_h$ in the $H^2(\Omega)$ norm is bounded by the square root of the distance between $\tilde{u}_h$ and $K$ (cf. [22]). In [14] we showed that there exist two nonnegative functions $\psi_1, \psi_2 \in C_0^\infty(\Omega)$ and a positive number $h_0$ such that for any $h \leq h_0$ we can find two positive numbers $\delta_{h,1}$ and $\delta_{h,2}$ so that

\[ \tilde{u}_h = u_h + \delta_{h,1} \psi_1 - \delta_{h,2} \psi_2 \in K \]  

(3.4)

and

\[ \delta_{h,1} \leq Ch^2. \]  

(3.5)

It follows from (3.4) and (3.5) that the distance between $\tilde{u}_h$ and $K$ is of order $O(h^2)$ and hence we have

\[ |u - \tilde{u}_h|_{H^2(\Omega)} \leq Ch. \]  

(3.6)

Details can be found in [14].

**Remark 3.1.** Even though we only consider convex domains and homogeneous Dirichlet boundary conditions in [14], the properties (3.4) and (3.5) are valid for general polygonal domains and general Dirichlet boundary conditions because the arguments in [14, Section 3] only require (i) $\tilde{K}_h$ is a closed convex subset of $H^2(\Omega)$, (ii) the separation of the obstacle functions and the boundary displacement (cf. (1.1)), (iii) the smoothness assumptions on the obstacle functions, and (iv) $u \in C^2(\Omega)$.

3.2. Enriching operator

We can connect the discrete obstacle problem (2.3) and the intermediate obstacle problem (3.1) through an enriching operator

\[ E_h : \tilde{V}_h \rightarrow W_h \cap H^2_0(\Omega), \]  

(3.7)

where $W_h$ is the Hsieh–Clough–Tocher macro element space [7,23] associated with $T_h$. The enriching operator $E_h$ is constructed by averaging and has the following properties (cf. [24]):

\[ (E_h v)(p) = v(p) \quad \forall p \in \mathcal{V}_h, v \in \tilde{V}_h, \]  

(3.8)

\[ \frac{1}{2} \sum_{m=0}^2 h_T^{2m} |v - E_h v|^2_{H^m(T)} \leq Ch_T^4 \sum_{T' \in T} |v|^2_{H^2(T')} \quad \forall v \in \tilde{V}_h, \]  

(3.9)

where $h_T = \text{diam} T$ and $T$ is the set of the triangles in $\mathcal{T}_h$ sharing a common vertex with $T$.

\[ \frac{1}{2} \sum_{m=0}^2 h_T^m |\xi - E_h \Pi_h \xi|^2_{H^m(\Omega)} \leq Ch_T^{2+\alpha} |\xi|^2_{H^2+\alpha(\Omega)} \quad \forall \xi \in H^{2+\alpha}(\Omega) \cap H_T^2(\Omega). \]  

(3.10)
It follows from (3.9) and standard inverse estimates [7,23] that
\[
\|v - E_h v\|_{L^2(\Omega)} + h \left( \sum_{T \in T_h} |v - E_h v|_{H^1(T)}^2 \right)^{\frac{1}{2}} \leq Ch^2 \|v\|_h.
\]  
(3.11)

Note that \(E_h\) does not map \(K_h\) to \(\tilde{K}_h\) (unless \(g = 0\)). In order to connect \(K_h\) to \(K\) through \(\tilde{K}_h\) for general \(g\), we consider a more general operator \(T_h : K_h \rightarrow H^2(\Omega)\) defined by
\[
T_h v = g + E_h(v - \Pi_h g) \quad \forall v \in K_h.
\]  
(3.12)

**Remark 3.2.** Since \(v - \Pi_h g \in \tilde{V}_h\) for \(v \in K_h\), the operator \(T_h\) is well-defined.

**Lemma 3.3.** We have
\[
T_h : K_h \rightarrow \tilde{K}_h,
\]
\[
|T_h w - T_h v|_{H^2(\Omega)} \leq C \|w - v\|_h \quad \forall v, w \in K_h.
\]  
(3.13)

\[
\sum_{m=0}^2 h^m |\zeta - T_h \Pi_h \zeta|_{H^m(\Omega)} \leq Ch^{2+\alpha} |\zeta - g|_{H^{2+\alpha}(\Omega)} \quad \forall \zeta \in H^{2+\alpha}(\Omega) \cap K.
\]  
(3.15)

