A SEQUENCE OF WEIGHTED BIRMAN–HARDY–RELLICH INEQUALITIES WITH LOGARITHMIC REFINEMENTS

FRITZ GESZTESY, LANCE L. LITTLEJOHN, ISAAC MICHAEL, AND MICHAEL M. H. PANG

Abstract. The principal aim of this paper is to extend Birman's sequence of integral inequalities

$$\int_0^\rho dx\, \big|f^{(m)}(x)\big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^\rho dx\, x^{-2m}|f(x)|^2,$$
$$f \in C_0^m((0,\rho)),\ m \in \mathbb{N}, \quad \rho \in (0,\infty) \cup \{\infty\},$$

originally obtained in 1961, and containing Hardy's and Rellich's inequality (i.e., $m = 1, 2$) as special cases, to a sequence of inequalities that incorporates power weights on either side and logarithmic refinements on the right-hand side of the inequality as well.

Introducing iterated logarithms given by

$$\ln_1(\,\cdot\,) = \ln(\,\cdot\,), \quad \ln_{j+1}(\,\cdot\,) = \ln(\ln_j(\,\cdot\,)), \quad j \in \mathbb{N},$$

and iterated exponentials,

$$e_0 = 0, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

a particular (but representative) extension of Birman's sequence we will prove then reads

$$\int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(\ell,\alpha) \int_0^\rho dx\, x^{\alpha-2\ell}\big|f^{(m-\ell)}(x)\big|^2$$
$$+ B(\ell,\alpha) \sum_{k=1}^N \int_0^\rho dx\, x^{\alpha-2\ell} \prod_{\ell=1}^k [\ln_\ell(\gamma/x)]^{-2}\big|f^{(m-\ell)}(x)\big|^2,$$
$$f \in C_0^\infty((0,\rho)),\ \ell, m, N \in \mathbb{N},\ 1 \leqslant \ell \leqslant m,\ \alpha \in \mathbb{R},\ \rho, \gamma \in (0,\infty),\ \gamma \geqslant e_N \rho.$$

Here the constants $A(p,\alpha)$ and $B(p,\alpha)$, $p \in \mathbb{N}$, are of the form

$$A(p,\alpha) = \prod_{j=1}^p \left(\frac{2j-1-\alpha}{2}\right)^2, \quad B(p,\alpha) = \frac{1}{4^p} \sum_{k=1}^p \prod_{\substack{j=1\\ j\neq k}}^p (2j-1-\alpha)^2.$$

The constants $A(\ell,\alpha)$ in the above extension of Birman's inequality are optimal, and so are the conditions on γ. Moreover, employing a new technique of proof relying on a combination of transforms originally due to Hartman and Müller-Pfeiffer, the parameter $\alpha \in \mathbb{R}$ in the power weights is now unrestricted, considerably improving on prior results in the literature.

We also indicate a vector-valued version of these inequalities, replacing complex-valued $f(\,\cdot\,)$ by $f(\,\cdot\,) \in \mathcal{H}$, with $\mathcal{H}$ a complex, separable Hilbert space.

Date: November 17, 2021.
2010 *Mathematics Subject Classification.* Primary: 26D10, 34A40; Secondary: 34L10.
Key words and phrases. Birman–Hardy–Rellich inequalities, logarithmic refinements.

1. Introduction

To be able to describe the content of this paper we start by recalling Birman's infinite sequence of integral inequalities [19], the sequence of Birman–Hardy–Rellich inequalities of the form

$$\begin{gathered}\int_a^b dx\,\big|f^{(m)}(x)\big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}}\int_a^b dx\,x^{-2m}|f(x)|^2,\\ f\in C_0^m((a,b)),\ m\in\mathbb{N},\quad 0\leqslant a<b\leqslant\infty,\end{gathered} \tag{1.1}$$

which appeared in 1961, and in English translation in 1966 (see also [47, pp. 83–84]). The case $m = 1$ in (1.1) represents Hardy's celebrated inequality [53], [54, Sect. 9.8] (see also [63, Chs. 1, 3, App.]), the case $m = 2$ is due to Rellich [86, Sect. II.7] (actually, in the multi-dimensional context). The inequalities (1.1) and their power weighted generalizations, that is, the first line in (1.10), are known to be strict, that is, equality holds in (1.1), resp., in the first line in (1.10) (in fact, for the entire inequality (1.10)) if and only if $f = 0$ on (a, b). Moreover, these inequalities are optimal, meaning, the constants $[(2m-1)!!]^2/2^{2m}$ in (1.1), respectively, the constants $A(m, \alpha)$ in (1.10) are sharp, although, this must be qualified as different authors frequently prove sharpness for different function spaces. In the present one-dimensional context at hand, sharpness of (1.1) (and typically, it's power weighted version, the first line in (1.10)), are often proved in an integral form (rather than the currently presented differential form) where $f^{(m)}$ on the left-hand side is replaced by F and f on the right-hand side by m repeated integrals over F. For pertinent one-dimensional sources, we refer, for instance, to [14, p. 3–5], [22], [25, p. 104–105], [44, 51, 53], [54, p. 240–243], [63, Ch. 3], [64, p. 5–11], [67, 76, 85]. We also note that higher-order Hardy inequalities, including various weight functions, are discussed in [62, Sect. 5], [63, Chs. 2–5], [64, Chs. 1–4], [65], and [84, Sect. 10] (however, Birman's sequence of inequalities (1.1) is not mentioned in these sources). In addition, there are numerous sources which treat multi-dimensional versions of these inequalities on various domains $\Omega \subseteq \mathbb{R}^n$, which, when specialized to radially symmetric functions (e.g., when Ω represents a ball), imply one-dimensional Birman–Hardy–Rellich-type inequalities with power weights under various restrictions on these weights (cf. Remarks 3.3 *(ii)* and A.3). However, none of the results obtained in this manner imply our principal result, (1.10), under optimal hypotheses on α and γ. We also mention that a large number of these references treat the L^p-setting, and in some references $x \in (a, b)$ is replaced by $d(x)$, the distance of x to the boundary of (a, b), respectively, Ω, but this represents quite a different situation (especially in the multi-dimensional context) and hence is not further discussed in this paper.

The primary aim in this paper is to prove optimal inequalities of the type (1.1) with additional weights (of power-type on either side of (1.1)) and logarithmic refinements (i.e., additional, only logarithmically weaker, singularities on the right-hand side of (1.1)). To describe our new results in detail, we need some preparations and introduce the iterated logarithms $\ln_j(\,\cdot\,)$, $j \in \mathbb{N}$ (cf. [55], [56, pp. 324–325])), given by

$$\ln_1(\,\cdot\,) = \ln(\,\cdot\,), \quad \ln_{j+1}(\,\cdot\,) = \ln(\ln_j(\,\cdot\,)), \quad j \in \mathbb{N}, \tag{1.2}$$

and also normalized iterated logarithms $L_j(\,\cdot\,)$, $j \in \mathbb{N}$ (see, e.g., [16]),

$$L_1(\,\cdot\,) = \big(1 - \ln(\,\cdot\,)\big)^{-1}, \quad L_{j+1}(\,\cdot\,) = L_1(L_j(\,\cdot\,)), \quad j \in \mathbb{N}. \tag{1.3}$$

In addition, we introduce iterated exponentials in the form,

$$e_0 = 0, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{1.4}$$

Moreover, for $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, we introduce the constants

$$A(m,\alpha) = \prod_{j=1}^{m} \left(\frac{2j-1-\alpha}{2}\right)^2, \tag{1.5}$$

$$B(m,\alpha) = \frac{1}{4^m} \sum_{k=1}^{m} \prod_{\substack{j=1 \\ j\neq k}}^{m} (2j-1-\alpha)^2. \tag{1.6}$$

One observes that

$$B(m,\alpha) = A(m,\alpha) \sum_{j=1}^{m} \frac{1}{(2j-1-\alpha)^2}, \quad m \in \mathbb{N},\ \alpha \in \mathbb{R}\backslash\{2j-1\}_{1\leqslant j\leqslant m}, \tag{1.7}$$

$$A(m,0) = \frac{[(2m-1)!!]^2}{2^{2m}}, \quad m \in \mathbb{N}, \tag{1.8}$$

in particular, $A(m,0)$ coincides with the constant in (1.1).

The improved Birman inequalities contain additional constants $c_\ell(m,\alpha)$, $\ell = 0,1,\dots,2m$, which are defined in terms of the polynomial

$$P_{m,\alpha}(\lambda) = \sum_{\ell=0}^{2m} c_\ell(m,\alpha)\lambda^\ell = \prod_{j=1}^{m} \left(\lambda^2 - \frac{(2j-1-\alpha)^2}{4}\right), \quad m \in \mathbb{N},\ \alpha \in \mathbb{R}. \tag{1.9}$$

Given the notation introduced in (1.2)–(1.9), we can now describe the principal results proved in this note: Let $\ell, m, N \in \mathbb{N}$, $1 \leqslant \ell \leqslant m$, $\alpha \in \mathbb{R}$, $\rho, \gamma \in (0,\infty)$, $\gamma \geqslant e_N \rho$, and $f \in C_0^\infty((0,\rho))$. Then the power-weighted Birman–Hardy–Rellich sequence with logarithmic refinements on the interior interval $(0,\rho)$ are of the form

$$\begin{aligned}
\int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 &\geqslant A(\ell,\alpha) \int_0^\rho dx\, x^{\alpha-2\ell} \big|f^{(m-\ell)}(x)\big|^2 \\
&+ B(\ell,\alpha) \sum_{k=1}^{N} \int_0^\rho dx\, x^{\alpha-2\ell} \prod_{\ell=1}^{k} [\ln_\ell(\gamma/x)]^{-2} \big|f^{(m-\ell)}(x)\big|^2 \\
&+ \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| A(j,0) \int_0^\rho dx\, x^{\alpha-2\ell} [\ln(\gamma/x)]^{-2j} \big|f^{(m-\ell)}(x)\big|^2 \\
&+ \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_0^\rho dx\, x^{\alpha-2\ell} [\ln(\gamma/x)]^{-2j} \\
&\qquad\qquad \times \prod_{p=1}^{k} [\ln_{p+1}(\gamma/x)]^{-2} \big|f^{(m-\ell)}(x)\big|^2.
\end{aligned} \tag{1.10}$$

Moreover, we prove the same sequence of inequalities on the exterior interval (ρ,∞) for $f \in C_0^\infty((\rho,\infty))$, and finally, both sets of inequalities (exterior and interior) are also proved with the iterated logarithms $\ln_j(\,\cdot\,)$ replaced by the normalized logarithms $L_j(\,\cdot\,)$, $j \in \mathbb{N}$. In the latter case an infinite series of logarithmic terms (i.e., the case $N = \infty$ in the analog of (1.10)) will be permitted. Furthermore, we show that all inequalities are strict, that is, equality holds if and only if $f = 0$

on $(0,\rho)$ (resp., (ρ,∞)). For brevity, a careful comparison of our result with the existing ones in the literature is postponed to Remarks 3.3 and A.3.

A multi-dimensional version of our approach, focusing on radial and logarithmic refinements of Birman–Hardy–Rellich-type inequalities, will appear in [42].

In Section 2 we introduce our principal tool, a combined Hartman–Müller-Pfeiffer transformation, our main results are then proved in Section 3. In Section 4 we derive the sequence of power-weighted Birman–Hardy–Rellich inequalities with logarithmic refinements in the vector-valued case, replacing complex-valued $f(\,\cdot\,)$ by $f(\,\cdot\,) \in \mathcal{H}$, with $\mathcal{H}$ a complex, separable Hilbert space. Finally, sharpness of the constants $A(m,\alpha)$ is derived in Appendix A.

