ON WEIGHTED HARDY-TYPE INEQUALITIES

CHIAN YEONG CHUAH, FRITZ GESZTESY, LANCE L. LITTLEJOHN, TAO MEI,
ISAAC MICHAEL, AND MICHAEL M. H. PANG

ABSTRACT. We revisit weighted Hardy-type inequalities employing an elementary ad hoc approach that yields explicit constants. We also discuss the infinite sequence of power weighted Birman–Hardy–Rellich-type inequalities and derive an operator-valued version thereof.

1. INTRODUCTION

To put the results derived in this paper into some perspective, we very briefly recall some of the history of Hardy's celebrated inequality. We will exclusively focus on the continuous case even though Hardy originally started to investigate the discrete case (i.e., sums instead of integrals).

Hardy's inequality, in its primordial version, is of the form

$$\int_0^\infty dx\, |f'(x)|^2 \geqslant 4^{-1} \int_0^\infty dx\, x^{-2}|f(x)|^2, \quad f \in C_0^\infty((0,\infty)), \qquad (1.1)$$

with the constant 4^{-1} being optimal and the inequality being a strict one for $f \neq 0$. (This extends to all $f \in AC([0,R])$ for all $R > 0$, $f' \in L^2((0,\infty);dx)$, with $f(0_+) = 0$, but we will not dwell on this improvement right now.) Hardy's work on his celebrated inequality started in 1915, [18], see also [19]–[21], and the historical comments in [26, Chs. 1, 3, App.]. Soon afterwards, Hardy also proved a weighted Hardy inequality (with power weights) of the form (cf. [22], [23, Sect. 9.8]),

$$\int_0^b dx\, x^\alpha |f'(x)|^p \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^\infty dx\, x^{\alpha - p}|f(x)|^p,$$
$$p \in [1,\infty),\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0,\infty)). \qquad (1.2)$$

Again, the constant $(|\alpha - p + 1|/p)^p$ is optimal and the inequality is strict for $f \neq 0$.

Equation (1.2) represents just the tip of an iceberg of weighted inequalities of Hardy-type. More generally, modern treatments of this subject are devoted to weighted inequalities of the form

$$\left(\int_a^b dx\, v(x)|f'(x)|^p\right)^{1/p} \geqslant C_{p,q}\left(\int_a^b dx\, w(x)|f(x)|^q\right)^{1/q}, \quad f \in C_0^\infty((a,b)), \quad (1.3)$$

for appropriate $a, b \in \mathbb{R} \cup \{\pm\infty\}$, $a < b$, $p, q \in [1,\infty) \cup \{\infty\}$, and appropriate weight functions $0 \leqslant v, w \in L^1_{loc}((a,b);dx)$. Again, this extends to certain optimal spaces for f, far beyond $f \in C_0^\infty((a,b))$. We refer to [26, Chs. 2–5], [27, Chs. 1,2], [34, Ch. 1], and the extensive literature cited therein. In particular, we mention

the following integral versions of the two-weighted Hardy-type inequality (1.3) (the former is sometimes referred to as the differential version),

$$\left(\int_a^b dx\, v(x)|F(x)|^p\right)^{1/p} \geqslant C_{p,q}\left(\int_a^b dx\, w(x)\left|\int_a^x dx'\, F(x')\right|^q\right)^{1/q},$$
$$F \in C_0^\infty((a,b)), \tag{1.4}$$

and its companion (or "dual") version

$$\left(\int_a^b dx\, v(x)|F(x)|^p\right)^{1/p} \geqslant C_{p,q}\left(\int_a^b dx\, w(x)\left|\int_x^b dx'\, F(x')\right|^q\right)^{1/q},$$
$$F \in C_0^\infty((a,b)). \tag{1.5}$$

We note that many authors make the additional assumption $F \geqslant 0$ in (1.4), (1.5).

Before describing the results obtained in this paper in some detail, we pause for a moment to introduce our notation: We start by briefly summarizing essentials on Bochner integrability and associated vector-valued L^p-spaces. Regarding details of the Bochner integral we refer, for instance, to [1, p. 6–21], [7, Ch. 1], [11, p. 44–50], [24, p. 71–86], [28, Sect. 4.2], [29, Ch. III], [38, Sect. V.5]. In particular, if $p \geqslant 1$, $(a,b) \subseteq \mathbb{R}$ is a finite or infinite interval, $0 \leqslant w \in L_{loc}^1((a,b); dx)$ is a weight function, and $\mathcal{B}$ a Banach space, the symbol $L^p((a,b); wdx; \mathcal{B})$ denotes the set of equivalence classes of strongly measurable $\mathcal{B}$-valued functions which differ at most on sets of Lebesgue measure zero, such that $\|f(\cdot)\|_{\mathcal{B}}^p \in L^1((a,b); wdx)$. The corresponding norm in $L^p((a,b); wdx; \mathcal{B})$ is given by

$$\|f\|_{L^p((a,b);wdx;\mathcal{B})} = \left(\int_{(a,b)} w(x)dx\, \|f(x)\|_{\mathcal{B}}^p\right)^{1/p} \tag{1.6}$$

and $L^p((a,b); wdx; \mathcal{B})$ is a Banach space. If $\mathcal{H}$ is a separable Hilbert space, then so is $L^2((a,b); wdx; \mathcal{H})$ (see, e.g., [3, Subsects. 4.3.1, 4.3.2], [5, Sect. 7.1]).

One recalls that by a result of Pettis [35], if $\mathcal{B}$ is separable, weak measurability of $\mathcal{B}$-valued functions implies their strong measurability.

A map $f : [c,d] \to \mathcal{B}$ (with $[c,d] \subset (a,b)$) is called *absolutely continuous on $[c,d]$*, denoted by $f \in AC([c,d]; \mathcal{B})$, if

$$f(x) = f(x_0) + \int_{x_0}^x dt\, g(t), \quad x_0, x \in [c,d], \tag{1.7}$$

for some $g \in L^1((c,d); dx; \mathcal{B})$. In particular, f is then strongly differentiable a.e. on (c,d) and

$$f'(x) = g(x) \text{ for a.e. } x \in (c,d). \tag{1.8}$$

Similarly, $f : [c,d] \to \mathcal{B}$ is called *locally absolutely continuous*, denoted by $f \in AC_{loc}([c,d]; \mathcal{B})$, if $f \in AC([c',d']; \mathcal{B})$ on any closed subinterval $[c',d'] \subset (c,d)$.

In the special case $\mathcal{B} = \mathbb{C}$, we omit $\mathbb{C}$ and just write $L^p((a,b); wdx)$, respectively, $L_{loc}^p((a,b); wdx)$, as usual.

For $p \in [1,\infty)$, its Hölder conjugate index p' is given in a standard manner by $p' = p/(p-1) \in (1,\infty) \cup \{\infty\}$.

If $\mathcal{H}$ represents a complex, separable Hilbert space, then $\mathcal{B}(\mathcal{H})$ denotes the Banach space (the C^*-algebra) of bounded, linear operators defined on all of $\mathcal{H}$, and $\mathcal{B}_p(\mathcal{H})$ denote the ℓ^p-based Schatten–von Neumann trace ideals, $p \in [1,\infty)$, with $\mathrm{tr}_{\mathcal{H}}(T)$ abbreviating the trace of a trace class operator $T \in \mathcal{B}_1(\mathcal{H})$.

Finally, we are in a position to briefly describe the principal result of our paper in Section 2. Assume that $-\infty \leqslant a < b \leqslant \infty$, $p \in [1, \infty)$, and suppose that $0 \leqslant w_1 \in AC_{loc}((a,b))$, $0 \leqslant [-w_1']$ a.e. on (a,b), $0 \leqslant w_2 \in L^1_{loc}((a,b); dx)$, and $[-w_1']^{1-p} w_2^p \in L^1_{loc}((a,b); dx)$. If $F \in C_0((a,b); \mathcal{B})$, then we prove that

$$
\int_a^b dx\, w_1(x)^p [-w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p
$$
$$
\geqslant p^{-p} \int_a^b dx\, [-w_1'(x)] \left(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \right)^p. \tag{1.9}
$$

Moreover, we prove the companion result with $\int_a^x dx' \dots$ replaced by $\int_x^b dx' \dots$. As an important special case of (1.9) one recovers the classical form of the power weighted Hardy inequality

$$
\int_0^b dx\, x^\alpha \|F(x)\|_{\mathcal{B}}^p \geqslant \left(\frac{|\alpha + 1 - p|}{p} \right)^p \int_0^b dx\, x^{\alpha-p} \left(\int_0^x dx'\, \|F(x')\|_{\mathcal{B}} \right)^p, \tag{1.10}
$$
$$
0 < b \leqslant \infty,\ p \in [1, \infty),\ \alpha < p - 1.
$$

As alluded to earlier, the constant $[(|\alpha - p + 1|)/p]^p$ on the right-hand side of (1.10) is best possible, and equality holds if and only if $F = 0$ a.e. on $(0, b)$. After describing appropriate iterations of (1.10) (again, including the companion results with $\int_a^x dx' \dots$ replaced by $\int_x^b dx' \dots$), we also recover as a special case the entire infinite sequence of the power weighted Birman–Hardy–Rellich-type inequalities (cf. [4, p. 48], [13], [14, pp. 83–84]) at the end of Section 2, namely,

$$
\int_0^b dx\, x^\alpha |f^{(n)}(x)|^p \geqslant \frac{\prod_{j=1}^k |\alpha - jp + 1|^p}{p^{kp}} \int_0^b dx\, x^{\alpha-kp} |f^{(n-k)}(x)|^p, \tag{1.11}
$$
$$
0 < b \leqslant \infty,\ p \in [1, \infty),\ 1 \leqslant k \leqslant n,\ n \in \mathbb{N},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0, b)).
$$

Replacing the restrictive hypothesis $F \in C_0((a,b); \mathcal{B})$ by the finiteness condition of the left-hand side in (1.9), and a detailed discussion of best possible constants in these inequalities are the principal subjects of Section 3.