**Proof.** Let \(v \in K_h\). Then \(T_h v - g = E_h(v - \Pi_h g)\) belongs to \(H^2(\Omega)\) by (3.7), and
\[
(T_h v)(p) = g(p) + (v - \Pi_h g)(p) = v(p)
\]
for any \(p \in V_h\) by (2.1a) and (3.8). It follows that
\[
\psi_1(p) \leq (T_h v)(p) = v(p) \leq \psi_2(p) \quad \forall p \in V_h
\]
and therefore \(T_h v \in \tilde{K}_h\) by (3.2).

From (3.12), we have
\[
T_h w - T_h v = E_h(w - v) \quad \forall v, w \in K_h.
\]  
(3.16)

\[
\zeta - T_h \Pi_h \zeta = (\zeta - g) - E_h \Pi_h (\zeta - g) \quad \forall \zeta \in H^{2+\alpha}(\Omega) \cap K.
\]  
(3.17)

Therefore the estimates (3.14) and (3.15) follow from (3.11) and (3.10) respectively. \(\square\)

4. **Convergence analysis**

Let \(u\) and \(u_h\) be the solutions of (1.2) and (2.3) respectively. In this section we will estimate the energy norm and the \(L_\infty\) norm of the error \(u - u_h\) and consider the convergence of the discrete free boundaries.

We have the following standard estimate [14, Lemma 2.6] that follows from (2.5) and the definition of \(\|\cdot\|_h\).

**Lemma 4.1.** There exist positive constants \(C_1\) and \(C_1\) such that
\[
\|u - u_h\|_h^2 \leq C_1 \|u - \Pi_h u\|_h^2 + C_1 [a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h)].
\]  
(4.1)

In view of (2.2), it only remains to estimate the second term on the right-hand side of (4.1). We will need some technical lemmas.

4.1. **Technical lemmas**

Recall the solution \(u\) of (1.2) belongs to \(H^{2+\alpha}(\Omega)\). Note also that \(H^2(\Omega)\) can be obtained from \(H^2(\Omega)\) and \(H^3(\Omega)\) by the real method of interpolation. More precisely, we have (cf. [25,26])
\[
H^{2+\alpha}(\Omega) = [H^2(\Omega), H^3(\Omega)]_{2,\alpha}.
\]  
(4.2)

**Lemma 4.2.** We have
\[
|a_h(u, v - E_h v)| \leq Ch^p \|u\|_{H^{2+\alpha}(\Omega)} \|v\|_h \quad \forall v \in \tilde{V}_h.
\]  
(4.3)

**Proof.** Given any \(v \in \tilde{V}_h\), we have, by (3.11),
\[
|a_h(\zeta, v - E_h v)| \leq \|\zeta\|_h \|v - E_h v\|_h \leq C \|\zeta\|_{H^{2+\alpha}(\Omega)} \|v\|_h \quad \forall \zeta \in H^2(\Omega).
\]  
(4.4)
Suppose $\zeta \in H^3(\Omega)$. Since the piecewise linear vector field $\nabla v$ is continuous at the midpoints of the interior edges in $E_h$ and vanishes at the midpoints of the boundary edges in $E_h$, and $E_h v \in C^1(\Omega) \cap H^2_0(\Omega)$, we have

$$a_h(\zeta, v - E_h v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla^2 \zeta : \nabla^2 (v - E_h v) \, dx$$

$$= - \sum_{T \in \mathcal{T}_h} \int_T \nabla (\Delta \zeta) \cdot \nabla (v - E_h v) \, dx + \sum_{T \in \mathcal{T}_h} \int_{\partial T} \nabla^2 \zeta : [\nabla (v - E_h v) \otimes n_T] \, ds$$

$$= - \sum_{T \in \mathcal{T}_h} \int_T \nabla (\Delta \zeta) \cdot \nabla (v - E_h v) \, dx + \sum_{e \in E_h} \int_{e} \int_{T} [\nabla^2 \zeta - \bar{(\nabla^2 \zeta)}] : [\nabla (v - E_h v) \otimes n_{e}] \, ds,$$  \hspace{1cm} (4.5)

where $\bar{(\nabla^2 \zeta)}$ is the average of $\nabla^2 \zeta$ along $e$ and $[\nabla (v - E_h v) \otimes n_{e}]$ is the sum of $\nabla (v - E_h v) \otimes n_{e}$ over the triangles that share a common edge $e$.