2. The Combined Hartman–Müeller-Pfeiffer Transformation

In this section we introduce an elementary, yet extremely useful, variable transformation, an appropriate combination of special cases of transformations considered by Hartman [55] (see also [56, p. 324–325]) and Müller-Pfeiffer [77, p. 200–207]. We now introduce an extension of these transformations by Hartman and Müller-Pfeiffer applicable to power weights and higher-order derivatives. This will be crucial in proving the power-weighted Birman–Hardy–Rellich inequalities with logarithmic refinements under most general conditions in our principal Section 3.

Let $m, N \in \mathbb{N}$ and suppose that

$$\alpha \in \mathbb{R}\backslash\{j \,|\, 1 \leqslant j \leqslant 2m-1\}. \tag{2.1}$$

Given $f \in C_0^\infty((e_N,\infty))$, the transformation

$$x = e^t,\ x \in (e_N,\infty), \quad dx = e^t dt, \qquad t \in (e_{N-1},\infty), \tag{2.2}$$

$$f(x) \equiv f(e^t) = e^{[(2m-1-\alpha)/2]t} w(t), \quad w \in C_0^\infty((e_{N-1},\infty)), \tag{2.3}$$

yields

$$\big(x^\alpha f^{(m)}(x)\big)^{(m)} = e^{-[(2m+1-\alpha)/2]t} \sum_{\ell=0}^{2m} c_\ell(m,\alpha) w^{(\ell)}(t), \tag{2.4}$$

for appropriate constants $c_\ell(m,\alpha)$, $\ell = 0,1,\dots,2m$ to be determined next.

The solutions of the differential equation

$$\big(x^\alpha f^{(m)}(x)\big)^{(m)} = 0, \tag{2.5}$$

are linear combinations of the following powers of x:

$$\begin{cases} x^j, & j = 0,1,\dots,m-1, \\ x^{k-\alpha}, & k = m,\dots,2m-1. \end{cases} \tag{2.6}$$

One notes that the solutions (2.6) are linearly independent due to (2.1).

Thus, recalling (2.2)–(2.4), it follows that the solutions of

$$\sum_{\ell=0}^{2m} c_\ell(m,\alpha) w^{(\ell)}(t) = 0, \quad t \in (e_{N-1},\infty), \tag{2.7}$$

are the functions

$$e^{(\frac{1+\alpha}{2}-m)t} x^j = e^{(j+\frac{1+\alpha}{2}-m)t}, \quad j = 0,1,\dots,m-1, \tag{2.8}$$

and

$$e^{(\frac{1+\alpha}{2}-m)t} x^{k-\alpha} = e^{(k+\frac{1-\alpha}{2}-m)t} \quad k = m,\dots,2m-1. \tag{2.9}$$

Observe that for $j = 0$ and $k = 2m - 1$,

$$\begin{aligned} e^{(j+\frac{1+\alpha}{2}-m)t} &= e^{(\frac{1+\alpha}{2}-m)t} \\ e^{(k+\frac{1-\alpha}{2}-m)t} &= e^{-(\frac{1+\alpha}{2}-m)t}. \end{aligned} \tag{2.10}$$

For $j = 1$ and $k = 2m - 2$,

$$\begin{aligned} e^{(j+\frac{1+\alpha}{2}-m)t} &= e^{(\frac{3+\alpha}{2}-m)t} \\ e^{(k+\frac{1-\alpha}{2}-m)t} &= e^{-(\frac{3+\alpha}{2}-m)t}. \end{aligned} \tag{2.11}$$

Continuing iteratively, one concludes that the linearly independent solutions of (2.7) are of the form

$$e^{\pm\frac{1}{2}(2j+1-2m+\alpha)t}, \quad j = 0, 1, \dots, m-1, \tag{2.12}$$

By a simple relabeling, given $\alpha \in \mathbb{R}\backslash\{j \,|\, 1 \leqslant j \leqslant 2m-1\}$, this is equivalent to

$$e^{\pm\frac{1}{2}(2j-1-\alpha)t}, \quad j = 1, \dots, m, \ t \in (e_{N-1}, \infty), \tag{2.13}$$

are linearly independent solutions of (2.7). The zeros of the characteristic polynomial of (2.7) are thus the constant factors in the exponents of (2.13). Hence, the characteristic polynomial is given by

$$\begin{aligned} P_{m,\alpha}(\lambda) &= \sum_{\ell=0}^{2m} c_\ell(m,\alpha)\lambda^\ell \\ &= \bigg(\lambda^2 - \frac{(1-\alpha)^2}{4}\bigg)\bigg(\lambda^2 - \frac{(3-\alpha)^2}{4}\bigg)\cdots\bigg(\lambda^2 - \frac{(2m-1-\alpha)^2}{4}\bigg) \\ &= \prod_{j=1}^{m}\bigg(\lambda^2 - \frac{(2j-1-\alpha)^2}{4}\bigg). \end{aligned} \tag{2.14}$$

Thus, the coefficients $c_\ell(m,\alpha)$, $\ell = 0, 1, \dots, 2m$, satisfy the following properties:

$$\begin{aligned} &(i)\ c_{2j-1}(m,\alpha) = 0, \quad j = 1, \dots, m; \\ &(ii)\ c_{2j}(m,\alpha) = (-1)^{m-j}|c_{2j}(m,\alpha)|, \quad j = 0, 1, \dots, m; \\ &(iii)\ |c_0(m,\alpha)| = A(m,\alpha); \\ &(iv)\ |c_2(m,\alpha)| = 4B(m,\alpha); \\ &(v)\ c_{2m}(m,\alpha) = 1. \end{aligned} \tag{2.15}$$

Turning our attention to the iterated logarithms, given $N \in \mathbb{N}$, the transformation (2.2) (i.e., $x = e^t$, $x \in (e_N, \infty)$, $t \in (e_{N-1}, \infty)$) yields

$$\sum_{k=1}^{N}\prod_{j=1}^{k}[\ln_j(x)]^{-2} = t^{-2} + t^{-2}\sum_{k=1}^{N-1}\prod_{j=1}^{k}[\ln_j(t)]^{-2}, \tag{2.16}$$

interpreting $\sum_{k=1}^{0}(\,\cdot\,) = 0$.

3. Power-Weighted Birman–Hardy–Rellich-type Inequalities with Logarithmic Refinements

In this section we now establish several improvements of existing power-weighted Birman–Hardy–Rellich inequalities in the literature by employing the combined Hartman–Müeller-Pfeiffer variable transformation from section 2 in a crucial (and new) manner. These weighted inequalities are proved for both types of iterated logarithms $\ln_j(\,\cdot\,)$, $j \in \mathbb{N}$ and $L_j(\,\cdot\,)$, $j \in \mathbb{N}$, and are given on both the exterior interval (ρ,∞) and interior interval $(0,\rho)$ for any $\rho \in (0,\infty)$.

The principal new result of this paper then reads as follows:

Theorem 3.1. *Let $\ell, m, N \in \mathbb{N}, \alpha \in \mathbb{R}$, and $\rho, \gamma, \tau \in (0,\infty)$. The following hold:*
(i) If $\rho \geqslant e_N\gamma$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((\rho,\infty))$,

$$\begin{aligned}
&\int_\rho^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(\ell,\alpha)\int_\rho^\infty dx\, x^{\alpha-2\ell}\big|f^{(m-\ell)}(x)\big|^2 \\
&\quad + B(\ell,\alpha)\sum_{k=1}^N \int_\rho^\infty dx\, x^{\alpha-2\ell}\prod_{p=1}^k [\ln_p(x/\gamma)]^{-2}\big|f^{(m-\ell)}(x)\big|^2 \\
&\quad + \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)| A(j,0)\int_\rho^\infty dx\, x^{\alpha-2\ell}[\ln(x/\gamma)]^{-2j}\big|f^{(m-\ell)}(x)\big|^2 \\
&\quad + \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)| B(j,0)\sum_{k=1}^{N-1}\int_\rho^\infty dx\, x^{\alpha-2\ell}[\ln(x/\gamma)]^{-2j} \\
&\qquad\qquad \times \prod_{p=1}^k [\ln_{p+1}(x/\gamma)]^{-2}\big|f^{(m-\ell)}(x)\big|^2.
\end{aligned} \tag{3.1}$$

(ii) If $\rho \geqslant \tau$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((\rho,\infty))$,

$$\begin{aligned}
&\int_\rho^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(\ell,\alpha)\int_\rho^\infty dx\, x^{\alpha-2\ell}\big|f^{(m-\ell)}(x)\big|^2 \\
&\quad + B(\ell,\alpha)\sum_{k=1}^N \int_\rho^\infty dx\, x^{\alpha-2\ell}\prod_{p=1}^k [L_p(\tau/x)]^{2}\big|f^{(m-\ell)}(x)\big|^2 \\
&\quad + \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)| A(j,0)\int_\rho^\infty dx\, x^{\alpha-2\ell}[L_1(\tau/x)]^{2j}\big|f^{(m-\ell)}(x)\big|^2 \\
&\quad + \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)| B(j,0)\sum_{k=1}^{N-1}\int_\rho^\infty dx\, x^{\alpha-2\ell}[L_1(\tau/x)]^{2j} \\
&\qquad\qquad \times \prod_{p=1}^k [L_{p+1}(\tau/x)]^{2}\big|f^{(m-\ell)}(x)\big|^2.
\end{aligned} \tag{3.2}$$

(iii) If $\gamma \geqslant e_N\rho$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((0,\rho))$,

$$\int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(\ell,\alpha)\int_0^\rho dx\, x^{\alpha-2\ell}\big|f^{(m-\ell)}(x)\big|^2$$

$$
\begin{aligned}
&+ B(\ell,\alpha) \sum_{k=1}^{N} \int_0^{\rho} dx\, x^{\alpha-2\ell} \prod_{p=1}^{k} [\ln_p(\gamma/x)]^{-2} \big|f^{(m-\ell)}(x)\big|^2 \qquad (3.3)\\
&+ \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| A(j,0) \int_0^{\rho} dx\, x^{\alpha-2\ell} [\ln(\gamma/x)]^{-2j} \big|f^{(m-\ell)}(x)\big|^2\\
&+ \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_0^{\rho} dx\, x^{\alpha-2\ell} [\ln(\gamma/x)]^{-2j}\\
&\qquad\qquad \times \prod_{p=1}^{k} [\ln_{p+1}(\gamma/x)]^{-2} \big|f^{(m-\ell)}(x)\big|^2.
\end{aligned}
$$

(iv) If $\tau \geqslant \rho$ *and* $1 \leqslant \ell \leqslant m$, *then for all* $f \in C_0^{\infty}((0,\rho))$,

$$
\begin{aligned}
&\int_0^{\rho} dx\, x^{\alpha} \big|f^{(m)}(x)\big|^2 \geqslant A(\ell,\alpha) \int_0^{\rho} dx\, x^{\alpha-2\ell} \big|f^{(m-\ell)}(x)\big|^2\\
&\quad + B(\ell,\alpha) \sum_{k=1}^{N} \int_0^{\rho} dx\, x^{\alpha-2\ell} \prod_{p=1}^{k} [L_p(x/\tau)]^2 \big|f^{(m-\ell)}(x)\big|^2 \qquad (3.4)\\
&\quad + \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| A(j,0) \int_0^{\rho} dx\, x^{\alpha-2\ell} [L_1(x/\tau)]^{2j} \big|f^{(m-\ell)}(x)\big|^2\\
&\quad + \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_0^{\rho} dx\, x^{\alpha-2\ell} [L_1(x/\tau)]^{2j} \prod_{p=1}^{k} [L_{p+1}(x/\tau)^2 \big|f^{(m-\ell)}(x)\big|^2.
\end{aligned}
$$

(v) Inequalities (3.1)–(3.4) *are strict for* $f \not\equiv 0$ *on* (ρ,∞), *respectively,* $(0,\rho)$.