Finally, in Section 4 we consider extensions of (1.10) and of the infinite sequence of Birman–Hardy–Rellich-type inequalities to the operator-valued context, extending some results of Hansen [16]. More specifically, assuming $F : (0, b) \to \mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $\|F(\,\cdot\,)\|_{\mathcal{B}_p(\mathcal{H})} \in L^p((0, b); x^\alpha dx)$, we derive the inequality

$$
\mathrm{tr}_{\mathcal{H}} \left(\int_0^b dx\, x^\alpha |F(x)|^p \right) \geqslant \left(\frac{|\alpha - p + 1|}{p} \right)^p \mathrm{tr}_{\mathcal{H}} \left(\int_0^b dx\, x^{\alpha-p} \left| \int_0^x dx'\, F(x') \right|^p \right),
$$
$$
0 < b \leqslant \infty,\ p \in [1, \infty),\ \alpha < p - 1. \tag{1.12}
$$

Again, the constant $[(|\alpha - p + 1|)/p]^p$ on the right-hand side of (1.12) is best possible, and equality holds if and only if $F = 0$ a.e. on $(0, b)$.

Moreover, for $p \in [1, 2]$, we remove the trace in inequality (1.12) as follows: Suppose that $F : (0, \infty) \to \mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $F(\,\cdot\,) \geqslant 0$ a.e. on $(0, \infty)$, and $\int_0^\infty dx\, x^\alpha F(x)^p \in \mathcal{B}(\mathcal{H})$, then we derive the operator-valued inequality

$$
\int_0^\infty dx\, x^\alpha F(x)^p \geqslant \left(\frac{|\alpha - p + 1|}{p} \right)^p \int_0^\infty dx\, x^{\alpha-p} \left(\int_0^x dx'\, F(x') \right)^p, \tag{1.13}
$$
$$
p \in [1, 2],\ \alpha < p - 1.
$$

Once again, the constant $[(|\alpha - p + 1|)/p]^p$ on the right-hand sides of (1.13) is best possible, and equality holds if and only if $F = 0$ a.e. on $(0, \infty)$. We also derive the corresponding companion results with $\int_a^x dx' \ldots$ replaced by $\int_x^b dx' \ldots$.

We emphasize that (1.12) and (1.13) with $\alpha = 0$ (and hence $p > 1$) were proved by Hansen in [16].

2. WEIGHTED HARDY-TYPE INEQUALITIES EMPLOYING AN AD HOC APPROACH

In this section we derive weighted Hardy inequalities employing an elementary ad hoc approach.

We begin by deriving a weighted Hardy inequality for $\mathcal{B}$-valued functions and hence make the following assumptions.

Hypothesis 2.1. *Let* $-\infty \leqslant a < b \leqslant \infty$, $p \in [1, \infty)$, *and* $0 \leqslant w_2 \in L^1_{loc}((a, b); dx)$.
(i) Suppose that $0 \leqslant w_1 \in AC_{loc}((a, b))$, $0 \leqslant [-w_1']$ *a.e. on* (a, b), $[-w_1']^{1-p} w_2^p \in L^1_{loc}((a, b); dx)$.
(ii) Suppose that $0 \leqslant w_1 \in AC_{loc}((a, b))$, $0 \leqslant w_1'$ *a.e. on* (a, b), $[w_1']^{1-p} w_2^p \in L^1_{loc}((a, b); dx)$.

The principal result of this section then reads as follows:

Theorem 2.2. *Let* $p \in [1, \infty)$, *and suppose that* $F \in C_0((a, b); \mathcal{B})$.
(i) Assume Hypothesis 2.1 (i), then

$$\int_a^b dx \, w_1(x)^p [-w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p$$
$$\geqslant p^{-p} \int_a^b dx \, [-w_1'(x)] \left(\int_a^x dx' \, w_2(x') \|F(x')\|_{\mathcal{B}} \right)^p. \tag{2.1}$$

(ii) Assume Hypothesis 2.1 (ii), then

$$\int_a^b dx \, w_1(x)^p [w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p$$
$$\geqslant p^{-p} \int_a^b dx \, w_1'(x) \left(\int_x^b dx' \, w_2(x') \|F(x')\|_{\mathcal{B}} \right)^p. \tag{2.2}$$

Proof. It suffices to prove (2.1) and then hint at the analogous proof of (2.2). Since

$$\frac{d}{dx} \left(w_1(x) \left(\int_a^x dx' \, w_2(x') \|F(x')\|_{\mathcal{B}} \right)^p \right) = w_1'(x) \left(\int_a^x dx' \, w_2(x') \|F(x')\|_{\mathcal{B}} \right)^p$$
$$+ p w_1(x) \left(\int_a^x dx' \, w_2(x') \|F(x')\|_{\mathcal{B}} \right)^{p-1} w_2(x) \|F(x)\|_{\mathcal{B}}, \tag{2.3}$$

one obtains

$$\int_a^b dx \, \frac{d}{dx} \left(w_1(x) \left(\int_a^x dx' \, w_2(x') \|F(x')\|_{\mathcal{B}} \right)^p \right)$$
$$= w_1(x) \left(\int_a^x dx' \, w_2(x') \|F(x')\|_{\mathcal{B}} \right)^p \Big|_{x=a}^b$$
$$= w_1(b) \left(\int_a^b dx' \, w_2(x') \|F(x')\|_{\mathcal{B}} \right)^p$$

$$= \int_a^b dx\, w_1'(x) \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p$$

$$+ p \int_a^b dx\, w_1(x) \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^{p-1} w_2(x) \|F(x)\|_{\mathcal{B}}. \tag{2.4}$$

Here we used that by hypothesis, $0 \leqslant w_1$ is monotonically decreasing, and that the right-hand side of (2.4) exists employing $F \in C_0((a,b);\mathcal{B})$. Thus, with $p^{-1} + [p']^{-1} = 1$, an application of Hölder's inequality yields

$$w_1(b) \bigg(\int_a^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p + \int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p$$

$$= p \int_a^b dx\, w_1(x) \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^{p-1} w_2(x) \|F(x)\|_{\mathcal{B}}$$

$$\leqslant p \bigg[\int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \bigg]^{1/p'}$$

$$\times \bigg[\int_a^b dx\, w_1(x)^p [-w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p \bigg]^{1/p}. \tag{2.5}$$

In particular,

$$\int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p$$

$$\leqslant p \bigg[\int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x') \|F(x')\| \bigg)^p \bigg]^{1/p'}$$

$$\times \bigg[\int_a^b dx\, w_1(x)^p [-w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p \bigg]^{1/p}. \tag{2.6}$$

and hence

$$\bigg[\int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \bigg]^{1/p}$$

$$\leqslant p \bigg[\int_a^b dx\, w_1(x)^p [-w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p \bigg]^{1/p}, \tag{2.7}$$

completing the proof of item (i).

For the proof of item (ii) one notes the identity

$$\frac{d}{dx} \bigg(w_1(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \bigg) = w_1'(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p$$

$$- p w_1(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^{p-1} w_2(x) \|F(x)\|_{\mathcal{B}}, \tag{2.8}$$

and then obtains upon integrating (2.8) with respect to x from a to b,

$$w_1(a) \bigg(\int_a^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p + \int_a^b dx\, w_1'(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p$$

$$= p \int_a^b dx\, w_1(x) w_2(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^{p-1} \|F(x)\|_{\mathcal{B}}. \tag{2.9}$$

In particular,

$$\int_a^b dx\, w_1'(x)\left(\int_x^b dx'\, w_2(x')\|F(x')\|_{\mathcal{B}}\right)^p$$

$$\leqslant p\int_a^b dx\, w_1(x)w_2(x)\left(\int_x^b dx'\, w_2(x')\|F(x')\|_{\mathcal{B}}\right)^{p-1}\|F(x)\|_{\mathcal{B}}, \tag{2.10}$$

and now one can repeat the Hölder inequality argument as in item (i). (Alternatively, if $b < \infty$, one can also prove item (ii) by the change of variable $x \mapsto a+(b-x)$, i.e., by reflecting the interval (a, b) at its midpoint). $\square$

We illustrate our general result with the following well-known special case, the power-weighted Hardy inequality. For pertinent references on inequalities (2.11), (2.12) below, we recall, for instance, [2, Theorem 1.2.1], [22], [23, p. 245–246], [26, p. 23, 43], [27, p. 9–11], [34, Lemma 1.3], and the references therein.