The two terms on the right-hand side of (4.5) can be estimated as follows:

$$\left| \sum_{T \in \mathcal{T}_h} \int_T \nabla (\Delta \zeta) \cdot \nabla (v - E_h v) \, dx \right| \leq |\Delta \zeta|_{H^1(\Omega)} \left( \sum_{T \in \mathcal{T}_h} |v - E_h v|_{H^1(T)}^2 \right)^{\frac{1}{2}}$$

$$\leq Ch |\zeta|_{H^3(\Omega)} \|v\|_h$$ \hspace{1cm} (4.6)

by the Cauchy–Schwarz inequality and (3.11);

$$\left| \sum_{e \in E_h} \int_{e} \int_{T} [\nabla^2 \zeta - \bar{(\nabla^2 \zeta)}] : [\nabla (v - E_h v) \otimes n_{e}] \, ds \right|$$

$$\leq \left( \sum_{e \in E_h} |e| \|\nabla^2 \zeta - \bar{(\nabla^2 \zeta)}\|_{L^2(e)}^2 \right)^{\frac{1}{2}} \left( \sum_{e \in E_h} |e| \|\nabla (v - E_h v) \otimes n_{e}\|_{L^2(e)}^2 \right)^{\frac{1}{2}}$$

$$\leq Ch |\zeta|_{H^3(\Omega)} \|v\|_h$$ \hspace{1cm} (4.7)

by the Cauchy–Schwarz inequality, the trace theorem with scaling, standard interpolation and inverse estimates, and (3.11). Combining (4.6) and (4.7), we find

$$|a_h(\zeta, v - E_h v)| \leq Ch |\zeta|_{H^3(\Omega)} \|v\|_h \hspace{1cm} \forall \zeta \in H^1(\Omega).$$ \hspace{1cm} (4.8)

It follows from (4.2), (4.4) and (4.8) that

$$|a_h(\zeta, v - E_h v)| \leq Ch^2 \|\zeta\|_{H^{3+\alpha}(\Omega)} \|v\|_h \hspace{1cm} \forall \zeta \in H^{3+\alpha}(\Omega),$$

and in particular the estimate (4.3) holds.

**Lemma 4.3.** We have

$$|a(u, T_h \Pi_h u - u)| \leq Ch^2 \|u\|_{H^{3+\alpha}(\Omega)} \|u - g\|_{H^{3+\alpha}(\Omega)}.$$ \hspace{1cm} (4.9)

**Proof.** The Cauchy–Schwarz inequality and (3.15) imply

$$|a(u, T_h \Pi_h u - u)| \leq Ch^2 \|\zeta\|_{H^{3}(\Omega)} \|u - g\|_{H^{3+\alpha}(\Omega)} \hspace{1cm} \forall \zeta \in H^{2}(\Omega).$$ \hspace{1cm} (4.10)

Since $T_h \Pi_h u - u \in H^2_0(\Omega)$ (cf. (3.17)), we also have, by integration by parts and (3.15),

$$|a(u, T_h \Pi_h u - u)| = \left| \int_{\Omega} \nabla (\Delta u) \cdot \nabla (T_h \Pi_h u - u) \, dx \right|$$

$$\leq Ch^{1+\alpha} \|\zeta\|_{H^{3}(\Omega)} \|u - g\|_{H^{3+\alpha}(\Omega)} \hspace{1cm} \forall \zeta \in H^{3}(\Omega).$$ \hspace{1cm} (4.11)

It follows from (4.2), (4.10) and (4.11) that

$$|a(u, T_h \Pi_h u - u)| \leq Ch^2 \|\zeta\|_{H^{3+\alpha}(\Omega)} \|u - g\|_{H^{3+\alpha}(\Omega)}.$$ \hspace{1cm} (4.12)