(vi) In the exceptional cases $\alpha \in \{2j-1\}_{1\leqslant j\leqslant \ell}$ *(i.e., if and only if* $A(\ell,\alpha)=0$*), the first terms containing* $A(\ell,\alpha)$ *on the right-hand sides of* (3.1)–(3.4) *are to be deleted.*

We break up the proof of Theorem 3.1 into four parts. For simplicity, we present the proof in the special case $\ell = m$; the general case follows upon replacing f by $f^{(m-\ell)}$ for $\ell = 1,\dots,m$.

Proof of Theorem 3.1 *(i)*. Let $\rho \geqslant e_N\gamma$, pick any $f \in C_0^{\infty}((\rho,\infty))$, and assume that $\alpha \in \mathbb{R}$ satisfies (2.1). The scaling

$$
x = \gamma y, \quad dx = \gamma dy, \quad g(y) = f(\gamma y), \quad y \in (\rho/\gamma,\infty) \subseteq (e_N,\infty), \tag{3.5}
$$

implies $g \in C_0^{\infty}((\rho/\gamma,\infty))$. Applying the transformation (2.2), (2.3) to g, that is, employing

$$
\begin{aligned}
&x/\gamma = y = e^t, \quad dx/\gamma = dy = e^t dt, \quad t \in (\ln(\rho/\gamma),\infty),\\
&f(x) = g(y) = e^{[(2m-1-\alpha)/2]t} w(t), \quad w \in C_0^{\infty}((\ln(\rho/\gamma),\infty)),
\end{aligned} \tag{3.6}
$$

then yields

$$
\big(y^{\alpha} g^{(m)}(y)\big)^{(m)} = e^{-[(2m+1-\alpha)/2]t} \sum_{j=0}^{m} (-1)^{m-j} |c_{2j}(m,\alpha)| w^{(2j)}(t), \tag{3.7}
$$

for $t \in (\ln(\rho/\gamma), \infty)) \subseteq (e_{N-1}, \infty)$, and $c_{2j}(m, \alpha)$ as in (2.15). Thus,

$$(-1)^m \big(y^\alpha g^{(m)}(y)\big)^{(m)} \overline{g(y)} = e^{-t} \sum_{j=0}^{m} (-1)^{2m-j} |c_{2j}(m, \alpha)| w^{(2j)}(t) \overline{w(t)}. \tag{3.8}$$

Furthermore, (2.2), (2.3), and (2.16) yield

$$y^{\alpha-2m} |g(y)|^2 = e^{-t} |w(t)|^2, \tag{3.9}$$
$$y^{\alpha-2m} \sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(y)]^{-2} |g(y)|^2 = e^{-t} \bigg\{ t^{-2} |w(t)|^2 + t^{-2} \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_p(t)]^{-2} |w(t)|^2 \bigg\},$$

and for $j = 2, \dots, m$,

$$y^{\alpha-2m} [\ln(y)]^{-2j} |g(y)|^2 = e^{-t} t^{-2j} |w(t)|^2, \tag{3.10}$$
$$y^{\alpha-2m} [\ln(y)]^{-2j} \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_{p+1}(y)]^{-2} |g(y)|^2 = e^{-t} t^{-2j} \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_p(t)]^{-2} |w(t)|^2.$$

Employing the elementary identity,

$$\begin{gathered} \int_a^b dx\, x^\alpha \big|f^{(m)}(x)\big|^2 = (-1)^m \int_a^b dx\, \big(x^\alpha f^{(m)}(x)\big)^{(m)} \overline{f(x)}, \\ m \in \mathbb{N},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((a,b)),\ 0 \leqslant a < b \leqslant \infty, \end{gathered} \tag{3.11}$$

and items (*iii*), (*iv*) of (2.15), it follows from (3.5)–(3.10) that

$$\begin{aligned} &\int_\rho^\infty dx\, \bigg\{ x^\alpha \big|f^{(m)}(x)\big|^2 - A(m,\alpha) x^{\alpha-2m} |f(x)|^2 \\ &\quad - B(m,\alpha) x^{\alpha-2m} \sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(x/\gamma)]^{-2} |f(x)|^2 \\ &\quad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| A(j,0) x^{\alpha-2m} [\ln(x/\gamma)]^{-2j} |f(x)|^2 \\ &\quad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| B(j,0) x^{\alpha-2m} [\ln(x/\gamma)]^{-2j} \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_{p+1}(x/\gamma)]^{-2} |f(x)|^2 \bigg\} \\ &= \gamma^{\alpha-2m+1} \int_{\rho/\gamma}^\infty dy\, \bigg\{ y^\alpha \big|g^{(m)}(y)\big|^2 - A(m,\alpha) y^{\alpha-2m} |g(y)|^2 \\ &\quad - B(m,\alpha) y^{\alpha-2m} \sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(y)]^{-2} |g(y)|^2 \\ &\quad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| A(j,0) y^{\alpha-2m} [\ln(y)]^{-2j} |g(y)|^2 \\ &\quad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| B(j,0) y^{\alpha-2m} [\ln(y)]^{-2j} \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_{p+1}(y)]^{-2} |g(y)|^2 \bigg\} \\ &= \gamma^{\alpha-2m+1} \bigg\{ \sum_{j=0}^{m} |c_{2j}(m,\alpha)| \int_{\ln(\rho/\gamma)}^\infty dt\, \big|w^{(j)}(t)\big|^2 - A(m,\alpha) \int_{\ln(\rho/\gamma)}^\infty dt\, |w(t)|^2 \end{aligned}$$

$$
\begin{aligned}
&\qquad - B(m,\alpha)\int_{\ln(\rho/\gamma)}^{\infty} dt\, t^{-2}|w(t)|^2 \\
&\qquad - B(m,\alpha)\sum_{k=1}^{N-1}\int_{\ln(\rho/\gamma)}^{\infty} dt\, t^{-2}\prod_{p=1}^{k}[\ln_p(t)]^{-2}|w(t)|^2 \\
&\qquad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|A(j,0)\int_{\ln(\rho/\gamma)}^{\infty} dt\, t^{-2j}|w(t)|^2 \\
&\qquad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|B(j,0)\sum_{k=1}^{N-1}\int_{\ln(\rho/\gamma)}^{\infty} dt\, t^{-2j}\prod_{p=1}^{k}[\ln_p(t)]^{-2}|w(t)|^2\bigg\} \\
&= \gamma^{\alpha-2m+1}\bigg\{\sum_{j=1}^{m}|c_{2j}(m,\alpha)|\int_{\ln(\rho/\gamma)}^{\infty} dt\,\big|w^{(j)}(t)\big|^2 \\
&\qquad - \sum_{j=1}^{m}|c_{2j}(m,\alpha)|A(j,0)\int_{\ln(\rho/\gamma)}^{\infty} dt\, t^{-2j}|w(t)|^2 \\
&\qquad - \sum_{j=1}^{m}|c_{2j}(m,\alpha)|B(j,0)\sum_{k=1}^{N-1}\int_{\ln(\rho/\gamma)}^{\infty} dt\, t^{-2j}\prod_{p=1}^{k}[\ln_p(t)]^{-2}|w(t)|^2\bigg\} \\
&= \gamma^{\alpha-2m+1}\sum_{j=1}^{m}|c_{2j}(m,\alpha)|\bigg\{\int_{\ln(\rho/\gamma)}^{\infty} dt\,\big|w^{(j)}(t)\big|^2 - A(j,0)\int_{\ln(\rho/\gamma)}^{\infty} dt\, t^{-2j}|w(t)|^2 \\
&\qquad\qquad - B(j,0)\sum_{k=1}^{N-1}\int_{\ln(\rho/\gamma)}^{\infty} dt\, t^{-2j}\prod_{p=1}^{k}[\ln_p(t)]^{-2}|w(t)|^2\bigg\}, \\
&\qquad\qquad\qquad w\in C_0^{\infty}((\ln(\rho/\gamma),\infty)), \qquad (3.12)
\end{aligned}
$$

interpreting $\sum_{k=1}^{0}(\,\cdot\,)=0$.

Hence, part (i), for $\alpha\in\mathbb{R}\backslash\{j\,|\,1\leqslant j\leqslant 2m-1\}$, follows via induction over $N\in\mathbb{N}$. Indeed, for $N=1$ equality (3.12) yields (cf. (2.3))

$$
\begin{aligned}
&\int_{\rho}^{\infty} dx\,\bigg\{x^{\alpha}\big|f^{(m)}(x)\big|^2 - A(m,\alpha)x^{\alpha-2m}|f(x)|^2 - B(m,\alpha)x^{\alpha-2m}[\ln(x/\gamma)]^{-2}|f(x)|^2 \\
&\qquad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|A(j,0)x^{\alpha-2m}[\ln(x/\gamma)]^{-2j}|f(x)|^2\bigg\} \\
&= \gamma^{\alpha-2m+1}\sum_{j=1}^{m}|c_{2j}(m,\alpha)|\bigg\{\int_{\ln(\rho/\gamma)}^{\infty} dt\,\big|w^{(j)}(t)\big|^2 - A(j,0)\int_{\ln(\rho/\gamma)}^{\infty} dt\, t^{-2j}|w(t)|^2\bigg\} \\
&\geqslant 0, \quad w\in C_0^{\infty}((\ln(\rho/\gamma),\infty)), \qquad (3.13)
\end{aligned}
$$

by (1.1) as a sum of unweighted Birman–Hardy–Rellich-type inequalities. Assuming (3.1) holds for $N-1\in\mathbb{N}$ then reapplying (3.12) proves (3.1) for $N\in\mathbb{N}$. Strictness also follows by induction over $N\in\mathbb{N}$ since $f\not\equiv 0$ implies $w\not\equiv 0$ by (2.2), (2.3) so that (3.13), and by induction, (3.12) is strictly positive.