Example 2.3. *Let $p \in [1, \infty)$, $a = 0$, $b \in (0, \infty) \cup \{\infty\}$, $w_2(x) = 1$, and suppose that the map $F : (0, \infty) \to \mathcal{B}$ is weakly measurable satisfying $\|F(\,\cdot\,)\|_{\mathcal{B}} \in L^p((0, b); x^\alpha dx)$.*
(i) If $w_1(x) = |\alpha - p + 1|^{-1}x^{-|\alpha-p+1|}$, $\alpha < p - 1$, then (2.1) reduces to the classical form

$$\int_0^b dx\, x^\alpha\, \|F(x)\|_{\mathcal{B}}^p \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^b dx\, x^{\alpha-p}\left(\int_0^x dx'\, \|F(x')\|_{\mathcal{B}}\right)^p. \tag{2.11}$$

(ii) If $w_1(x) = [|\alpha - p + 1|]^{-1}x^{|\alpha-p+1|}$, $\alpha > p - 1$, then (2.2) reduces to the complementary classical form

$$\int_0^b dx\, x^\alpha\, \|F(x)\|_{\mathcal{B}}^p \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^b dx\, x^{\alpha-p}\left(\int_x^b dx'\, \|F(x')\|_{\mathcal{B}}\right)^p. \tag{2.12}$$

In both cases (i) and (ii), the constant $[(|\alpha - p + 1|)/p]^p$ is best possible and equality holds if and only if $F = 0$ a.e. on $(0, b)$.

The case $F \in C_0((0, b); \mathcal{B})$ in Example 2.3 is a corollary of Theorem 2.2 and optimality of the constants on the right-hand sides of (2.11), (2.12), and the fact that equality is only attained in the trivial case $F = 0$ a.e. on $(0, b)$, is a classical result (see, e.g., [2, Theorem 1.2.1]). The extension of Example 2.3 to the case $F \in L^p((0, b); x^\alpha dx; \mathcal{B})$, $p \in [1, \infty)$, follows along the lines in [34, Theorem 1.14, Sects. 1.3, 1.5]. We will briefly return to this issue after Theorem 3.4

Iterating the weighted Hardy inequality yields the sequence of vector-valued Birman inequalities as follows. Consider the iterated Hardy-type operators,

$$(H_{-,1}F)(x) = \int_a^x dt_1\, F(t_1),$$

$$(H_{-,\ell}F)(x) = H_{-,1}\left(\int_a^\bullet dt_2 \cdots \int_a^{t_{\ell-1}} dt_\ell\, F(t_\ell)\right)(x)$$

$$= \int_a^x dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{\ell-1}} dt_\ell\, F(t_\ell)$$

$$= [(\ell-1)!]^{-1}\int_a^x dt\,(x-t)^{\ell-1}F(t), \quad \ell \in \mathbb{N},\ \ell \geqslant 2, \tag{2.13}$$

$$F \in L^p((a, c); dx) \text{ for all } c \in (a, b),$$

$$(H_{+,1}F)(x) = \int_x^b dt_1\, F(t_1),$$

$$(H_{+,\ell}F)(x) = H_{+,1}\left(\int_{\bullet}^b dt_2 \cdots \int_{t_{\ell-1}}^b dt_\ell\, F(t_\ell) \right)(x)$$

$$= \int_x^b dt_1 \int_{t_1}^b dt_2 \cdots \int_{t_{\ell-1}}^b dt_\ell\, F(t_\ell)$$

$$= [(\ell-1)!]^{-1} \int_x^b dt\, (x-t)^{\ell-1} F(t), \quad \ell \in \mathbb{N},\ \ell \geqslant 2, \tag{2.14}$$

$$F \in L^p((c,b); dx) \text{ for all } c \in (a,b).$$

Applying (2.11) and (2.12) iteratively in the form (with $a = 0$)

$$\int_0^b dx\, x^{\gamma-p}[(H_{\mp,1}(G_\ell(\,\cdot\,))(x)]^p$$
$$\leqslant \left(\frac{p}{|\gamma - p + 1|} \right)^p \int_0^b dx\, x^\gamma G_\ell(x)^p, \quad p \in [1,\infty),\ \gamma \lessgtr p - 1, \tag{2.15}$$

for appropriate $0 \leqslant G_\ell \in L^p((0,b); x^\gamma dx)$, $p \in [1,\infty)$, then yields for $F : (0,\infty) \to \mathcal{B}$ a weakly measurable map satisfying $\|F(\,\cdot\,)\|_{\mathcal{B}} \in L^p((0,b); x^\alpha dx)$

$$\int_0^b dx\, x^\alpha \|F(x)\|_{\mathcal{B}}^p$$
$$\geqslant \prod_{k=1}^\ell \left(\frac{|\alpha - kp + 1|}{p} \right)^p \int_0^b dx\, x^{\alpha-\ell p}[(H_{\mp,\ell}\|F(\,\cdot\,)\|_{\mathcal{B}})(x)]^p, \tag{2.16}$$

$$0 < b \leqslant \infty,\ p \in [1,\infty),\ \alpha \lessgtr \begin{cases} p-1, \\ \ell p - 1, \end{cases} \quad \ell \in \mathbb{N}.$$

It is well-known that the constants in (2.11), (2.12) and (2.16) are all optimal and that, in fact, these inequalities are all strict unless $F = 0$ on $(0,b)$.

Turning to the differential form of the iterated (integral) Hardy inequalities (2.16), and adding appropriate boundary conditions for F at both endpoints a, b, permits one to avoid the gap $(p-1, \ell p - 1)$ for α in (2.16) as follows: Assuming $F \in C_0^\infty((a,b); \mathcal{B})$ for simplicity, and introducing

$$\widetilde{f}(x) = \int_0^x dx' \|F(x')\|_{\mathcal{B}},\ x \in (0,b), \quad \widetilde{f}^{(n)}(a) = 0,\ n \in \mathbb{N},$$
$$\widetilde{g}(x) = \int_x^b dx' \|F(x')\|_{\mathcal{B}},\ x \in (0,b), \quad \widetilde{g}^{(n)}(b) = 0,\ n \in \mathbb{N}, \tag{2.17}$$

inequalities (2.11) and (2.12) become

$$\int_0^b dx\, x^\alpha [\widetilde{f}'(x)]^p \geqslant \left(\frac{|\alpha - p + 1|}{p} \right)^p \int_0^b dx\, x^{\alpha-p}\widetilde{f}(x)^p, \quad \alpha < p - 1, \tag{2.18}$$

$$\int_0^b dx\, x^\alpha \big[-\widetilde{g}'(x)\big]^p \geqslant \left(\frac{|\alpha - p + 1|}{p} \right)^p \int_0^b dx\, x^{\alpha-p}\widetilde{g}(x)^p, \quad \alpha > p - 1. \tag{2.19}$$

As a special case one obtains

$$\int_0^b dx\, x^\alpha |f'(x)|^p \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^b dx\, x^{\alpha-p}|f(x)|^p, \tag{2.20}$$
$$0 < b \leqslant \infty,\ p \in [1, \infty),\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0,b)).$$

Iterating (2.20) yields the well-known result

$$\int_0^b dx\, x^\alpha |f^{(n)}(x)|^p \geqslant \frac{\prod_{j=1}^k |\alpha - jp + 1|^p}{p^{kp}} \int_0^b dx\, x^{\alpha-kp}|f^{(n-k)}(x)|^p, \tag{2.21}$$
$$0 < b \leqslant \infty,\ p \in [1, \infty),\ 1 \leqslant k \leqslant n,\ n \in \mathbb{N},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0,b)).$$

For additional results on higher-order (overdetermined) Hardy-type inequalities see also [27, Ch. 4], [31], [32], [33].

3. More on Weighted Hardy-Type Inequalities

To remove the assumption $F \in C_0((0,b); \mathcal{B})$ in Theorem 2.2 and to take a closer look at the issue of best possible constants in the inequality, we next recall (a generalization of) a celebrated 1969 result due to Talenti [36], Tomaselli [37], and shortly afterwards by Chisholm and Everitt [8] and Muckenhoupt [30], followed by Chisholm, Everitt, and Littlejohn [9]. For exhaustive textbook presentations we refer, for instance, to [2, Sect. 1.2], [10, Sect. 5.3], [12, Sect. 2.2], [23, Sects. 9.8, 9.9], [26, Chs. 3, 4], [27, Chs. 1, 3, 4], [34, Sects. 1.1–1.3, 1.5, 1.6, 1.10].

In addition to $H_{\mp,1}$ in (2.13), (2.14), we now also introduce the generalized (weighted) Hardy operators as follows.