Taking $\zeta = u$ in (4.12), we obtain (4.9). \hspace{1cm} \square

**Lemma 4.4.** We have

$$a(u, E_h (\Pi_h u - u_h)) - (f, E_h (\Pi_h u - u_h)) \leq C(h^{2\alpha} + h \|\Pi_h u - u_h\|_h).$$ \hspace{1cm} (4.13)
Proof. Since $E_h(\Pi_h u - u_h) = T_h \Pi_h u - T_h u_h$ by (3.16), we have
\[
a(u, E_h(\Pi_h u - u_h)) = a(u, T_h \Pi_h u - T_h u_h) - (f, T_h \Pi_h u - T_h u_h)
\]
and it follows from (3.15) and Lemma 4.3 that
\[
a(u, T_h \Pi_h u - u) - (f, T_h \Pi_h u - u) \leq Ch^{2\alpha}.
\]
(4.15)

From (1.6), (3.4) and (3.5) we have
\[
a(u, u - T_h u_h) - (f, u - T_h u_h) = |a(u, u - \hat{u}_h) - (f, u - \hat{u}_h)| + |a(\hat{u}_h, \hat{u}_h - T_h u_h) - (f, \hat{u}_h - T_h u_h)|
\]
\[
\leq a(u, \hat{u}_h - T_h u_h) - (f, \hat{u}_h - T_h u_h)
\]
\[
= a(u, \hat{u}_h - T_h u_h) - (f, \hat{u}_h - T_h u_h) + a(u, \hat{u}_h - T_h u_h) - (f, \hat{u}_h - T_h u_h)
\]
\[
\leq Ch^2 + a(u, \hat{u}_h - T_h u_h) - (f, \hat{u}_h - T_h u_h).
\]
(4.16)

Note that
\[
|u - T_h u_h|_{H^2(\Omega)} \leq |u - T_h \Pi_h u|_{H^2(\Omega)} + |T_h \Pi_h u - T_h u_h|_{H^2(\Omega)}
\]
\[
\leq C(h^{2\alpha} + \|\Pi_h u - u_h\|_h)
\]
(4.17)

by (3.14) and (3.15).

In view of (3.13), we can combine (3.3), (3.6) and (4.17) to obtain
\[
a(u, \hat{u}_h - T_h u_h) - (f, \hat{u}_h - T_h u_h) = a(u, \hat{u}_h - T_h u_h) - (f, \hat{u}_h - T_h u_h)
\]
\[
+ a(u, \hat{u}_h - T_h u_h) - (f, \hat{u}_h - T_h u_h)
\]
\[
\leq Ch^2 + Ch|u - T_h u_h|_{H^2(\Omega)} \leq C(h^{1+\alpha} + h\|\Pi_h u - u_h\|_h).
\]
(4.18)

The estimate (4.13) follows from (4.14) to (4.16) and (4.18). □

4.2. Estimates for $u - u_h$

We can now bound the second term on the right-hand side of (4.1).

Lemma 4.5. We have
\[
a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h) \leq C(h^{2\alpha} + h^\alpha \|\Pi_h u - u_h\|_h).
\]
(4.19)

Proof. Since $a_h(u, E_h(\Pi_h u - u_h)) = a(u, E_h(\Pi_h u - u_h))$ and $\Pi_h u - u_h \in V_h$, we have
\[
a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h) = a_h(u, (\Pi_h u - u_h) - E_h(\Pi_h u - u_h)) - (f, (\Pi_h u - u_h) - E_h(\Pi_h u - u_h))
\]
\[
+ a(u, E_h(\Pi_h u - u_h)) - (f, E_h(\Pi_h u - u_h))
\]
\[
\leq C(h^\alpha \|\Pi_h u - u_h\|_h + h^{2\alpha})
\]
by (3.11), Lemmas 4.2 and 4.4. □

Theorem 4.6. There exists a positive constant $C$ independent of $h$ such that
\[
\|u - u_h\|_h \leq Ch^\alpha.
\]
(4.20)

Proof. It follows from (2.2), Lemmas 4.1, 4.5, the triangle inequality and the arithmetic–geometric mean inequality that
\[
\|u - u_h\|^2_h \leq C(h^{2\alpha} + h^\alpha \|\Pi_h u - u_h\|_h) \leq C(h^{2\alpha} + h^\alpha \|u - u_h\|_h) \leq Ch^{2\alpha} + \frac{1}{2} \|u - u_h\|^2_h,
\]
which implies the estimate (4.20). □

We can also derive an error estimate in the $L^\infty$ norm.