The case $\alpha\in\{j\,|\,1\leqslant j\leqslant 2m-1\}$ then follows by taking the limits $\alpha\to k\in\{j\,|\,1\leqslant j\leqslant 2m-1\}$, noting that $A(m,\alpha)$, $B(m,\alpha)$, and $c_{2j}(m,\alpha)$ are continuous as polynomials in $\alpha\in\mathbb{R}$. This completes the proof of part (i). $\square$

Proof of Theorem 3.1 (ii). By taking limits as in part (i), it suffices once more to consider $\alpha \in \mathbb{R}\backslash\{2j-1\}_{1\leqslant j\leqslant m}$. Let $\rho \geqslant \tau$ and pick any $f \in C_0^\infty((\rho,\infty))$. The scaling

$$x = \tau y, \quad dx = \tau dy, \quad g(y) = f(\tau y), \quad y \in (\rho/\tau, \infty), \tag{3.14}$$

yields $g \in C_0^\infty((\rho/\tau,\infty)) \subseteq C_0^\infty((1,\infty))$. One modifies the transformation (2.2), (2.3) applied to g by

$$\begin{aligned} &y = e^{t-1}, \quad dy = e^{t-1}dt, \quad t \in (1,\infty), \\ &g(y) \equiv g(e^{t-1}) = e^{[(2m-1-\alpha)/2](t-1)}v(t), \quad v \in C_0^\infty((1,\infty)), \end{aligned} \tag{3.15}$$

where v is given by

$$v(t) := w(t-1), \quad t \in (1,\infty), \tag{3.16}$$

with $w \in C_0^\infty((0,\infty))$. Setting

$$s = t-1, \quad ds = dt, \tag{3.17}$$

and noting

$$\frac{d}{dt}v(t) = \frac{d}{ds}w(s), \tag{3.18}$$

yields, similarly to (3.7),

$$\begin{aligned} \big(y^\alpha g^{(m)}(y)\big)^{(m)} &= e^{-[(2m+1-\alpha)/2]s}\sum_{\ell=0}^{2m} c_\ell(m,\alpha)w^{(\ell)}(s) \\ &= e^{-[(2m+1-\alpha)/2](t-1)}\sum_{\ell=0}^{2m} c_\ell(m,\alpha)v^{(\ell)}(t). \end{aligned} \tag{3.19}$$

Hence, an analogous argument as in section 2 shows the constants $c_\ell(m,\alpha)$ satisfy (i)–(v) in (2.15) as before. Therefore by (3.19),

$$(-1)^m\big(y^\alpha g^{(m)}(y)\big)^{(m)}\overline{g(y)} = e^{1-t}\sum_{j=0}^{m}(-1)^{2m-j}|c_{2j}(m,\alpha)|v^{(2j)}(t)\overline{v(t)}. \tag{3.20}$$

Now, (3.15) yields

$$L_1(1/y) = \big(1-\ln(1/y)\big)^{-1} = \big(1-\ln(e^{1-t})\big)^{-1} = t^{-1}, \tag{3.21}$$

and

$$L_2(1/y) = L_1(L_1(1/y)) = L_1(1/t). \tag{3.22}$$

Inductively, we see that

$$L_1(1/y) = t^{-1}, \quad L_j(1/y) = L_{j-1}(1/t), \quad j = 2,3,\dots \tag{3.23}$$

Hence,

$$\begin{aligned} &y^{\alpha-2m}|g(y)|^2 = e^{1-t}|v(t)|^2, \\ &y^{\alpha-2m}\sum_{k=1}^{N}\prod_{p=1}^{k}L_p^2(1/y)|g(y)|^2 = e^{1-t}\bigg\{t^{-2}|v(t)|^2 + t^{-2}\sum_{k=1}^{N-1}\prod_{p=1}^{k}L_p^2(1/t)|v(t)|^2\bigg\}, \end{aligned} \tag{3.24}$$

and for $j = 2,\dots,m$,

$$y^{\alpha-2m}L_1^{2j}(1/y)|g(y)|^2 = e^{1-t}t^{-2j}|v(t)|^2, \tag{3.25}$$

$$y^{\alpha-2m}L_1^{2j}(1/y)\sum_{k=1}^{N-1}\prod_{p=1}^{k}L_{p+1}^2(1/y)|g(y)|^2 = e^{1-t}t^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k}L_p^2(1/t)|v(t)|^2.$$

Again recalling (3.11) and (*iii*)–(*iv*) of (2.15), (3.20), (3.24), and (3.25) yield (cf. (3.15))

$$\begin{aligned}
&\int_\rho^\infty dx\,\bigg\{x^\alpha\big|f^{(m)}(x)\big|^2 - A(m,\alpha)x^{\alpha-2m}|f(x)|^2\\
&\qquad - B(m,\alpha)x^{\alpha-2m}\sum_{k=1}^{N}\prod_{p=1}^{k}L_p^2(\tau/x)|f(x)|^2\\
&\qquad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|A(j,0)x^{\alpha-2m}L_1^{2j}(\tau/x)|f(x)|^2\\
&\qquad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|B(j,0)x^{\alpha-2m}L_1^{2j}(\tau/x)\sum_{k=1}^{N-1}\prod_{p=1}^{k}L_{p+1}^2(\tau/x)|f(x)|^2\bigg\}\\
&= \tau^{\alpha-2m+1}\sum_{j=1}^{m}|c_{2j}(m,\alpha)|\bigg\{\int_1^\infty dt\,\big|v^{(j)}(t)\big|^2 - A(j,0)\int_1^\infty dt\,t^{-2j}|v(t)|^2\\
&\qquad - B(j,0)\sum_{k=1}^{N-1}\int_1^\infty dt\,t^{-2j}\prod_{p=1}^{k}L_p^2(1/t)|v(t)|^2\bigg\},\\
&\qquad v\in C_0^\infty((1,\infty)), \qquad (3.26)
\end{aligned}$$

and the proof again follows by induction over $N\in\mathbb{N}$. $\square$

Proof of Theorem 3.1 (*iii*). Consider again $\alpha\in\mathbb{R}\backslash\{j\,|\,1\leqslant j\leqslant 2m-1\}$. Let $\gamma\geqslant e_N\rho$ and pick any $f\in C_0^\infty((0,\rho))$. The scaling

$$x=\gamma y,\quad dx=\gamma dy,\quad y\in(0,\rho/\gamma),\quad g(y)=f(\gamma y), \tag{3.27}$$

yields $g\in C_0^\infty((0,\rho/\gamma))$. Slightly modifying the transformation (2.2), (2.3) applied to g leads to

$$\begin{aligned}
&y=e^{-t},\quad dy=-e^{-t}dt,\quad t\in(\ln(\gamma/\rho),\infty),\\
&g(y)\equiv g(e^{-t})=e^{-[(2m-1-\alpha)/2]t}u(t),\quad u\in C_0^\infty((\ln(\gamma/\rho),\infty)).
\end{aligned}\tag{3.28}$$

This implies

$$\big(y^\alpha g^{(m)}(y)\big)^{(m)} = e^{[(2m+1-\alpha)/2]t}\sum_{\ell=0}^{2m}(-1)^\ell c_\ell(m,\alpha)u^{(\ell)}(t), \tag{3.29}$$

and hence, (*i*)–(*v*) in (2.15) still hold. Thus,

$$(-1)^m\big(y^\alpha g^{(m)}(y)\big)^{(m)}\overline{g(y)} = e^t\sum_{j=0}^{m}(-1)^{2m-j}|c_{2j}(m,\alpha)|u^{(2j)}(t)\overline{u(t)}. \tag{3.30}$$

Furthermore,

$$y^{\alpha-2m}|g(y)|^2 = e^t|u(t)|^2, \tag{3.31}$$

$$y^{\alpha-2m}\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(1/y)]^{-2}|g(y)|^2 = e^t\bigg\{t^{-2}|u(t)|^2 + t^{-2}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(t)]^{-2}|u(t)|^2\bigg\},$$

and for $j = 2, \dots, m$,

$$y^{\alpha-2m}[\ln(1/y)]^{-2j}|g(y)|^2 = e^t t^{-2j}|u(t)|^2, \tag{3.32}$$
$$y^{\alpha-2m}[\ln(1/y)]^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_{p+1}(1/y)]^{-2}|g(y)|^2 = e^t t^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(t)]^{-2}|u(t)|^2.$$

Applying (3.30)–(3.32) yields

$$\begin{aligned}
&\int_0^\rho dx\,\bigg\{x^\alpha\big|f^{(m)}(x)\big|^2 - A(m,\alpha)x^{\alpha-2m}|f(x)|^2\\
&\quad - B(m,\alpha)x^{\alpha-2m}\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(\gamma/x)]^{-2}|f(x)|^2\\
&\quad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|A(j,0)x^{\alpha-2m}[\ln(\gamma/x)]^{-2j}|f(x)|^2\\
&\quad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|B(j,0)x^{\alpha-2m}[\ln(\gamma/x)]^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_{p+1}(\gamma/x)]^{-2}|f(x)|^2\bigg\}\\
&= \gamma^{\alpha-2m+1}\sum_{j=1}^{m}|c_{2j}(m,\alpha)|\bigg\{\int_{\ln(\gamma/\rho)}^\infty dt\,\big|u^{(j)}(t)\big|^2 - A(j,0)\int_{\ln(\gamma/\rho)}^\infty dt\,t^{-2j}|u(t)|^2\\
&\quad - B(j,0)\sum_{k=1}^{N-1}\int_{\ln(\gamma/\rho)}^\infty dt\,t^{-2j}\prod_{p=1}^{k}[\ln_p(t)]^{-2}|u(t)|^2\bigg\},\\
&\qquad u \in C_0^\infty((\ln(\gamma/\rho),\infty)),
\end{aligned} \tag{3.33}$$

and the proof follows by induction over $N \in \mathbb{N}$, as before. $\square$

Proof of Theorem 3.1 *(iv)*. Once more, consider $\alpha \in \mathbb{R}\backslash\{j \,|\, 1 \leqslant j \leqslant 2m-1\}$. Suppose $\tau \geqslant \rho$, $f \in C_0^\infty((0,\rho))$, and use the scaling

$$x = \tau y, \quad dx = \tau dy, \quad y \in (0,\rho/\tau), \quad g(y) = f(\tau y), \tag{3.34}$$

so that $g \in C_0^\infty((0,\rho/\tau)) \subseteq C_0^\infty((0,1))$. Next, one applies the modified transformation

$$\begin{aligned}
&y = e^{-t+1}, \quad dy = -e^{-t+1}dt, \quad t \in (1,\infty),\\
&g(y) \equiv g(e^{-t+1}) = e^{[(2m-1-\alpha)/2](1-t)}v(t), \quad v \in C_0^\infty((1,\infty)),
\end{aligned} \tag{3.35}$$

where v is given by

$$v(t) := w(1-t), \quad t \in (1,\infty), \tag{3.36}$$

with $w \in C_0^\infty((-\infty,0))$. Therefore

$$(-1)^m\big(y^\alpha g^{(m)}(y)\big)^{(m)}\overline{g(y)} = e^{t-1}\sum_{j=0}^{m}(-1)^{2m-j}|c_{2j}(m,\alpha)|v^{(2j)}(t)\overline{v(t)}. \tag{3.37}$$

Also,

$$y^{\alpha-2m}|g(y)|^2 = e^{t-1}|v(t)|^2, \tag{3.38}$$
$$y^{\alpha-2m}\sum_{k=1}^{N}\prod_{p=1}^{k}L_p^2(y)|g(y)|^2 = e^{t-1}\bigg\{t^{-2}|v(t)|^2 + t^{-2}\sum_{k=1}^{N-1}\prod_{p=1}^{k}L_p^2(1/t)|v(t)|^2\bigg\},$$

and for $j = 2, \dots, m$,

$$y^{\alpha-2m} L_1^{2j}(y) |g(y)|^2 = e^{t-1} t^{-2j} |v(t)|^2, \tag{3.39}$$
$$y^{\alpha-2m} L_1^{2j}(y) \sum_{k=1}^{N-1} \prod_{p=1}^{k} L_{p+1}^2(y) |g(y)|^2 = e^{t-1} t^{-2j} \sum_{k=1}^{N-1} \prod_{p=1}^{k} L_p^2(1/t) |v(t)|^2.$$

Hence,

$$\begin{aligned}
&\int_0^\rho dx \bigg\{ x^\alpha \big|f^{(m)}(x)\big|^2 - A(m,\alpha) x^{\alpha-2m} |f(x)|^2 \\
&\qquad - B(m,\alpha) x^{\alpha-2m} \sum_{k=1}^{N} \prod_{p=1}^{k} L_p^2(x/\tau) |f(x)|^2 \\
&\qquad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| A(j,0) x^{\alpha-2m} L_1^{2j}(x/\tau) |f(x)|^2 \\
&\qquad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| B(j,0) x^{\alpha-2m} L_1^{2j}(x/\tau) \sum_{k=1}^{N-1} \prod_{p=1}^{k} L_{p+1}^2(x/\tau) |f(x)|^2 \bigg\} \\
&= \tau^{\alpha-2m+1} \sum_{j=1}^{m} |c_{2j}(m,\alpha)| \bigg\{ \int_1^\infty dt \, \big|v^{(j)}(t)\big|^2 - A(j,0) \int_1^\infty dt \, t^{-2j} |v(t)|^2 \\
&\qquad - B(j,0) \sum_{k=1}^{N-1} \int_1^\infty dt \, t^{-2j} \prod_{p=1}^{k} L_p^2(1/t) |v(t)|^2 \bigg\}, \\
&\qquad\qquad v \in C_0^\infty((1,\infty)),
\end{aligned} \tag{3.40}$$

and the proof follows again by induction over $N \in \mathbb{N}$. $\square$

Theorem 3.1 (ii), (iv) can be further improved by replacing the N-th sum with an infinite series. See, for example, [16, 50, 95] for similar results and discussions of the convergence of the series $\sum_{k=1}^\infty \prod_{j=1}^k L_j^2(s)$ for $s \in (0,1)$.