Hypothesis 3.1. *Let* $-\infty \leqslant a < b \leqslant \infty$ *and* $p \in [1, \infty)$.
(i) Assume that v *and* w *are weight functions satisfying* v, w *measurable on* (a,b), $v > 0,\ w > 0$ *a.e. on* (a,b).
(ii) Suppose that $\phi_\mp, \psi_\mp$ *satisfy for all* $c \in (a,b)$,

$$0 < \phi_\mp \text{ a.e. on } (a,b),\ 0 < \psi_\mp \text{ a.e. on } (a,b),$$
$$\phi_- \in L^p((c,b); v\,dx),\quad \psi_- \in L^{p'}((a,c); w^{-p'/p}dx), \tag{3.1}$$
$$\phi_+ \in L^p((a,c); v\,dx),\quad \psi_+ \in L^{p'}((c,b); w^{-p'/p}dx).$$

Given Hypothesis 3.1 we introduce

$$(H_{-,\phi_-,\psi_-}F)(x) = \phi_-(x)\int_a^x dx'\,\psi_-(x')F(x'),\quad x \in (a,b), \tag{3.2}$$
$$F \in L^p((a,c); w\,dx) \text{ for all } c \in (a,b),$$

$$(H_{+,\phi_+,\psi_+}F)(x) = \phi_+(x)\int_x^b dx'\,\psi_+(x')F(x'),\quad x \in (a,b), \tag{3.3}$$
$$F \in L^p((c,b); w\,dx) \text{ for all } c \in (a,b).$$

In particular, $H_{\mp,1,1} = H_{\mp,1}$.

The following result, Theorem 3.2, is well-known and a special case of more general situations recorded in the literature. For instance, we refer to [6], [15], [26, p. 38–40], [27, Theorem 2.3] (after specializing to the case $\varphi_1 = \psi_1 = 1$), and [34, Theorem 1.14, Lemma 5.4 in Ch. 1] (choosing $q = p$ in their results).

Theorem 3.2. *Assume Hypothesis 3.1 (i).*

(i) There exists a constant $C_- \in (0, \infty)$ such that

$$C_- \left(\int_a^b dx \, w(x) F(x)^p \right)^{1/p} \geqslant \left(\int_a^b dx \, v(x) [(H_{-,1} F)(x)]^p \right)^{1/p} \qquad (3.4)$$

for all F measurable on (a, b) and $F \geqslant 0$ a.e. on (a, b), if and only if

$$A_- := \sup_{c \in (a,b)} \left(\int_c^b dx \, v(x) \right)^{1/p} \left(\int_a^c dx \, w(x)^{-p'/p} \right)^{1/p'} < \infty. \qquad (3.5)$$

(If $p = 1$ and hence $p' = \infty$, the second factor in the right-hand side of (3.5) is interpreted as $\|1/w\|_{L^\infty((a,c);dx)}$.) Moreover, the smallest constant $C_{0,-} \in (0, \infty)$ in (3.4) satisfies

$$\begin{aligned} A_- &\leqslant C_{0,-} \leqslant p^{1/p}(p')^{1/p'} A_-, \quad p \in (1, \infty), \\ C_{0,-} &= A_-, \quad p = 1. \end{aligned} \qquad (3.6)$$

(ii) There exists a constant $C_+ \in (0, \infty)$ such that

$$C_+ \left(\int_a^b dx \, w(x) F(x)^p \right)^{1/p} \geqslant \left(\int_a^b dx \, v(x) [(H_{+,1} F)(x)]^p \right)^{1/p} \qquad (3.7)$$

for all F measurable on (a, b) and $F \geqslant 0$ a.e. on (a, b), if and only if

$$A_+ := \sup_{c \in (a,b)} \left(\int_a^c dx \, v(x) \right)^{1/p} \left(\int_c^b dx \, w(x)^{-p'/p} \right)^{1/p'} < \infty. \qquad (3.8)$$

(If $p = 1$ and hence $p' = \infty$, the second factor in the right-hand side of (3.8) is interpreted as $\|1/w\|_{L^\infty((c,b);dx)}$.) Moreover, the smallest constant $C_{0,+} \in (0, \infty)$ in (3.7) satisfies

$$\begin{aligned} A_+ &\leqslant C_{0,+} \leqslant p^{1/p}(p')^{1/p'} A_+, \quad p \in (1, \infty), \\ C_{0,+} &= A_+ \quad p = 1. \end{aligned} \qquad (3.9)$$

We emphasize that items (i) and (ii) in Theorem 3.2 do not exclude the trivial case where the left-hand sides of (3.4) and (3.7) are infinite.

We also note that Theorem 3.2 naturally extends to $p = \infty$, but as we will not use this in this note we omit further details (cf. [34, Sect. 1.5]). Moreover, [34, Sects. 1.3, 1.5] actually discuss the more general case with p replaced by $q \in [1, \infty) \cup \{\infty\}$ on the right-hand sides of (3.5), (3.8).

To extend the considerations in Theorem 3.2 to the case where $H_{\mp,1}$ is replaced by the weighted Hardy operator $H_{\mp,\phi_\mp,\psi_\mp}$ one recalls the following elementary fact, still assuming $F \geqslant 0$ a.e. on (a, b).

$$\begin{aligned} \|H_{-,\phi_-,\psi_-} F\|_{L^p((a,b);v dx)} &= \|H_{-,1}(\psi_- F)\|_{L^p((a,b);v\phi_-^p dx)} \\ &= \left(\int_a^b dx \, v(x) \phi_-(x)^p \left| \int_a^x dx' \, \psi_-(x') F(x') \right|^p \right)^{1/p} \\ &\leqslant \widetilde{C}_- \left(\int_a^b dx \, w(x) \psi_-(x)^{-p} [\psi_-(x) F(x)]^p \right)^{1/p} \\ &= \widetilde{C}_- \|F\|_{L^p((a,b);w dx)} \\ &= \widetilde{C}_- \|\psi_- F\|_{L^p((a,b);w\psi_-^{-p} dx)}, \end{aligned} \qquad (3.10)$$

as well as

$$\|H_{+,\phi_+,\psi_+}F\|_{L^p((a,b);vdx)} = \|H_{+,1}(\psi_+F)\|_{L^p((a,b);v\phi_+^p dx)}$$

$$= \left(\int_a^b dx\, v(x)\phi_+(x)^p \left|\int_x^b dx'\, \psi_+(x')F(x')\right|^p\right)^{1/p}$$

$$\leqslant \widetilde{C}_+ \left(\int_a^b dx\, w(x)\psi_+(x)^{-p}[\psi_+(x)F(x)]^p\right)^{1/p}$$

$$= \widetilde{C}_+\|F\|_{L^p((a,b);wdx)}$$

$$= \widetilde{C}_+\|\psi_+F\|_{L^p((a,b);w\psi_+^{-p}dx)}, \tag{3.11}$$

are equivalent to

$$\left\|H_{-,1}\widetilde{F}_-\right\|_{L^p((a,b);v\phi_-^p dx)} \leqslant \widetilde{C}_- \left\|\widetilde{F}_-\right\|_{L^p((a,b);w\psi_-^{-p}dx)}, \tag{3.12}$$

$$\left\|H_{+,1}\widetilde{F}_+\right\|_{L^p((a,b);v\phi_+^p dx)} \leqslant \widetilde{C}_- \left\|\widetilde{F}_+\right\|_{L^p((a,b);w\psi_+^{-p}dx)}, \tag{3.13}$$

upon identifying $\widetilde{F}_\mp = \psi_\mp F \geqslant 0$ and replacing the original weights v and w by $\widetilde{v} = v\phi_\mp^p$ and $\widetilde{w} = w\psi_\mp^{-p}$, respectively.

Thus, one obtains the following consequence of Theorem 3.2, (3.10)–(3.13) (see also [27, Theorem 2.3]):

Corollary 3.3. *Assume Hypothesis 3.1.*

(i) There exists a constant $\widetilde{C}_- \in (0,\infty)$ such that

$$(\widetilde{C}_-)^p \int_a^b dx\, w(x)G(x)^p \geqslant \int_a^b dx\, v(x)[(H_{-,\phi_-,\psi_-}G)(x)]^p \tag{3.14}$$

for all G measurable on (a,b) and $G \geqslant 0$ a.e. on (a,b), if and only if

$$\widetilde{A}_- := \sup_{c\in(a,b)} \left(\int_c^b dx\, v(x)\phi_-(x)^p\right)^{1/p} \left(\int_a^c dx\, w(x)^{-p'/p}\psi_-(x)^{p'}\right)^{1/p'} < \infty. \tag{3.15}$$

(If $p=1$ and hence $p'=\infty$, the second factor in the right-hand side of (3.15) is interpreted as $\|\psi_-/w\|_{L^\infty((a,c);dx)}$.) Moreover, the smallest constant $\widetilde{C}_{0,-} \in (0,\infty)$ in (3.14) satisfies

$$\widetilde{A}_- \leqslant \widetilde{C}_{0,-} \leqslant p^{1/p}(p')^{1/p'}\widetilde{A}_-, \quad p \in (1,\infty),$$
$$\widetilde{C}_{0,-} = \widetilde{A}_-, \quad p=1. \tag{3.16}$$