Theorem 4.7. There exists a positive constant $C$ independent of $h$ such that
\[
\|u - u_h\|_{L^\infty(\Omega)} \leq Ch^\alpha.
\]
(4.21)

Proof. We start with
\[
\|u - u_h\|_{L^\infty(\Omega)} \leq \|u - \Pi_h u\|_{L^\infty(\Omega)} + \|\Pi_h u - u_h\|_{L^\infty(\Omega)} + \|E_h(\Pi_h u - u_h)\|_{L^\infty(\Omega)}.
\]
(4.22)
The right-hand side of (4.22) can be estimated as follows:
\[ \|u - \Pi_h u\|_{L^\infty(\Omega)} \leq Ch^{1+\alpha} \|u\|_{H^{2+\alpha}(\Omega)} \]  
(4.23)
by a standard interpolation error estimate:
\[ \|(\Pi_h u - u_h) - E_h(\Pi_h u - u_h)\|_{L^\infty(\Omega)} = \max_{T \subseteq T_h} \|(\Pi_h u - u_h) - E_h(\Pi_h u - u_h)\|_{L^\infty(T)} \leq \max_{T \subseteq T_h} h_T^{-1} \|(\Pi_h u - u_h) - E_h(\Pi_h u - u_h)\|_{L^2(T)} \leq C h \|\Pi_h u - u_h\|_h \]  
(4.24)
by a standard inverse estimate and (3.9);
\[ \|E_h(\Pi_h u - u_h)\|_{L^\infty(\Omega)} \leq C \|E_h(\Pi_h u - u_h)\|_{H^2(\Omega)} \leq C \|\Pi_h u - u_h\|_h \]  
(4.25)
by the Sobolev inequality [25], a Poincaré–Friedrichs inequality [27], and (3.11).
The estimate (4.21) follows from (2.2), Theorem 4.6 and (4.22)–(4.25). \(\Box\)

**Remark 4.8.** Numerical results in Section 5 indicate that the estimate (4.21) is not sharp.

### 4.3. Convergence of discrete free boundaries

Next we consider the convergence of the discrete free boundaries to the continuous free boundary. Let \(l_i \ (i = 1, 2)\) be the coincidence sets of the obstacle problem defined by
\[ l_i = \{x \in \Omega : u(x) = \psi_i(x)\}, \]
and let \(F_i = \partial l_i \ (i = 1, 2)\) be the free boundaries.

We define the discrete coincidence sets \(l_{h,i} \ (i = 1, 2)\) by
\[ l_{h,1} = \{x \in \Omega : u_h(x) - \psi_1(x) \leq \tau_h\} \quad \text{and} \quad l_{h,2} = \{x \in \Omega : \psi_2(x) - u_h(x) \leq \tau_h\}, \]
where \(\tau_h = \rho \|u - u_h\|_{L^\infty(\Omega)}\) and \(\rho\) can be any number \(> 1\).

It follows from Theorem 4.7 that the discrete coincidence sets are disjoint compact subsets of \(\Omega\) for \(h\) sufficiently small, which is assumed to be the case. The discrete free boundaries are then defined to be \(F_{h,i} = \partial l_{h,i} \ (i = 1, 2)\).

The convergence of the discrete free boundaries can be established under the following non-degeneracy assumption (cf. [28–30]): There exist constants \(\mu_i > 0 \ (i = 1, 2), \epsilon_0 > 0\) and \(C > 0\) such that
\[ \{x \in \Omega : 0 < |u(x) - \psi_i(x)| \leq \epsilon\} \subseteq \{x \in \Omega : \text{dist}(x, F_i) \leq C \epsilon^{\mu_i}\} \quad \forall 0 < \epsilon < \epsilon_0. \]  
(4.26)
Under assumption (4.26) we can show the convergence of the discrete free boundaries in the following sense:
\[ F_{h,i} \subseteq \{x \in \Omega : \text{dist}(x, F_i) \leq C \epsilon^{\mu_i}\} \]  
(4.27)
for \(h\) sufficiently small. The proof of (4.27) can be found in [19, Section 4].