Corollary 3.2. *Let $\ell, m \in \mathbb{N}, \alpha \in \mathbb{R}$, and $\rho, \tau \in (0,\infty)$. Then* (3.2) *and* (3.4) *extend to $N = \infty$.*

Proof. It suffices to discuss the proof of (3.2). Given $f \in C_0^\infty((\rho,\infty))$, Theorem 3.1 (ii) implies that (3.2) holds for any $N \in \mathbb{N}$. Thus, by taking $N \uparrow \infty$ and recalling that increasing sequences bounded above are convergent, (3.2) holds with $N = \infty$. $\square$

To put our results in perspective and to compare with existing results in the literature, we offer some comments next.

Remark 3.3. (i) Theorem 3.1 (i),(ii) (resp., Theorem 3.1 (iii),(iv)) extends to $N = \rho = 0$ (resp., $N = 0$, $\rho = \infty$) upon disregarding all logarithmic terms (i.e., upon putting $B(\ell,\alpha) = c_{2j}(\ell,\alpha) = 0$, $2 \leqslant j \leqslant \ell$, $1 \leqslant \ell \leqslant m$), we omit the details.

(ii) Originally, logarithmic refinements of Hardy's inequality started with oscillation theoretic considerations going back to Hartman [55] (see also [56, p. 324–325]) and have been used in connection with Hardy's inequality in [39, 45], and more recently in [40, 41]. Since then there has been enormous activity in this context and we mention, for instance, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12], [14, Chs. 3, 5], [16, 17, 18, 21, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 40, 46, 48, 49], [50, Chs. 2,6,7], [58, 59, 69, 70, 71, 72, 74, 75, 79, 81, 82], [86, Sect. 2.7], [87, 88, 89, 93, 94, 95, 96]. The vast majority of these references deals with analogous multi-dimensional settings (relevant to our setting in particular in the case of radially symmetric functions), most also in the L^p-context.

(iii) For $m \geqslant 2$ these inequalities are new in the following sense: The weight parameter $\alpha \in \mathbb{R}$ is now unrestricted (as opposed to prior results, see item (ii) of this remark) and at the same time the conditions on the logarithmic parameters γ and τ are sharp. Moreover, the two integral terms containing $c_{2j}(m,\alpha)$ are new in this generality (we note that a single term of the type $x^{-2m}[\ln(\gamma/x)]^{-4}$ appeared in [4] and [21]; and [77, Ch. 6] discusses sums involving even powers of $[\ln(\gamma/x)]^{-1}$). We also note that the inequalities are proved for both iterated logarithms $\ln_j(\,\cdot\,)$ and $L_j(\,\cdot\,)$, $j \in \mathbb{N}$, and finally they are proved on both the exterior interval (ρ,∞) and interior interval $(0,\rho)$ for any $\rho \in (0,\infty)$. $\diamond$

We conclude this section by extending Theorem 3.1 from C_0^∞-functions to functions in appropriately weighted Sobolev spaces.

To this end we introduce the norms on $C_0^\infty((a,b))$,

$$\|f\|_{m,\alpha}^2 = \sum_{k=0}^m \int_a^b dx\, x^\alpha \big|f^{(k)}(x)\big|^2, \quad |||f|||_{m,\alpha}^2 = \int_a^b dx\, x^\alpha \big|f^{(m)}(x)\big|^2, \\ 0 \leqslant a < b \leqslant \infty,\ m \in \mathbb{N},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((a,b)). \tag{3.41}$$

and define the weighted Sobolev spaces

$$H_0^m\big((a,b); x^\alpha dx\big) = \overline{C_0^\infty((a,b))}^{\|\,\cdot\,\|_{m,\alpha}}, \quad m \in \mathbb{N},\ \alpha \in \mathbb{R}, \tag{3.42}$$

and the corresponding homogeneous weighted Sobolev spaces

$$\dot{H}_0^m\big((a,b); x^\alpha dx\big) = \overline{C_0^\infty((a,b))}^{|||\,\cdot\,|||_{m,\alpha}}, \quad m \in \mathbb{N},\ \alpha \in \mathbb{R}. \tag{3.43}$$

In the case $b < \infty$ we also note the following (higher-order) weighted Poincaré-type inequality:

Lemma 3.4. *Let $\rho \in (0,\infty)$, $k,m \in \mathbb{N}$, $0 \leqslant k \leqslant m-1$, $\alpha \in \mathbb{R}$. Then there exists $C_{k,m} = C(k,m,\alpha,\rho) \in (0,\infty)$ such that*

$$C_{k,m}\big\|f^{(k)}\big\|_{L^2((0,\rho);x^\alpha dx)}^2 = C_{k,m}|||f|||_{k,\alpha}^2 \leqslant |||f|||_{m,\alpha}^2 = \big\|f^{(m)}\big\|_{L^2((0,\rho);x^\alpha dx)}^2, \\ f \in C_0^\infty((0,\rho)). \tag{3.44}$$

Proof. If $A(m-k,\alpha) \neq 0$, one can use the simplest inequality in Theorem 3.1 (iii) to conclude for $f \in C_0^\infty((0,\rho))$,

$$\begin{split} \int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 &\geqslant A(m-k,\alpha) \int_0^\rho dx\, x^{\alpha-2(m-k)} \big|f^{(k)}(x)\big|^2 \\ &\geqslant A(m-k,\alpha)\rho^{-2(m-k)} \int_0^\rho dx\, x^\alpha \big|f^{(k)}(x)\big|^2. \end{split} \tag{3.45}$$

If $A(m-k,\alpha) = 0$, one uses the next simplest inequality in Theorem 3.1 (iii) to infer

$$\int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant B(m-k,\alpha) \int_0^\rho dx\, x^{\alpha-2(m-k)} [\ln(\gamma/x)]^{-2} \big|f^{(k)}(x)\big|^2$$

$$\begin{aligned}&\geqslant B(m-k,\alpha)\bigg(\int_0^\eta+\int_\eta^\rho\bigg)dx\,x^{\alpha-2(m-k)}[\ln(\gamma/x)]^{-2}\big|f^{(k)}(x)\big|^2\\&\geqslant C_{k,m}\int_0^\rho dx\,x^\alpha\big|f^{(k)}(x)\big|^2,\end{aligned}\tag{3.46}$$

where $\eta\in(0,\rho)$ is chosen such that $x^{-2(m-k)}[\ln(\gamma/x)]^{-2}$ is strictly monotonically decreasing on the interval $(0,\eta)$. $\square$

Thus, for $\rho\in(0,\infty)$, (3.44) implies equivalence of the norms $\|\cdot\|_{m,\alpha}$ and $|||\cdot|||_{m,\alpha}$ on $C_0^\infty((0,\rho))$ since repeated application of (3.44) yields,

$$\begin{aligned}|||f|||_{m,\alpha}^2\leqslant\|f\|_{m,\alpha}^2=|||f|||_{m,\alpha}^2+\sum_{k=0}^{m-1}|||f|||_{k,\alpha}^2\leqslant C|||f|||_{m,\alpha}^2,&\\ f\in C_0^\infty((0,\rho)),\ m\in\mathbb{N},&\end{aligned}\tag{3.47}$$

with $C=C(m,\alpha,\rho)\in(0,\infty)$. In particular,

$$H_0^m\big((0,\rho);x^\alpha dx\big)=\overset{\bullet}{H}{}_0^m\big((0,\rho);x^\alpha dx\big),\quad\rho\in(0,\infty),\ m\in\mathbb{N},\ \alpha\in\mathbb{R}.\tag{3.48}$$

Of course, since x^α is bounded from above and from below near $x=\rho$,

$$\begin{aligned}f\in\overset{\bullet}{H}{}_0^m\big((0,\rho);x^\alpha dx\big)=H_0^m\big((0,\rho);x^\alpha dx\big),\quad\rho\in(0,\infty),\\ \text{implies }\ f(\rho)=f'(\rho)=\cdots=f^{(m-1)}(\rho)=0.\end{aligned}\tag{3.49}$$

Given these preparations, we can now extend Theorem 3.1 as follows:

Theorem 3.5. *Under the hypotheses in Theorem* 3.1*, items* (i) *and* (ii) *extend from* $f\in C_0^\infty((\rho,\infty))$ *to* $f\in\overset{\bullet}{H}{}_0^m\big((\rho,\infty);x^\alpha dx\big)$ *and items* (iii) *and* (iv) *extend from* $f\in C_0^\infty((0,\rho))$ *to* $f\in\overset{\bullet}{H}{}_0^m\big((0,\rho);x^\alpha dx\big)=H_0^m\big((0,\rho);x^\alpha dx\big)$.

Proof. Since the proofs of items (i)–(iv) follow the same route based on combining Theorem 3.1 with Fatou's lemma, it suffices to focus on cases (i) and (iii).