(ii) There exists a constant $\widetilde{C}_+ \in (0,\infty)$ such that

$$(\widetilde{C}_+)^p \int_a^b dx\, w(x)G(x)^p \geqslant \int_a^b dx\, v(x)[(H_{+,\phi_+,\psi_+}G)(x)]^p \tag{3.17}$$

for all measurable G on (a,b) and $G \geqslant 0$ a.e. on (a,b), if and only if

$$\widetilde{A}_+ := \sup_{c\in(a,b)} \left(\int_a^c dx\, v(x)\phi_+(x)^p\right)^{1/p} \left(\int_c^b dx\, w(x)^{-p'/p}\psi_+(x)^{p'}\right)^{1/p'} < \infty. \tag{3.18}$$

(*If $p = 1$ and hence $p' = \infty$, the second factor in the right-hand side of* (3.18) *is interpreted as* $\|\psi_+/w\|_{L^\infty((c,b);dx)}$.) *Moreover, the smallest constant* $\widetilde{C}_{0,+} \in (0,\infty)$ *in* (3.17) *satisfies*

$$\widetilde{A}_+ \leqslant \widetilde{C}_{0,+} \leqslant p^{1/p}(p')^{1/p'}\widetilde{A}_+, \quad p \in (1,\infty),$$
$$\widetilde{C}_{0,+} = \widetilde{A}_+, \quad p = 1. \tag{3.19}$$

An application of Corollary 3.3 then permits one to remove the hypothesis $F \in C_0((0,b);\mathcal{B})$ in Theorem 2.2, Example 2.3, and (2.16) as follows.

Theorem 3.4. *Let $p \in [1,\infty)$.*
(i) *In addition to Hypothesis 2.1 (i), assume that $w_j > 0$ a.e. on (a,b), $j = 1,2$, and that*

$$F \in L^p\big((a,b); w_1^p[-w_1']^{1-p}w_2^p \, dx; \mathcal{B}\big). \tag{3.20}$$

Then

$$\int_a^b dx \, w_1(x)^p[-w_1'(x)]^{1-p}w_2(x)^p\|F(x)\|_{\mathcal{B}}^p$$
$$\geqslant p^{-p}\int_a^b dx \, [-w_1'(x)]\bigg(\int_a^x dx' \, w_2(x')\|F(x')\|_{\mathcal{B}}\bigg)^p. \tag{3.21}$$

(ii) *In addition to Hypothesis 2.1 (ii), assume that $w_j > 0$ a.e. on (a,b), $j = 1,2$, and that*

$$F \in L^p\big((a,b); w_1^p[w_1']^{1-p}w_2^p \, dx; \mathcal{B}\big). \tag{3.22}$$

Then

$$\int_a^b dx \, w_1(x)^p[w_1'(x)]^{1-p}w_2(x)^p\|F(x)\|_{\mathcal{B}}^p$$
$$\geqslant p^{-p}\int_a^b dx \, w_1'(x)\bigg(\int_x^b dx' \, w_2(x')\|F(x')\|_{\mathcal{B}}\bigg)^p. \tag{3.23}$$

Proof. It suffices to consider item (i) as item (ii) is proved analogously. Identifying

$$G(\,\cdot\,) = \|F(\,\cdot\,)\|_{\mathcal{B}}, \quad w = w_1^p[-w_1']^{1-p}w_2^p, \quad v = [-w_1'], \quad \phi_- = 1, \quad \psi_- = w_2 \tag{3.24}$$

in Corollary 3.3 (i), the estimate (3.14) proves boundedness of the weighted Hardy operator

$$H_{-,1,w_2} \in \mathcal{B}\big(L^p\big((a,b); w_1^p[-w_1']^{1-p}w_2^p \, dx\big), L^p\big((a,b); [-w_1'] \, dx\big)\big). \tag{3.25}$$

if and only if

$$\widetilde{A}_- = \sup_{c \in (a,b)} \left[\bigg(\int_c^b dx \, [-w_1'(x)]\bigg)^{1/p}\right.$$
$$\times \bigg(\int_a^c dx \, \big\{w_1(x)^p\big[-w_1'(x)\big]^{1-p}w_2(x)^p\big\}^{-p'/p}w_2(x)^{p'}\bigg)^{1/p'}\bigg]$$
$$= \sup_{c \in (a,b)} \left[\bigg(\int_c^b dx \, [-w_1'(x)]\bigg)^{1/p}\bigg(\int_a^c dx \, w_1(x)^{-p'}[-w_1'(x)]\bigg)^{1/p'}\right] < \infty, \tag{3.26}$$

employing $-p'/p = 1 - p'$, $-(1-p)p'/p = 1$, temporarily assuming $p \in (1, \infty)$. The constant $\widetilde{A}_-$ is easily estimated and one obtains

$$\widetilde{A}_- = \sup_{c \in (a,b)} \left[[w_1(c) - w_1(b)]^{1/p} \left[\frac{w_1(c)^{1-p'} - w_1(a)^{1-p'}}{p'-1} \right]^{1/p'} \right]$$

$$\leqslant (p'-1)^{-1/p'} \sup_{c \in (a,b)} \left[w_1(c)^{(1/p)+[(1-p')/p']} \right]$$

$$= (p'-1)^{-1/p'} = (p/p')^{1/p'} < \infty, \quad p \in (1, \infty). \tag{3.27}$$

Thus, $\widetilde{C}_- \in (0, \infty)$ as in (3.14) exists, implying (3.25). Given the estimate (3.27), The smallest constant $\widetilde{C}_{0,-}$ as in (3.14), (3.16) satisfies

$$\widetilde{C}_{0,-} \leqslant p^{1/p}(p')^{1/p'} \widetilde{A}_- \leqslant p^{1/p}(p')^{1/p'}(p/p')^{1/p'} = p, \tag{3.28}$$

proving the estimate (3.21).

In the case $p = 1$, $p' = \infty$, the analog of (3.26) becomes

$$\widetilde{A}_- = \sup_{c \in (a,b)} \left[\left(\int_c^b dx \, [-w_1'(x)] \right) \|1/w_1\|_{L^\infty((a,c);dx)} \right]$$

$$= \sup_{c \in (a,b)} \left[[w_1(c) - w_1(b)] \|1/w_1\|_{L^\infty((a,c);dx)} \right]$$

$$= \sup_{c \in (a,b)} \left[[w_1(c) - w_1(b)] w_1(c)^{-1} \right]$$

$$= \sup_{c \in (a,b)} \left[1 - [w_1(b)/w_1(c)] \right]$$

$$= \left[1 - [w_1(b)/w_1(a)] \right] \leqslant 1, \quad p = 1, \tag{3.29}$$

and hence the fact $\widetilde{C}_{0,-} = \widetilde{A}_-$, according to (3.15), also yields (3.21) for $p = 1$. $\quad\square$

In particular, we now removed the hypothesis $F \in C_0((0, b); \mathcal{B})$ in Theorem 2.2 and replaced it by (3.20), (3.22). Consequently, this illustrates that Example 2.3 and (2.16) now extend from $F \in C_0((0, b); \mathcal{B})$ to $F \in L^p((a, b); x^\alpha dx; \mathcal{B})$.

Due to the fundamental importance of the constants $\widetilde{A}_\mp$ in connection with smallest constants $\widetilde{C}_{0,\mp}$ in Hardy-type inequalities (as detailed in (3.16), (3.19)), we now take a second look at them.

Lemma 3.5. *Let $p \in [1, \infty)$.*

(i) Assume Hypothesis 2.1 (i), then

$$\widetilde{A}_- = \begin{cases} (p/p')^{1/p'} \left[1 - \left[w_1(b)^{1/p}/w_1(a)^{1/p} \right] \right], & p \in (1, \infty), \\ \left[1 - [w_1(b)/w_1(a)] \right], & p = 1. \end{cases} \tag{3.30}$$

In particular, if $w_1(b) = 0$, or $1/w_1(a) = 0$, then

$$\widetilde{A}_- = \begin{cases} (p/p')^{1/p'}, & p \in (1, \infty), \\ 1, & p = 1. \end{cases} \tag{3.31}$$

(ii) Assume Hypothesis 2.1 (ii), then

$$\widetilde{A}_+ = \begin{cases} (p/p')^{1/p'} \left[1 - \left[w_1(a)^{1/p}/w_1(b)^{1/p} \right] \right], & p \in (1, \infty), \\ \left[1 - [w_1(a)/w_1(b)] \right], & p = 1. \end{cases} \tag{3.32}$$

In particular, if $w_1(a) = 0$, or $1/w_1(b) = 0$, then

$$\widetilde{A}_+ = \begin{cases} (p/p')^{1/p'}, & p \in (1, \infty), \\ 1, & p = 1. \end{cases} \tag{3.33}$$