**Remark 4.9.** We can also consider a weaker non-degeneracy assumption (cf. [28–30]): There exist constants \(\mu_i > 0 \ (i = 1, 2), \epsilon_0 > 0\) and \(C > 0\) such that
\[ \{x \in \Omega : 0 < |u(x) - \psi_i(x)| \leq \epsilon\} \subseteq C \epsilon^{\mu_i}\]  
(4.28)
Under assumption (4.28) we have the following approximation result for the discrete coincidence sets:
\[ |l_i \Delta l_{h,i}| \leq C \epsilon^{\mu_i} \]  
for \(h\) sufficiently small, where \(l_i \Delta l_{h,i} = (l_i \setminus l_{h,i}) \cup (l_{h,i} \setminus l_i)\). Details can be found in [19].

**Remark 4.10.** Unlike the second order case (cf. [28–30]), the assumptions (4.26) and (4.28) have to be verified for individual obstacle problems for plates.

### 5. Numerical results

For simplicity we only consider one-obstacle problems. The algorithm we use to solve the discrete problems is an active set algorithm developed in [31]. We denote the lower obstacle function by \(\psi(x)\) and solve the discrete obstacle problems on uniform triangulations, where the length \(h_j\) of the horizontal/vertical edge in \(T_j\) is \(2^{-j}\) for Examples 1–3 and \(2^{-j+i}\) for Example 4.
Table 5.1
Energynormerrorsand\(l_\infty\)errorsforExample 1.

<table>
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<tr>
<th>j</th>
<th>(|e_j|/|u_8|_8)</th>
<th>(\beta_h)</th>
<th>(|e_j|_\infty)</th>
<th>(\beta_\infty)</th>
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<td>0.0000E+00</td>
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<td>-0.4377 1.6530E-03</td>
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</tr>
</tbody>
</table>

Fig. 5.1. Discrete coincidenceset\(I_j\)and exact free boundary for Example 1.

Example 1. We begin with the obstacle problem on the disc \(\{|x| < 2\}\) for the data \(\psi(x) = 1 - |x|^2, f(x) = 0\) and \(g(x) = 0\). Because of the rotational symmetry of this problem we can find the exact solution \(u(x)\) of this obstacle problem:

\[
u(x) = \begin{cases} 
C_1|x|^2 \ln |x| + C_2|x|^2 + C_3 \ln |x| + C_4 & |x| > r_0 \\
1 - |x|^2 & |x| \leq r_0 
\end{cases}
\]  

where \(r_0 \approx 0.181345, C_1 \approx 0.525041, C_2 \approx -0.628609, C_3 \approx 0.017266, C_4 \approx 1.046746\).

Now we restrict the problem to \(\Omega = (-0.5, 0.5)^2\) with the same \(\psi\) and \(f\) so that the exact solution is the restriction of \(u\) on \(\Omega\) with Dirichlet boundary conditions given by (5.1). Let \(u_j\) be the \(j\)-th level discrete solution and \(\Pi_j\) be the interpolation operator for the \(j\)-th level Morley finite element space. We consider the error defined by \(e_j = \Pi_j u - u\) and evaluate the error in the energy norm (denoted by \(\| \cdot \|_j\)) and in the \(l_\infty\) norm (denoted by \(\| \cdot \|_\infty\)), where

\[
\|e_j\|_\infty = \max_{p \in V_j} |e_j(p)|.
\]

The rates of convergence in these norms are computed by

\[
\beta_h = \ln(\|e_{j-1}\|_{j-1}/\|e_j\|_j)/\ln(2) \quad \text{and} \quad \beta_\infty = \ln(\|e_{j-1}\|_\infty/\|e_j\|_\infty)/\ln(2).
\]

The numerical results are presented in Table 5.1. The error estimate in the energy norm is of order \(O(h)\), which agrees with the theoretical result in Theorem 4.6. However, the \(l_\infty\) error estimate is \(O(h^2)\), which is better than the theoretical result in Theorem 4.7.