(i). We start with the finite interval case (iii). Since $C_0^\infty((0,\rho))$ is dense in $H_0^m\big((0,\rho);x^\alpha dx\big)$ (in the norm $\|\cdot\|_{m,\alpha}$), given $f\in H_0^m\big((0,\rho);x^\alpha dx\big)$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}\subset C_0^\infty((0,\rho))$ such that $\lim_{n\to\infty}\big\|f_n-f\big\|_{m,\alpha}^2=0$, explicitly,

$$\lim_{n\to\infty}\int_0^\rho dx\,x^\alpha\big|f_n^{(k)}(x)-f^{(k)}(x)\big|^2=0,\quad 0\leqslant k\leqslant m,\ \alpha\in\mathbb{R}.\tag{3.50}$$

Hence, for each $0\leqslant k\leqslant m$, one can find a subsequence $\{f_{n_p,k}\}_{p\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ such that

$$x^{\alpha/2}f_{n_p,k}^{(k)}\underset{p\to\infty}{\longrightarrow}x^{\alpha/2}f^{(k)}\ \text{ pointwise a.e. on }(0,\rho),\tag{3.51}$$

equivalently,

$$f_{n_p,k}^{(k)}\underset{p\to\infty}{\longrightarrow}f^{(k)}\ \text{ pointwise a.e. on }(0,\rho).\tag{3.52}$$

Hence, abbreviating

$$\begin{aligned}w_{\ell,\alpha,N}(x)&=A(\ell,\alpha)+B(\ell,\alpha)\sum_{k=1}^N\prod_{p=1}^k[\ln_p(\gamma/x)]^{-2}\\&\quad+\sum_{j=2}^{\ell}|c_{2j}(\ell,\alpha)|A(j,0)[\ln(\gamma/x)]^{-2j}\end{aligned}\tag{3.53}$$

$$+\sum_{j=2}^{\ell}|c_{2j}(\ell,\alpha)|B(j,0)\sum_{k=1}^{N-1}[\ln(\gamma/x)]^{-2j}\prod_{p=1}^{k}[\ln_{p+1}(\gamma/x)]^{-2}, \quad x\in(0,\rho),$$

(a well-known consequence of) Fatou's lemma (cf., e.g., [37, Corollary 2.19]) and inequality (3.3) imply

$$\begin{aligned}
&\int_0^\rho dx\, x^{\alpha-2\ell} w_{\ell,\alpha,N}(x)\big|f^{(m-\ell)}(x)\big|^2 \\
&\quad \leqslant \liminf_{p\to\infty}\int_0^\rho dx\, x^{\alpha-2\ell} w_{\ell,\alpha,N}(x)\big|f_{n_p,m-\ell}^{(m-\ell)}(x)\big|^2 \quad \text{(by Fatou's lemma)} \\
&\quad = \lim_{p\to\infty}\int_0^\rho dx\, x^{\alpha-2\ell} w_{\ell,\alpha,N}(x)\big|f_{n_p,m-\ell}^{(m-\ell)}(x)\big|^2 \\
&\quad \leqslant \lim_{p\to\infty}\int_0^\rho dx\, x^{\alpha}\big|f_{n_p,m-\ell}^{(m)}(x)\big|^2 \quad \text{(by (3.3))} \\
&\quad = \int_0^\rho dx\, x^{\alpha}\big|f^{(m)}(x)\big|^2 \quad \text{(by (3.50) with } k=m).
\end{aligned} \tag{3.54}$$

(ii). To treat the interval (ρ,∞) one can argue as follows. Using arguments analogous to those in the proof of [44, Proposition 3.1], one shows that the space

$$\begin{aligned}
H_{m,\alpha}([\rho,\infty)) = \big\{f:[\rho,\infty)\to\mathbb{C}\,\big|\,&\text{for all } R>\rho,\, f^{(k)}\in AC([\rho,R]),\ 0\leqslant k\leqslant m-1; \\
&f^{(k)}(\rho)=0,\ 0\leqslant k\leqslant m-1;\ f^{(m)}\in L^2\big((\rho,\infty);x^\alpha dx\big)\big\},
\end{aligned} \tag{3.55}$$

is a Hilbert space space with respect to the norm $|||\cdot|||_{m,\alpha}$ associated with the inner product

$$\langle f,g\rangle_{m,\alpha} = \int_\rho^\infty x^\alpha dx\, \overline{f^{(m)}(x)}g^{(m)}(x), \quad f,g\in H_{m,\alpha}([\rho,\infty)). \tag{3.56}$$

The fact $C_0^\infty((\rho,\infty))\subset H_{m,\alpha}([\rho,\infty))$ naturally leads to the introduction of the space $\dot{H}_0^m\big((\rho,\infty);x^\alpha dx\big)$ as the closure of $C_0^\infty((\rho,\infty))$ in the norm $|||\cdot|||_{m,\alpha}$ in accordance with (3.43). Then a routine argument (see Appendix B for details) shows that if $f\in\dot{H}_0^m\big((\rho,\infty);x^\alpha dx\big)$ then there exists a sequence $\{f_n\}_{n\in\mathbb{N}}\subset C_0^\infty((\rho,\infty))$ such that for $0\leqslant k\leqslant m$,

$$\lim_{n\to\infty} f_n^{(k)}(x) = f^{(k)}(x) \text{ for a.e. } x\geqslant\rho. \tag{3.57}$$

At this point one can follow the Fatou-type argument in (3.54). □

4. The Vector-Valued Case

In our final section, we establish that all previous inequalities extend line by line to the vector-valued case in which f is $\mathcal{H}$-valued, with $\mathcal{H}$ a separable, complex Hilbert space. The relevance of such a generalization is briefly mentioned at the end of this section.

We start by stating a power-weighted extension of (1.1) for vector-valued functions, which is derived from the more general Hardy result [23, Example 1] by simple iteration (see also [44, Theorem 8.1] for the special case $\alpha=0$, $a=0$, $b=\infty$). Inequality (4.1) will replace (1.1) in the base step of each induction proof.

Lemma 4.1. *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}\backslash\{2j-1\}_{1\leqslant j\leqslant m}$, $0 \leqslant a < b \leqslant \infty$. Then for all $f \in C_0^\infty((a,b);\mathcal{H})$,*

$$\int_a^b dx\, x^\alpha \big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 \geqslant A(m,\alpha) \int_a^b dx\, x^{\alpha-2m}\|f(x)\|_{\mathcal{H}}^2. \tag{4.1}$$

The constant $A(m,\alpha)$ is sharp and equality holds if and only if $f=0$ on (a,b).

In addition, the combined Hartman–Müeller-Pfeiffer transformation extends to the $\mathcal{H}$-valued context. Indeed, given $m, N \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\alpha \neq 1,\dots,2m-1$, and $f \in C_0^\infty((e_N,\infty);\mathcal{H})$, one sets

$$\begin{aligned} &x = e^t, \quad dx = e^t dt, \quad t \in (e_{N-1},\infty), \\ &f(x) \equiv f(e^t) = e^{(m-\frac{1+\alpha}{2})t} w(t), \quad w \in C_0^\infty((e_{N-1},\infty);\mathcal{H}), \end{aligned} \tag{4.2}$$

so that

$$\big(x^\alpha f^{(m)}(x)\big)^{(m)} = e^{-(m+\frac{1-\alpha}{2})t} \sum_{\ell=0}^{2m} c_\ell(m,\alpha) w^{(\ell)}(t). \tag{4.3}$$

Combining (4.2) and (4.3) yields

$$(-1)^m\Big(\big(x^\alpha f^{(m)}(x)\big)^{(m)}, f(x)\Big)_{\mathcal{H}} = e^{-t}\sum_{j=0}^{m}(-1)^{2m-j}|c_{2j}(m,\alpha)|\big(w^{(2j)}(t), w(t)\big)_{\mathcal{H}}. \tag{4.4}$$

Furthermore,

$$\begin{aligned} &x^{\alpha-2m}\|f(x)\|_{\mathcal{H}}^2 = e^{-t}\|w(t)\|_{\mathcal{H}}^2, \\ &x^{\alpha-2m}\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(x)]^{-2}\|f(x)\|_{\mathcal{H}}^2 \\ &\quad = e^{-t}\bigg\{t^{-2}\|w(t)\|_{\mathcal{H}}^2 + t^{-2}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(t)]^{-2}\|w(t)\|_{\mathcal{H}}^2\bigg\}, \end{aligned} \tag{4.5}$$

and for $j = 2,\dots,m$,

$$\begin{aligned} &x^{\alpha-2m}[\ln(x)]^{-2j}\|f(x)\|_{\mathcal{H}}^2 = e^{-t}t^{-2j}\|w(t)\|_{\mathcal{H}}^2, \\ &x^{\alpha-2m}[\ln(x)]^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_{p+1}(x)]^{-2}\|f(x)\|_{\mathcal{H}}^2 \\ &\quad = e^{-t}t^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(t)]^{-2}\|w(t)\|_{\mathcal{H}}^2. \end{aligned} \tag{4.6}$$

The modified variable transformations (3.15), (3.28), (3.35), generalize analogously.

Finally, we note that (3.11) extends to the vector-valued situation in the form

$$\int_a^b dx\, x^\alpha \big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 = (-1)^m \int_a^b dx\, \Big(\big(x^\alpha f^{(m)}(x)\big)^{(m)}, f(x)\Big)_{\mathcal{H}}, \tag{4.7}$$

for $f \in C_0^\infty((a,b);\mathcal{H})$, where $0 \leqslant a < b \leqslant \infty, m \in \mathbb{N}, \alpha \in \mathbb{R}$.

Given these preliminaries, the vector-valued case becomes completely analogous to the scalar situation treated in Section 3:

Theorem 4.2. *Let $\ell, m, N \in \mathbb{N}, \alpha \in \mathbb{R}$, and $\rho, \gamma, \tau \in (0,\infty)$. The following hold:*

(i) If $\rho \geqslant e_N \gamma$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((\rho,\infty);\mathcal{H})$,

$$\begin{aligned}
\int_\rho^\infty dx\, x^\alpha \big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 &\geqslant A(\ell,\alpha) \int_\rho^\infty dx\, x^{\alpha-2\ell} \big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\
&\quad + B(\ell,\alpha) \sum_{k=1}^N \int_\rho^\infty dx\, x^{\alpha-2\ell} \prod_{p=1}^k [\ln_p(x/\gamma)]^{-2} \big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\
&\quad + \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| A(j,0) \int_\rho^\infty dx\, x^{\alpha-2\ell} [\ln(x/\gamma)]^{-2j} \big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\
&\quad + \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_\rho^\infty dx\, x^{\alpha-2\ell} [\ln(x/\gamma)]^{-2j} \\
&\qquad\qquad \times \prod_{p=1}^k [\ln_{p+1}(x/\gamma)]^{-2} \big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2.
\end{aligned} \tag{4.8}$$

(ii) If $\rho \geqslant \tau$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((\rho,\infty);\mathcal{H})$,

$$\begin{aligned}
\int_\rho^\infty dx\, x^\alpha \big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 &\geqslant A(\ell,\alpha) \int_\rho^\infty dx\, x^{\alpha-2\ell} \big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\
&\quad + B(\ell,\alpha) \sum_{k=1}^N \int_\rho^\infty dx\, x^{\alpha-2\ell} \prod_{p=1}^k L_p^2(\tau/x) \big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\
&\quad + \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| A(j,0) \int_\rho^\infty dx\, x^{\alpha-2\ell} L_1^{2j}(\tau/x) \big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\
&\quad + \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_\rho^\infty dx\, x^{\alpha-2\ell} L_1^{2j}(\tau/x) \prod_{p=1}^k L_{p+1}^2(\tau/x) \big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2.
\end{aligned} \tag{4.9}$$

(iii) If $\gamma \geqslant e_N \rho$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((0,\rho);\mathcal{H})$,

$$\begin{aligned}
\int_0^\rho dx\, x^\alpha \big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 &\geqslant A(\ell,\alpha) \int_0^\rho dx\, x^{\alpha-2\ell} \big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\
&\quad + B(\ell,\alpha) \sum_{k=1}^N \int_0^\rho dx\, x^{\alpha-2\ell} \prod_{p=1}^k [\ln_p(\gamma/x)]^{-2} \big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\
&\quad + \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| A(j,0) \int_0^\rho dx\, x^{\alpha-2\ell} [\ln(\gamma/x)]^{-2j} \big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\
&\quad + \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_0^\rho dx\, x^{\alpha-2\ell} [\ln(\gamma/x)]^{-2j} \\
&\qquad\qquad \times \prod_{p=1}^k [\ln_{p+1}(\gamma/x)]^{-2} \big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2.
\end{aligned} \tag{4.10}$$

(iv) If $\tau \geqslant \rho$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((0,\rho);\mathcal{H})$,

$$\begin{aligned}
&\int_0^\rho dx\, x^\alpha \big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 \geqslant A(\ell,\alpha)\int_0^\rho dx\, x^{\alpha-2\ell}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\
&\quad + B(\ell,\alpha)\sum_{k=1}^N \int_0^\rho dx\, x^{\alpha-2\ell}\prod_{p=1}^k L_p^2(x/\tau)\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\
&\quad + \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)|A(j,0)\int_0^\rho dx\, x^{\alpha-2\ell}L_1^{2j}(x/\tau)\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\
&\quad + \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)|B(j,0)\sum_{k=1}^{N-1}\int_0^\rho dx\, x^{\alpha-2\ell}L_1^{2j}(x/\tau)\prod_{p=1}^k L_{p+1}^2(x/\tau)\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2.
\end{aligned} \tag{4.11}$$

(v) Inequalities (4.8)–(4.11) are strict for $f \not\equiv 0$ on (ρ,∞), respectively, $(0,\rho)$.