Proof. Again, we prove item (i) only. Starting with the case $p \in (1, \infty)$, we first prove (3.31) directly (even though that is not necessary). Suppose that $w_1(b) = 0$, then

$$\widetilde{A}_- = \sup_{c \in (a,b)} w_1(c)^{p'/(pp')} \left[\frac{w_1(c)^{1-p'} - w_1(a)^{1-p'}}{p' - 1} \right]^{1/p'}$$

$$= (p' - 1)^{-1/p'} \sup_{c \in (a,b)} \left[1 - \frac{w_1(c)^{p'-1}}{w_1(a)^{p'-1}} \right]^{1/p'}$$

$$= (p' - 1)^{-1/p'} = (p/p')^{1/p'}, \tag{3.34}$$

as the supremum is attained for $c = b$. Similarly, if $1/w_1(a) = 0$, then

$$\widetilde{A}_- = \sup_{c \in (a,b)} [w_1(c) - w_1(b)]^{1/p} \left[\frac{w_1(c)^{1-p'}}{p' - 1} \right]^{1/p'}$$

$$= (p' - 1)^{-1/p'} \sup_{c \in (a,b)} \left[1 - \frac{w_1(b)}{w_1(c)} \right]^{1/p}$$

$$= (p' - 1)^{-1/p'} = (p/p')^{1/p'}, \tag{3.35}$$

as the supremum is attained for $c = a$. To deal with the general case (3.30) (which of course, directly yields (3.34), (3.35)) one can proceed as follows.

$$\widetilde{A}_- = \sup_{c \in (a,b)} \left\{ [w_1(c) - w_1(b)]^{1/p} \left[\frac{w_1(c)^{1-p'} - w_1(a)^{1-p'}}{p' - 1} \right]^{1/p'} \right\}$$

$$= (p' - 1)^{-1/p'} \sup_{c \in (a,b)} \left\{ \left[1 - \frac{w_1(b)}{w_1(c)} \right]^{1/p} \left[1 - \frac{w_1(c)^{p'-1}}{w_1(a)^{p'-1}} \right]^{1/p'} \right\}. \tag{3.36}$$

To maximize the right-hand side of (3.36), we introduce the absolutely continuous function

$$\eta(c) := \left[1 - \frac{w_1(b)}{w_1(c)} \right]^{1/p} \left[1 - \frac{w_1(c)^{p'-1}}{w_1(a)^{p'-1}} \right]^{1/p'}, \quad c \in (a, b), \tag{3.37}$$

and note that $\eta'(c) = 0$ is equivalent to

$$w_1(c) = w_1(a)^{1/p} w_1(b)^{1/p'}. \tag{3.38}$$

Relation (3.38) yields a maximum of η on (a, b) ($c \in \{a, b\}$ being excluded as a maximum since $\eta(a) = \eta(b) = 0$ if $p \in (1, \infty)$) and insertion of (3.38) into the right-hand side of (3.36) then yields (3.30) for $p \in (1, \infty)$.

The case $p = 1$ (and hence, $p' = \infty$) follows from

$$\widetilde{A}_- = \sup_{c \in (a,b)} \left\{ [w_1(c) - w_1(b)] \| 1/w_1 \|_{L^\infty((a,c);dx)} \right\}$$

$$= \sup_{c \in (a,b)} \left\{ [w_1(c) - w_1(b)]/w_1(c) \right\}$$

$$= \left[1 - \frac{w_1(b)}{w_1(a)} \right], \tag{3.39}$$

as the supremum is attained at $c = a$. $\qquad\square$

Remark 3.6. (*i*) One observes that the first lines on the right-hand sides of (3.30)–(3.33) indeed converge to the second lines on the right-hand sides of (3.30)–(3.33) as $p \downarrow 1$ and $p' \uparrow \infty$.

(*ii*) If $w_1(b) \neq 0$ and $1/w_1(a) \neq 0$ (resp., if $w_1(a) \neq 0$ and $1/w_1(b) \neq 0$), then (3.30) (resp., (3.32)) proves in conjunction with (3.16) (resp., (3.19)) that inequality (3.21) (resp., (3.23)), and hence our ad hoc inequality (2.1) (resp., (2.2)) is not optimal, that is, the constant p^{-p} in (3.21) and (3.23) is not optimal.

4. Some Applications to the Operator-Valued Case

The principal purpose of this section is to extend Example 2.3 to the operator-valued situation.

We start with a few preparations. Given a separable, complex Hilbert space $\mathcal{H}$, we recall that we denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of linear, bounded operators $T \colon \mathcal{H} \to \mathcal{H}$ defined on all of $\mathcal{H}$. Similarly, $\mathcal{B}_p(\mathcal{H})$ denote the ℓ^p-based Schatten–von Neumann trace ideals, $p \in [1, \infty)$.

The eigenvalues of a bounded linear operator $B \in \mathcal{B}(\mathcal{H})$ are abbreviated by $\lambda_j(B)$, $j \in \mathcal{J}$, with $\mathcal{J} \subseteq \mathbb{N}$ an appropriate index set, and the trace of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$ is denoted by $\mathrm{tr}_{\mathcal{H}}(A)$ and computed via Lidskii's theorem as

$$\mathrm{tr}_{\mathcal{H}}(A) = \sum_{j \in \mathcal{J}} \lambda_j(A). \tag{4.1}$$

In particular, if $T \in \mathcal{B}_p(\mathcal{H})$ for some $p \in [1, \infty)$, and $|T|$ is defined by $|T| := (T^*T)^{1/2}$, one recalls the fact,

$$\|T\|_{\mathcal{B}_p(\mathcal{H})}^p = \mathrm{tr}_{\mathcal{H}}\big(|T|^p\big). \tag{4.2}$$

Moreover, if $A : (0, \infty) \to \mathcal{B}(\mathcal{H})$ is weakly measurable, $0 \leqslant A(\,\cdot\,) \in \mathcal{B}_1(\mathcal{H})$ a.e. on $(0, \infty)$, and $\mathrm{tr}_{\mathcal{H}}(A(\,\cdot\,)) \in L^1((a, b); dt)$, then by an application of the monotone convergence theorem,

$$\left\| \int_a^b dt\, A(t) \right\|_{\mathcal{B}_1(\mathcal{H})} = \mathrm{tr}_{\mathcal{H}}\left(\int_a^b dt\, A(t) \right)$$

$$= \sum_{n \in \mathcal{N}} \int_a^b dt\, (e_n, A(t)e_n)_{\mathcal{H}} = \int_a^b dt \sum_{n \in \mathcal{N}} (e_n, A(t)e_n)_{\mathcal{H}} \tag{4.3}$$

$$= \int_a^b dt\, \mathrm{tr}_{\mathcal{H}}(A(t)) = \int_a^b dt\, \|A(t)\|_{\mathcal{B}_1(\mathcal{H})},$$

where $\{e_n\}_{n \in \mathcal{N}}$ represents a complete orthonormal system in $\mathcal{H}$, with $\mathcal{N} \subseteq \mathbb{N}$ an appropriate index set. In this context we also recall the well-known fact,

$$\left\| \int_a^b dt\, A(t) \right\|_{\mathcal{B}_p(\mathcal{H})} \leqslant \int_a^b dt\, \|A(t)\|_{\mathcal{B}_p(\mathcal{H})}, \quad p \in [1, \infty), \tag{4.4}$$

and similarly with $\mathcal{B}_p(\mathcal{H})$ replaced by $\mathcal{B}(\mathcal{H})$.

Given these preparations, one can restate Example 2.3 in the case where $\mathcal{B} = \mathcal{B}_p(\mathcal{H})$ as follows.

Corollary 4.1. *Let $p \in [1, \infty)$, $b \in (0, \infty) \cup \{\infty\}$, and suppose that $F : (0, b) \to \mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $\|F(\cdot)\|_{\mathcal{B}_p(\mathcal{H})} \in L^p((0, b); x^\alpha dx)$, with $\alpha \in \mathbb{R}$ chosen according to items (i) and (ii) below:*

(i) If $\alpha < p - 1$, then (2.11) implies

$$\mathrm{tr}_{\mathcal{H}}\left(\int_0^b dx \ x^\alpha |F(x)|^p \right) \geqslant \left(\frac{|\alpha - p + 1|}{p} \right)^p \mathrm{tr}_{\mathcal{H}}\left(\int_0^b dx \ x^{\alpha - p} \left| \int_0^x dx' \ F(x') \right|^p \right).$$
(4.5)

(ii) If $\alpha > p - 1$, then (2.12) implies

$$\mathrm{tr}_{\mathcal{H}}\left(\int_0^b dx \ x^\alpha |F(x)|^p \right) \geqslant \left(\frac{|\alpha - p + 1|}{p} \right)^p \mathrm{tr}_{\mathcal{H}}\left(\int_0^b dx \ x^{\alpha - p} \left| \int_x^b dx' \ F(x') \right|^p \right).$$
(4.6)

In both cases (i) and (ii), the constant $[(|\alpha - p + 1|)/p]^p$ is best possible and equality holds if and only if $F = 0$ a.e. on $(0, b)$.