We plot the discrete coincidence set \(I_j\) for level 8 in Fig. 5.1 where

\[
I_j = \{p \in V_j : u_j(p) - \psi(p) \leq \|e_j\|_\infty\}.
\]  

The black circle in this figure represents the exact free boundary \(F = \{x \in \Omega : |x| = r_0\}\).
In this example, we take Example 2. The convergence rate for the free boundary is computed by
\[ \beta_h = \ln(d_{j-1}/d_j)/\ln(2), \]
where \( d_j = \text{dist}(f_j, F) \). For this example, Taylor’s theorem implies that \( \mu = 1/3 \) in (4.26), hence the convergence rate for the free boundary should be \( O(h^{2/3}) \). This is confirmed by the numerical results in Table 5.2.

**Example 2.** In this example, we take \( \Omega = (-0.5, 0.5)^2, f = g = 0 \) and \( \psi(x) = 1 - 5|x|^2 + |x|^4 \). Since the exact solution is unknown, we take \( \tilde{e}_j = u_{j-1} - u_j \) and the rates of convergence in the energy norm and the \( l_{\infty} \) norm are computed by
\[ \tilde{p}_h = \ln(\|\tilde{e}_j\|_{l1}/\|\tilde{e}_{j-1}\|_{l1})/\ln(2) \quad \text{and} \quad \tilde{p}_\infty = \ln(\|\tilde{e}_j\|_{l\infty}/\|\tilde{e}_{j-1}\|_{l\infty})/\ln(2), \]
where \( \|\tilde{e}_j\|_{l\infty} = \max_{p \in \Omega \setminus I} |\tilde{e}_j(p)|. \)

The numerical results are given in Table 5.3. The asymptotic convergence rate is \( O(h) \) in the energy norm as predicted by Theorem 4.6. However, the \( l_{\infty} \) norm errors is of order \( O(h^2) \) for this example.

We replace \( e_j \) with \( \tilde{e}_j \) in (5.2) and depict the discrete coincidence sets for levels 5–8 in Fig. 5.2. In this example, the non-coincidence set \( \Omega \setminus I \) is connected since \( \Delta^2 \psi > 0 \) (cf. [12]). This fact is confirmed by Fig. 5.2.

**Example 3.** In this example, we take \( \Omega = (-0.5, 0.5)^2, f = g = 0 \) and \( \psi(x) = 1 - 5|x|^2 - |x|^4 \). The numerical results are presented in Table 5.4. The energy norm error is of order \( O(h) \), which agrees with Theorem 4.6. But the \( l_{\infty} \) norm error is of order \( O(h^2) \), which is better than the \( l_{\infty} \) error estimate in Theorem 4.7.

We display the discrete coincidence sets for levels 7–8 in Fig. 5.3. The negative sign of \( |x|^4 \) in \( \psi(x) \) implies \( \Delta^2 \psi < 0 \) and hence there is no interior point for the coincidence set (cf. [12]). The pictures in Fig. 5.3 confirm this fact.

**Example 4.** In this example, we consider an L-shaped domain \( \Omega = (-0.5, 0.5)^2 \setminus [0, 0.5]^2 \) with \( f = g = 0 \) and
\[ \psi(x) = 1 - \left( \frac{(x_1 + 0.25)^2}{0.2^2} + \frac{x_2^2}{0.35^2} \right). \]
The numerical results are presented in Table 5.5. Since $\Omega$ is nonconvex in this example, we have $\alpha < 1$ (in fact, $\alpha \approx 0.5445$). The energy norm errors in Table 5.5 indicate it has not reached the asymptotic region, while the $l_\infty$ norm errors suggest the convergence rate for the $l_\infty$ norm is $O(h^{2\alpha})$ for this example.

Since $\Delta^2 \psi = 0$ in this example, the non-coincidence set is connected (cf. [12]). This is confirmed by Fig. 5.4 where we plot the discrete coincidence sets for levels 4–7.
Table 5.5
Energy norm errors and $l_{\infty}$ errors for Example 4.

<table>
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<tr>
<th>$j$</th>
<th>$|\tilde{e}_j|/|u_7|$</th>
<th>$\bar{\beta}_h$</th>
<th>$|\tilde{e}<em>j|</em>{\infty}$</th>
<th>$\bar{\beta}_{\infty}$</th>
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Fig. 5.4. Discrete coincidence sets for Example 4.

References