(vi) In the exceptional cases $\alpha \in \{2\ell-1\}_{1\leqslant \ell\leqslant m}$ (i.e., if and only if $A(\ell,\alpha)=0$), the first terms containing $A(\ell,\alpha)$ on the right-hand sides of (4.8)–(4.11) are to be deleted.

Corollary 4.3. *Let $\ell, m \in \mathbb{N}, \alpha \in \mathbb{R}$, and $\rho,\tau \in (0,\infty)$. Then (4.9) and (4.11) extend to $N=\infty$.*

Using Lemma 4.1 and identity (4.7) for the base step in the induction proof over $N \in \mathbb{N}$, one can follow the special scalar case treated in the proof of Theorem 3.1, and Corollary 3.2 line by line.

Remark 4.4. Theorem 4.2 can also be proved using the following alternative consideration. We will illustrate the case of Theorem 4.2 (i): Since $\mathcal{H}$ is separable, let $\{\phi_r\}_{r\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$ and let f_r, $r \in \mathbb{N}$, be the "coordinate functions" of f with respect to $\{\phi_r\}_{r\in\mathbb{N}}$, that is, $f_r(x) = (\phi_r, f(x))_{\mathcal{H}}$, $x \in (\rho,\infty)$, $r \in \mathbb{N}$. Then $f_r \in C_0^\infty((\rho,\infty))$ and $f_r^{(m)}(x) = \big(\phi_r, f^{(m)}(x)\big)_{\mathcal{H}}$, $x \in (\rho,\infty)$, $m,r \in \mathbb{N}$. Applying Theorem 3.1 (i) to f_r one obtains

$$\begin{aligned}
&\int_\rho^\infty dx\, x^\alpha \big|f_r^{(m)}(x)\big|^2 \geqslant A(\ell,\alpha)\int_\rho^\infty dx\, x^{\alpha-2\ell}\big|f_r^{(m-\ell)}(x)\big|^2 \\
&\quad + B(\ell,\alpha)\sum_{k=1}^N \int_\rho^\infty dx\, x^{\alpha-2\ell}\prod_{p=1}^k [\ln_p(x/\gamma)]^{-2}\big|f_r^{(m-\ell)}(x)\big|^2 \\
&\quad + \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)|A(j,0)\int_\rho^\infty dx\, x^{\alpha-2\ell}[\ln(x/\gamma)]^{-2j}\big|f_r^{(m-\ell)}(x)\big|^2 \\
&\quad + \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)|B(j,0)\sum_{k=1}^{N-1}\int_\rho^\infty dx\, x^{\alpha-2\ell}[\ln(x/\gamma)]^{-2j} \\
&\qquad\qquad \times \prod_{p=1}^k [\ln_{p+1}(x/\gamma)]^{-2}\big|f_r^{(m-\ell)}(x)\big|^2.
\end{aligned} \tag{4.12}$$

Summing over $r \in \mathbb{N}$, one obtains

$$\int_\rho^\infty dx\, x^\alpha \sum_{r=1}^\infty \big|f_r^{(m)}(x)\big|^2 \geqslant A(\ell,\alpha)\int_\rho^\infty dx\, x^{\alpha-2\ell}\sum_{r=1}^\infty \big|f_r^{(m-\ell)}(x)\big|^2$$

$$
\begin{aligned}
&+ B(\ell,\alpha)\sum_{k=1}^{N}\int_\rho^\infty dx\, x^{\alpha-2\ell}\prod_{p=1}^{k}[\ln_p(x/\gamma)]^{-2}\sum_{r=1}^{\infty}\big|f_r^{(m-\ell)}(x)\big|^2 \\
&+ \sum_{j=2}^{\ell}|c_{2j}(\ell,\alpha)|A(j,0)\int_\rho^\infty dx\, x^{\alpha-2\ell}[\ln(x/\gamma)]^{-2j}\sum_{r=1}^{\infty}\big|f_r^{(m-\ell)}(x)\big|^2 \\
&+ \sum_{j=2}^{\ell}|c_{2j}(\ell,\alpha)|B(j,0)\sum_{k=1}^{N-1}\int_\rho^\infty dx\, x^{\alpha-2\ell}[\ln(x/\gamma)]^{-2j} \\
&\qquad\qquad \times \prod_{p=1}^{k}[\ln_{p+1}(x/\gamma)]^{-2}\sum_{r=1}^{\infty}\big|f_r^{(m-\ell)}(x)\big|^2,
\end{aligned} \tag{4.13}
$$

which is (4.8). Theorem 4.2 *(ii)*–*(iv)* can be proved in a similar manner. ◇

As in the scalar case, the constants $A(m,\alpha)$, $\alpha\in\mathbb{R}\backslash\{2j-1\}_{1\leqslant j\leqslant m}$, are sharp and the inequalities extend to the associated weighted Sobolev spaces of $\mathcal{H}$-valued functions; we omit the details.

We conclude with the observation that the vector-valued Hardy case (i.e., $m=1$) without logarithmic refinements (i.e., $N=0$), played an important role in the spectral theory of n-dimensional Schrödinger operators ($n\in\mathbb{N}$, $n\geqslant 2$) as detailed, for instance in [66, Chs. IV, V]. In this context one employs polar coordinates and $\mathcal{H}$ is then naturally identified with $L^2(S^{n-1};d^{n-1}\omega)$. This aspect will also play a crucial role in the multi-dimensional generalizations of the results presented in this note, see [42].

Appendix A. Optimality of $A(m,\alpha)$

In this appendix we demonstrate sharpness of the constants $A(\ell,\alpha)$, $1\leqslant\ell\leqslant m$.

Theorem A.1. *The constants $A(\ell,\alpha)$, $1\leqslant\ell\leqslant m$, $\alpha\in\mathbb{R}\backslash\{2j-1\}_{1\leqslant j\leqslant\ell}$, in Theorems* 3.1 *and* 3.5 *are sharp.*

Proof. For simplicity, we consider the interval $(0,\rho)$ (the case (ρ,∞) being completely analogous).

To simplify notation we assume, without loss of generality, that $\rho>2$ for the remainder of this proof.

We first present the proof for the case $\ell=m$ and near the end indicate the necessary changes to treat the analogous cases $1\leqslant\ell\leqslant m-1$, $m\geqslant 2$. Introducing

$$
y_0(x)=x^{(2\ell-1-\alpha)/2},\quad x>0,\ \ell\in\mathbb{N},\ \alpha\in\mathbb{R}, \tag{A.1}
$$

one notes the facts

$$
y_0^{(\ell)}(x)=2^{-\ell}(2\ell-1-\alpha)(2\ell-3-\alpha)\cdots(3-\alpha)(1-\alpha)x^{-(1+\alpha)/2}, \tag{A.2}
$$

$$
x^\alpha\big[y_0^{(\ell)}\big]^2=A(\ell,\alpha)x^{\alpha-2\ell}[y_0(x)]^2=A(\ell,\alpha)x^{-1}, \tag{A.3}
$$

$$
(-1)^\ell\big(x^\alpha y_0^{(\ell)}(x)\big)^{(\ell)}-A(\ell,\alpha)x^{\alpha-2\ell}y_0(x)=0. \tag{A.4}
$$

Next, we also introduce the cutoff functions

$$
\phi\in C^\infty(\mathbb{R}),\quad 0\leqslant\phi(x)\leqslant 1,\ x\in\mathbb{R},\quad \phi(x)=\begin{cases}0, & x\leqslant 1,\\ 1, & x\geqslant 2,\end{cases} \tag{A.5}
$$

$$
\phi_\varepsilon(x)=\phi(x/\varepsilon),\ x\in\mathbb{R},\ \ 0<\varepsilon \text{ sufficiently small}, \tag{A.6}
$$

$$\psi \in C^\infty(\mathbb{R}), \quad 0 \leqslant \psi(x) \leqslant 1,\, x \in \mathbb{R}, \quad \psi(x) = \begin{cases} 1, & x \leqslant \rho - 2, \\ 0, & x \geqslant \rho - 1, \end{cases} \tag{A.7}$$

and mollify y_0 as follows,

$$y_{0,\varepsilon}(x) = y_0(x)\phi_\varepsilon(x)\psi(x),\ 0 \leqslant x \leqslant \rho, \quad y_{0,\varepsilon} \in C_0^\infty((0,\rho)). \tag{A.8}$$

Then one verifies

$$\begin{aligned} A(\ell,\alpha)\int_0^\rho dx\, x^{\alpha-2\ell}[y_{0,\varepsilon}(x)]^2 &= A(\ell,\alpha)\int_0^\rho dx\, x^{-1}\phi(x/\varepsilon)^2\psi(x)^2 \\ &= A(\ell,\alpha)\int_\varepsilon^{\rho-2} dx\, x^{-1}\phi(x/\varepsilon)^2 + A(\ell,\alpha)\int_{\rho-2}^{\rho-1} dx\, x^{-1}\psi(x)^2 \\ &= A(\ell,\alpha)\int_2^{(\rho-2)/\varepsilon} d\xi\, \xi^{-1}\phi(\xi)^2 + A(\ell,\alpha)\int_1^2 d\xi\, \xi^{-1}\phi(\xi)^2 \\ &\quad + A(\ell,\alpha)\int_{\rho-2}^{\rho-1} dx\, x^{-1}\psi(x)^2 \\ &\underset{\varepsilon\downarrow 0}{=} A(\ell,\alpha)\ln(1/\varepsilon) + O(1), \end{aligned} \tag{A.9}$$

and

$$\begin{aligned} \int_0^\rho dx\, x^\alpha\big[y_{0,\varepsilon}^{(\ell)}(x)\big]^2 &= \int_\varepsilon^{\rho-2} dx\, x^\alpha\big[y_{0,\varepsilon}^{(\ell)}(x)\big]^2 + \int_{\rho-2}^{\rho-1} dx\, x^\alpha\big[y_{0,\varepsilon}^{(\ell)}(x)\big]^2 \\ &= \int_\varepsilon^{\rho-2} dx\, x^\alpha\big\{[(y_0(x)\phi(x/\varepsilon)]^{(\ell)}\big\}^2 + \int_{\rho-2}^{\rho-1} dx\, x^\alpha\big\{[y_0(x)\psi(x)]^{(\ell)}\big\}^2. \end{aligned} \tag{A.10}$$

Next, one employs

$$\begin{aligned} [y_0(x)\phi(x/\varepsilon)]^{(\ell)} &= \sum_{k=0}^{\ell}\binom{\ell}{k} y_0^{(\ell-k)}(x)\frac{d^k}{dx^k}\phi(x/\varepsilon) \\ &= x^{-(1+\alpha)/2}\sum_{k=0}^{\ell} c_{\ell,k,\alpha}(x/\varepsilon)^k\phi^{(k)}(x/\varepsilon), \end{aligned} \tag{A.11}$$

where

$$\begin{aligned} &c_{\ell,0,\alpha} = 2^{-\ell}(2\ell-1-\alpha)(2\ell-3-\alpha)\cdots(3-\alpha)(1-\alpha), \\ &c_{\ell,0,\alpha}^2 = A(\ell,\alpha). \end{aligned} \tag{A.12}$$