Proof. It suffices to consider item (i). Then an application of (4.1)–(4.4) yields

$$\left(\frac{|\alpha - p + 1|}{p} \right)^p \mathrm{tr}_{\mathcal{H}}\left(\int_0^b dx \ x^{\alpha - p} \left| \int_0^x dx' \ F(x') \right|^p \right)$$

$$= \left(\frac{|\alpha - p + 1|}{p} \right)^p \int_0^b dx \ x^{\alpha - p} \mathrm{tr}_{\mathcal{H}}\left(\left| \int_0^x dx' \ F(x') \right|^p \right) \quad \text{(by (4.3))}$$

$$= \left(\frac{|\alpha - p + 1|}{p} \right)^p \int_0^b dx \ x^{\alpha - p} \left\| \int_0^x dx' \ F(x') \right\|_{\mathcal{B}_p(\mathcal{H})}^p \quad \text{(by (4.2))}$$

$$\leqslant \left(\frac{|\alpha - p + 1|}{p} \right)^p \int_0^b dx \ x^{\alpha - p} \left(\int_0^x dx' \ \|F(x')\|_{\mathcal{B}_p(\mathcal{H})} \right)^p \quad \text{(by (4.4))}$$

$$\leqslant \int_0^b dx \ x^\alpha \|F(x)\|_{\mathcal{B}_p(\mathcal{H})}^p \quad \text{(by (2.11))}$$

$$= \int_0^b dx \ x^\alpha \mathrm{tr}_{\mathcal{H}}\left(|F(x)|^p \right) \quad \text{(by (4.2))}$$

$$= \mathrm{tr}_{\mathcal{H}}\left(\int_0^b dx \ x^\alpha |F(x)|^p \right) \quad \text{(by (4.3))}.$$
(4.7)

The final part about optimality of the constant on the right-hand side in (4.5) and (4.6), and the equality part, then follow as in Example 2.3. $\qquad\square$

We note that the case $\alpha = 0$, $b = \infty$, $F(\cdot) \geqslant 0$ a.e. on $(0, \infty)$ in (4.5) was proved by Hansen [16, Theorem 2.4] on the basis of a convexity argument (see also [17], [25]). Our strategy of proof is different and based on that in Theorem 2.2.

Next, following Hansen [16], we will remove the trace in Corollary 4.1 in the case where $p \in [1, 2]$.

We start by recalling [16, Lemma 2.1]:

Lemma 4.2. *Let $p \in [1, 2]$, and suppose that $F : (0, \infty) \to \mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $F(\cdot) \geqslant 0$ a.e. on $(0, \infty)$, and $\int_0^\infty dx \ x^{-1} F(x)^p \in \mathcal{B}(\mathcal{H})$. Then,*

$$\int_0^\infty dx \ x^{-1} F(x)^p \geqslant \int_0^\infty dx \ x^{-1-p} \left(\int_0^x dx' \ F(x') \right)^p.$$
(4.8)

The constant 1 on the right-hand side of the inequality (4.8) is best possible.

Employing Lemma 4.2 we can prove the principal result of this section.

Theorem 4.3. *Let $p \in [1,2]$, and suppose that $F : (0,\infty) \to \mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $F(\cdot) \geqslant 0$ a.e. on $(0,\infty)$, and $\int_0^\infty dx\, x^\alpha F(x)^p \in \mathcal{B}(\mathcal{H})$, with $\alpha \in \mathbb{R}$ chosen according to items (i) and (ii) below:*

(i) If $\alpha < p - 1$, then

$$\int_0^\infty dx\, x^\alpha\, F(x)^p \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^\infty dx\, x^{\alpha - p}\left(\int_0^x dx'\, F(x')\right)^p. \qquad (4.9)$$

(ii) If $\alpha > p - 1$, then

$$\int_0^\infty dx\, x^\alpha\, F(x)^p \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^\infty dx\, x^{\alpha - p}\left(\int_x^\infty dx'\, F(x')\right)^p. \qquad (4.10)$$

In both cases (i) and (ii), the constant $[(|\alpha + 1 - p|)/p]^p$ is best possible and equality holds if and only if $F = 0$ a.e. on $(0,b)$.

Proof. We start by proving item (i). Closely following the strategy of proof in [16, Theorem 2.3], we introduce

$$G(x) = F\left(x^{p/|\alpha - p + 1|}\right)x^{(1+\alpha)/|\alpha - p + 1|}, \quad x > 0, \qquad (4.11)$$

and the change of variables

$$y = x^{p/|\alpha - p + 1|}, \quad dy = [p/|\alpha - p + 1|]x^{(1+\alpha)/|\alpha - p + 1|}dx. \qquad (4.12)$$

Then Lemma 4.2 applied to G yields

$$\int_0^\infty dx\, x^{-1} G(x)^p = \int_0^\infty dx\, x^{-1} F\left(x^{p/|\alpha - p + 1|}\right)^p x^{p(1+\alpha)/|\alpha - p + 1|}$$

$$\geqslant \int_0^\infty dx\, x^{-1-p}\left(\int_0^x dx' G(x')\right)^p \quad \text{(by (4.8))}$$

$$= \int_0^\infty dx\, x^{-1-p}\left(\int_0^x dt\, F\left((t^{p/|\alpha - p + 1|}\right)t^{(1+\alpha)/|\alpha - p + 1|}\right)^p$$

$$= \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^\infty dx\, x^{-1-p}\left(\int_0^{x^{p/|\alpha - p + 1|}} dy\, F(y)\right)^p. \qquad (4.13)$$

Introducing another change of variables

$$w = x^{p/|\alpha - p + 1|}, \quad dw\, w^{-1} = [p/|\alpha - p + 1|]dx\, x^{-1}, \qquad (4.14)$$

then implies

$$\left(\frac{|\alpha - p + 1|}{p}\right)\int_0^\infty dw\, w^\alpha F(w)^p$$

$$= \left(\frac{|\alpha - p + 1|}{p}\right)\int_0^\infty dw\, w^{-1} F(w)^p w^{1+\alpha}$$

$$= \int_0^\infty dx\, x^{-1} F\left(x^{p/|\alpha - p + 1|}\right)^p x^{p(1+\alpha)/|\alpha - p + 1|}$$

$$\geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^\infty dx\, x^{-1-p}\left(\int_0^{x^{p/|\alpha - p + 1|}} dy\, F(y)\right)^p \quad \text{(by (4.13))}$$

$$= \left(\frac{|\alpha - p + 1|}{p}\right)^{p+1} \int_0^\infty dw\, w^{-1-(p-1-\alpha)}\left(\int_0^w dy\, F(y)\right)^p \quad \text{(by (4.14))}$$

$$= \left(\frac{|\alpha - p + 1|}{p} \right)^{p+1} \int_0^\infty dw \, w^{\alpha - p} \left(\int_0^w dy \, F(y) \right)^p, \tag{4.15}$$

proving (4.9).

While $\Phi(F) = x^{-1} \int_0^x dx' \, F(x')$ represents a positive, unital map (i.e., $\Phi(F) \geqslant 0$ if $F \geqslant 0$ and $\Phi(I_{\mathcal{H}}) = I_{\mathcal{H}}$), $\int_x^\infty dx' \, F(x')$ cannot possibly be of this type and hence one cannot simply follow the proof of [16, Theorem 2.3] to derive (4.10). Fortunately, the following elementary alternative approach applies. Introducing the change of variables,

$$y = 1/x, \quad G(y) = F(1/y)y^{-2}, \tag{4.16}$$

in (4.9) (w.r.t. x on either side in (4.9) and, especially, w.r.t. x' on the right-hand side of (4.9)) results in

$$\int_0^\infty dy \, y^\beta G(y)^p \geqslant \left(\frac{|\beta - p + 1|}{p} \right)^p \int_0^\infty dx \, x^{\beta - p} \left(\int_x^\infty dy \, G(y) \right)^p, \tag{4.17}$$

where $\beta = 2p - 2 - \alpha$, and hence $\alpha < p - 1$ is equivalent to $\beta > p - 1$.

The final part about optimality of the constant on the right-hand side in (4.9) and (4.10), and the equality part, then follow as in Corollary 4.1 from Example 2.3 upon taking the trace on either side of (4.9) and (4.10). $\qquad \square$

Again we note that the case $\alpha = 0$ in (4.9) was proved by Hansen [16, Theorem 2.3]; he also proved that Theorem 4.3 does not extend to $p > 2$.

While we focused on the underlying interval $(0, \infty)$ in Theorem 4.3, the analogous case $(0, b)$, $b \in (0, \infty)$ follows upon employing the variable transformations discussed in [27, p. 36–38].

Extending the definition of $(H_{\mp, \ell} F)(x)$, $x \in (0, \infty)$, $\ell \in \mathbb{N}$, in (2.13), (2.14) to the operator-valued context where $F : (0, \infty) \to \mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $F(\cdot) \geqslant 0$ a.e. on $(0, \infty)$, and for all $c \in (0, \infty)$, $\int_0^c dx \, F(x)^p \in \mathcal{B}(\mathcal{H})$ in connection with $H_{-, \ell}$ and $\int_c^\infty dx \, F(x)^p \in \mathcal{B}(\mathcal{H})$ in connection with $H_{+, \ell}$, the facts (4.9), (4.10) can be rewritten as

$$\begin{aligned} &\int_0^\infty dx \, x^{\gamma - p} [H_{\mp, 1}(F(\cdot))(x)]^p \\ &\leqslant \left(\frac{p}{|\gamma - p + 1|} \right)^p \int_0^\infty dx \, x^\gamma F(x)^p, \quad p \in [1, \infty), \ \gamma \lessgtr p - 1. \end{aligned} \tag{4.18}$$

Thus one obtains the following result.

Corollary 4.4. *Let $p \in [1, 2]$, and suppose that $F : (0, \infty) \to \mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $F(\cdot) \geqslant 0$ a.e. on $(0, \infty)$, and $\int_0^\infty dx \, x^\alpha F(x)^p \in \mathcal{B}(\mathcal{H})$, with $\alpha \in \mathbb{R}$ chosen according to (4.19) below. Then*

$$\begin{aligned} &\int_0^\infty dx \, x^\alpha F(x)^p \\ &\geqslant \prod_{k=1}^\ell \left(\frac{|\alpha - kp + 1|}{p} \right)^p \int_0^\infty dx \, x^{\alpha - \ell p} [H_{\mp, \ell}(F(\cdot))(x)]^p, \tag{4.19} \end{aligned}$$

$$\alpha \lessgtr \begin{cases} p - 1, \\ \ell p - 1, \end{cases} \quad \ell \in \mathbb{N}.$$

Proof. It suffices to iterate (4.18) by applying it to appropriate $F = F_\ell$ as in the derivation of (2.16). □

Replacing $F \geqslant 0$ by $|F| = (F^*F)^{1/2}$ and mimicking the differential version of the Hardy inequalities at the end of Section 2 yields

$$\int_0^\infty dx\, x^\alpha |f'(x)|^p \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^\infty dx\, x^{\alpha-p}|f(x)|^p, \qquad (4.20)$$
$$p \in [1,2], \ \alpha \in \mathbb{R}, \ f \in C_0^\infty((0,\infty); \mathcal{B}(\mathcal{H})).$$

Iterating (4.20) the yields as in (2.21)

$$\int_0^\infty dx\, x^\alpha |f^{(n)}(x)|^p \geqslant \frac{\prod_{j=1}^k |\alpha - jp + 1|^p}{p^{kp}} \int_0^\infty dx\, x^{\alpha-kp}|f^{(n-k)}(x)|^p, \qquad (4.21)$$
$$p \in [1,2], \ 1 \leqslant k \leqslant n, \ n \in \mathbb{N}, \ \alpha \in \mathbb{R}, \ f \in C_0^\infty((0,\infty); \mathcal{B}(\mathcal{H})).$$

REFERENCES

[1] W. Arendt, C. K. Batty, M. Hieber, F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Transforms*, Monographs in Mathematics, Vol. 96, Birkhäuser, Basel, 2001.

[2] A. A. Balinsky, W. D. Evans, and R. T. Lewis, *The Analysis and Geometry of Hardy's Inequality*, Universitext, Springer, 2015.

[3] H. Baumgärtel and M. Wollenberg, *Mathematical Scattering Theory*, Operator Theory: Advances and Applications, Vol. 9, Birkhäuser, Boston, 1983.

[4] M. S. Birman, *The spectrum of singular boundary problems*, Mat. Sb. (N.S.) **55** (97), 125–174 1961 (Russian). Engl. transl. in Amer. Math. Soc. Transl., Ser. 2, **53**, 23–80 (1966).

[5] M. S. Birman and M. Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, Reidel, Dordrecht, 1987.

[6] J. S. Bradley, *Hardy inequalities with mixed norms*, Canad. Math. Bull. **21**, 405–408 (1978).

[7] P. Cembranos and J. Mendoza, *Banach Spaces of Vector-Valued Functions*, Lecture Notes in Mathematics, Vol. 1676, Springer, Berlin, 1997.

[8] R. S. Chisholm and W. N. Everitt, *On bounded integral operators in the space of integrable-square functions*, Proc. Roy. Soc. Edinb. (A), **69**, 199–204 (1970/71).

[9] R. S. Chisholm, W. N. Everitt, and L. L. Littlejohn, *An integral operator inequality with applications*, J. of Inequal. Appls. **3**, 245–266 (1999).

[10] E. B. Davies, *Spectral Theory and Differential Operators*, Cambridge Studies in Advanced Mathematics, Vol. 42, Cambridge University Press, Cambridge, UK, 1995.

[11] J. Diestel and J. J. Uhl, *Vector Measures*, Mathematical Surveys, Vol. 15, Amer. Math. Soc., Providence, RI, 1977.

[12] D. E. Edmunds and W. D. Evans, *Hardy Operators, Function Spaces, and Embeddings*, Springer, Berlin, 2004.

[13] F. Gesztesy, L. Littlejohn, I. Michael, and R. Wellman, *On Birman's Sequence of Hardy–Rellich-Type Inequalities*, J. Diff. Eq. **264(4)**, 2761–2801 (2018).

[14] I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, Israel Program for Scientific Translations, Jerusalem, 1965, Daniel Davey & Co., Inc., New York, 1966.

[15] A. Gogatishvili, A. Kufner, L.-E. Persson, and A. Wedestig, *An equivalence theorem for integral conditions related to Hardy's inequality*, Real Anal. Exchange **29**, 867–880 (2003/04).

[16] F. Hansen, *Non-commutative Hardy inequalities*, Bull. Lond. Math. Soc. **41**, 1009–1016 (2009).

[17] F. Hansen, K. Krulić, J. Pečarić, and L.-E. Persson, *Generalized noncommutative Hardy and Hardy–Hilbert type inequalities*, Intl. J. Math. **21**, 1283–1295 (2010).

[18] G. H. Hardy, *Notes on some points in the integral calculus, XLI. On the convergence of certain integrals and series*, Messenger Math. **45**, 163–166 (1915).

[19] G. H. Hardy, *Notes on some points in the integral calculus, LI. On Hilbert's double-series theorem, and some connected theorems concerning the convergence of infinite series and integrals*, Messenger Math. **48**, 107–112 (1919).

[20] G. H. Hardy, *Note on a theorem of Hilbert*, Math. Z. **6**, 314–317 (1920).

[21] G. H. Hardy, *Notes on some points in the integral calculus, LX. An inequality between integrals*, Messenger Math. **54**, 150–156 (1925).

[22] G. H. Hardy, *Notes on some points in the integral calculus LXIV*, Messenger Math. **57**, 12–16 (1928).

[23] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, reprinted, 1988.

[24] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Colloquium Publications, Vol. 31, rev. ed., Amer. Math. Soc., Providence, RI, 1985.

[25] M. Kian, *On a Hardy operator inequality*, Positivity **22**, 773–781 (2018).

[26] A. Kufner, L. Maligranda and L.-E. Persson, *The Hardy Inequality: About its History and Some Related Results*, Vydavatelský Servis, Pllsen, 2007.

[27] A. Kufner, L.-E. Persson, and N. Samko, *Weighted Inequalities of Hardy Type*, 2nd ed., World Scientific, Singapore, 2017.

[28] S. T. Kuroda, *An Introduction to Scattering Theory*, Aarhus University Lecture Notes Series, No. 51, 1978.

[29] J. Mikusiński, *The Bochner Integral*, Academic Press, New York, 1978.

[30] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math. **44**, 31–38 (1972).

[31] M. Nassyrova, *Weighted Inequalities Involving Hardy-type and Limiting Geometric Mean Operators*, doctoral dissertation, Dept. of Math., Luleå University of Technology, March 2002.

[32] M. Nasyrova and V. Stepanov, *On weighted Hardy inequalities on semiaxis for functions vanishing at the endpoints*, J. Inequal. Appl. **1**, 223–238 (1997).

[33] M. Nasyrova and V. Stepanov, *On maximal overdetermined Hardy's inequality of second order on a finite interval*, Math. Bohem. **124**, 293–302 (1999).

[34] B. Opic and A. Kufner, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series, Vol. 219. Longman Scientific & Technical, Harlow, 1990.

[35] B. J. Pettis *On integration in vector spaces*, Trans. Am. Math. Soc. **44**, 277–304, (1938).

[36] G. Talenti, *Osservazioni sopra una classe di disuguaglianze*, Rend. Sem. Mat. Fis. Milano **39**, 171–185 (1969).

[37] G. Tomaselli, *A class of inequalities*, Boll. Un. Mat. Ital. (4) **2**, 622–631 (1969).

[38] K. Yosida, *Functional Analysis*, 6th ed., Springer, Berlin, 1980.

Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798-7328, USA
  E-mail address: Chian_Chuah@baylor.edu

Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798-7328, USA
  E-mail address: Fritz_Gesztesy@baylor.edu
  URL: https://www.baylor.edu/math/index.php?id=935340

Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798-7328, USA
  E-mail address: Lance_Littlejohn@baylor.edu
  URL: https://www.baylor.edu/math/index.php?id=53980

Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798-7328, USA
  E-mail address: Tao_Mei@baylor.edu
  URL: https://www.baylor.edu/math/index.php?id=925779

Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798-7328, USA
  E-mail address: Isaac_Michael@baylor.edu
  URL: http://blogs.baylor.edu/isaac_michael/

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
  E-mail address: pangm@missouri.edu
  URL: https://www.math.missouri.edu/people/pang