Thus, one can continue (A.10) as follows:

$$\begin{aligned} (\text{A.10}) &= \int_\varepsilon^{\rho-2} dx\, x^{-1}\left[\sum_{k=0}^{\ell} c_{\ell,k,\alpha}(x/\varepsilon)^k\phi^{(k)}(x/\varepsilon)\right]^2 \\ &\quad + \int_{\rho-2}^{\rho-1} dx\, x^\alpha\big\{[y_0(x)\psi(x)]^{(\ell)}\big\}^2 \\ &= \int_1^{(\rho-2)/\varepsilon} d\xi\, \xi^{-1}\left[\sum_{k=0}^{\ell} c_{\ell,k,\alpha}\xi^k\phi^{(k)}(\xi)\right]^2 + \int_{\rho-2}^{\rho-1} dx\, x^\alpha\big\{[y_0(x)\psi(x)]^{(\ell)}\big\}^2 \\ &= \int_1^{(\rho-2)/\varepsilon} d\xi\, \xi^{-1}\left[c_{\ell,0,\alpha}\phi(\xi) + \sum_{k=1}^{\ell} c_{\ell,k,\alpha}\xi^k\phi^{(k)}(\xi)\right]^2 \end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{\rho-2}^{\rho-1} dx\, x^{\alpha} \big\{[y_0(x)\psi(x)]^{(\ell)}\big\}^2 \\
&= A(\ell,\alpha) \int_1^{(\rho-2)/\varepsilon} d\xi\, \xi^{-1} \phi(\xi)^2 \\
&\quad + \int_1^2 d\xi\, \xi^{-1} \Bigg\{ 2c_{\ell,0,\alpha}\phi(\xi) \sum_{k=1}^{\ell} c_{\ell,k,\alpha} \xi^k \phi^{(k)}(\xi) \\
&\qquad\qquad + \Bigg[\sum_{k=1}^{\ell} c_{\ell,k,\alpha} \xi^k \phi^{(k)}(\xi)\Bigg]^2 \Bigg\} + \int_{\rho-2}^{\rho-1} dx\, x^{\alpha} \big\{[y_0(x)\psi(x)]^{(\ell)}\big\}^2 \\
&\underset{\varepsilon\downarrow 0}{=} A(\ell,\alpha) \int_1^{(\rho-2)/\varepsilon} d\xi\, \xi^{-1}\phi(\xi)^2 + O(1) \\
&\underset{\varepsilon\downarrow 0}{=} A(\ell,\alpha) \int_2^{(\rho-2)/\varepsilon} d\xi\, \xi^{-1} + A(\ell,\alpha) \int_1^2 d\xi\, \xi^{-1}\phi(\xi)^2 + O(1) \\
&\underset{\varepsilon\downarrow 0}{=} A(\ell,\alpha) \ln(1/\varepsilon) + O(1), \qquad \text{(A.13)}
\end{aligned}
$$

employing the fact that supp $\big(\phi^{(k)}\big) \subseteq [1,2]$, $k \geqslant 1$. Thus,(A.9) and (A.13) yield

$$
\frac{\int_0^{\rho} dx\, x^{\alpha} \big[y_{0,\varepsilon}^{(\ell)}(x)\big]^2}{A(\ell,\alpha) \int_0^{\rho} dx\, x^{\alpha-2\ell} [y_{0,\varepsilon}(x)]^2} \underset{\varepsilon\downarrow 0}{=} 1 + O(1/\ln(1/\varepsilon)), \qquad \text{(A.14)}
$$

proving sharpness of $A(\ell,\alpha)$ for $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{R}\backslash\{2j-1\}_{1\leqslant j\leqslant \ell}$ on the function space $C_0^{\infty}((0,\rho))$.

For $1 \leqslant \ell \leqslant m-1$, $m \geqslant 2$, one replaces y_0 by

$$
\begin{aligned}
& f_0(x) = [\widetilde{A}(\ell,\alpha)/\widetilde{A}(m,\alpha)] x^{(2m-\alpha-1)/2}, \quad x > 0,\ \alpha \in \mathbb{R}, \\
& \widetilde{A}(\ell,\alpha) = 2^{-\ell}(2\ell-1-\alpha)(2\ell-3-\alpha)\cdots(3-\alpha)(1-\alpha), \quad \alpha \in \mathbb{R},
\end{aligned} \qquad \text{(A.15)}
$$

and observes the facts,

$$
f_0^{m-\ell}(x) = x^{(2\ell-1-\alpha)/2}, \qquad \text{(A.16)}
$$

$$
f_0^{(m)}(x) = \widetilde{A}(\ell,\alpha) x^{-(\alpha+1)/2}, \qquad \text{(A.17)}
$$

$$
x^{\alpha}\big[f_0^{(m)}(x)\big]^2 = A(\ell,\alpha) x^{\alpha-2\ell} \big[f_0^{(m-\ell)}(x)\big]^2 = A(\ell,\alpha) x^{-1}, \qquad \text{(A.18)}
$$

and then mollifies f_0 as before via

$$
f_{0,\varepsilon}(x) = f_0(x)\phi_{\varepsilon}(x)\psi(x),\ 0 \leqslant x \leqslant \rho, \quad f_{0,\varepsilon} \in C_0((0,\infty)). \qquad \text{(A.19)}
$$

At this point one can follow the above proof step by step arriving at

$$
\frac{\int_0^{\rho} dx\, x^{\alpha} \big[f_{0,\varepsilon}^{(m)}(x)\big]^2}{A(\ell,\alpha) \int_0^{\rho} dx\, x^{\alpha-2\ell} \big[f_{0,\varepsilon}^{(m-\ell)}(x)\big]^2} \underset{\varepsilon\downarrow 0}{=} 1 + O(1/\ln(1/\varepsilon)), \qquad \text{(A.20)}
$$

once more proving sharpness of $A(\ell,\alpha)$ for $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{R}\backslash\{2j-1\}_{1\leqslant j\leqslant \ell}$.

Since Theorem 3.5 exhibits the same constant $A(\ell,\alpha)$, the latter is sharp also for the larger function space $H_0^m((0,\rho); x^{\alpha}dx)$. □

Remark A.2. (*i*) Once more we recall that $A(\ell,\alpha) = 0$ if and only if $\alpha \in \{2j-1\}_{1\leqslant j\leqslant \ell}$. Thus, the inequality

$$
\int_0^{\rho} dx\, x^{\alpha} \big|f^{(m)}(x)\big|^2 \geqslant A(\ell,\alpha) \int_0^{\rho} dx\, x^{\alpha-2\ell} \big|f^{(m-\ell)}(x)\big|^2, \quad f \in C_0^{\infty}((0,\rho)), \qquad \text{(A.21)}
$$

is rendered trivial if $\alpha \in \{2j-1\}_{1\leqslant j\leqslant \ell}$, with the right-hand side of (A.21) being zero. The same observation applies of course to the remaining three cases (i), (ii), and (iv) in Theorem 3.1. However, we emphasize that inequalities (3.1)–(3.4) remain valid and nontrivial with just the first terms on their right-hand sides removed.

(ii) For $\alpha \in \mathbb{R}\backslash\{2j-1\}_{1\leqslant j\leqslant \ell}$, inequality (A.21) extends to $\rho = \infty$, again with $A(\ell, \alpha)$ being the sharp constant for $f \in C_0^\infty((0,\infty))$. In particular, the proof of Theorem A.1, suitably adapted, extends to the case $\rho = \infty$. (This observation applies of course to cases (i), (ii) (if $\rho = 0$), and (iii), (iv) (if $\rho = \infty$) in Theorem 3.1). This is of course in accordance with the fact that $C_0^\infty((0,\rho))$-functions extended by zero beyond ρ, $\rho \in (0,\infty)$, can be viewed as a subset of $C_0^\infty((0,\infty))$. ⋄

Remark A.3. Regarding sharpness (optimality) of constants, we first note that the smaller the underlying function space, the larger the efforts needed to prove optimality. In particular, in connection with the proof presented in Theorem A.1, assuming $f \in C_0^\infty((0,\rho))$ requires mollification of y_0 in (A.1) near $x = 0$ and $x = \rho$ and of course analogously in the case $f \in C_0^\infty((\rho,\infty))$. Many of the results cited in the remainder of this remark, under particular restrictions on the weight parameter α, establish sharpness for larger classes of functions f which do not automatically continue to hold in the $C_0^\infty((0,\rho))$-context. It is this simple observation that adds considerable complexity to sharpness proofs for the space $C_0^\infty((0,\rho))$. (By the same token, optimality proofs obtained for C_0^∞ function spaces automatically hold for larger function spaces as long as the inequalities have already been established for the larger function spaces with the same constants $A(m,\alpha), B(m,\alpha)$.) This comment applies, in particular, to many papers that prove sharpness results in multi-dimensional situations for larger function spaces such as $C_0^\infty(B(0;\rho))$ or (homogeneous, weighted) Sobolev spaces rather than $C_0^\infty(B(0;\rho)\backslash\{0\})$, $B(0;\rho) \subseteq \mathbb{R}^n$ the open ball in $\mathbb{R}^n$, $n \geqslant 2$, with center at the origin $x = 0$ and radius $\rho > 0$. Unless $C_0^\infty(B(0;\rho)\backslash\{0\})$ is dense in the appropriate norm (cf. the discussion preceding Theorem 3.5 in the one-dimensional context), one cannot *a priori* assume that the optimal constants $A(m,\widetilde{\alpha})$ and $B(m,\widetilde{\alpha})$ (with $\widetilde{\alpha}$ appropriately depending on n, e.g., $\widetilde{\alpha} = \alpha + n - 1$) remain the same for $C_0^\infty(B(0;\rho))$ and $C_0^\infty(B(0;\rho)\backslash\{0\})$, say. At least in principle, they could actually increase for the space $C_0^\infty(B(0;\rho)\backslash\{0\})$. In this context we emphasize that the multi-dimensional results then naturally lead to one-dimensional results for $C_0^\infty((0,\rho))$ upon specializing to radially symmetric functions in $C_0^\infty(B(0;\rho)\backslash\{0\})$.

Sharpness of the constant $A(m,0)$, $m \in \mathbb{N}$ (i.e., in the unweighted case, $\alpha = 0$), in connection with the space $C_0^\infty((0,\infty))$ has been shown by Yafaev [96]. In fact, he also established this result for fractional m (in this context we also refer to appropriate norm bounds in $L^p(\mathbb{R}^n; d^n x)$ of operators of the form $|x|^{-\beta}|-i\nabla|^{-\beta}$, $1 < p < n/\beta$, see [13, Sect. 1.7], [14, 57, 60, 61, 83, 91, 92, Sects. 1.7, 4.2]). Sharpness of $A(2,0)$ (i.e., in the unweighted Rellich case) was shown by Rellich [86, p. 91–101] in connection with the space $C_0^\infty((0,\infty))$; his multi-dimensional results also yield sharpness of $A(2,n-1)$ for $n \in \mathbb{N}$, $n \geqslant 3$, again for $C_0^\infty((0,\infty))$. In this context see also [14, Corollary 6.3.5]. An exhaustive study of optimality of $A(2,\widetilde{\alpha})$ (i.e., Rellich inequalities with power weights) for the space $C_0^\infty(\Omega\backslash\{0\})$ for cones $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, appeared in Caldiroli and Musina [21]. The authors, in particular, describe situations where $A(2,\widetilde{\alpha})$ has to be replaced by other constants and also treat the special case of radially symmetric functions in detail. Additional results for power weighted Rellich inequalities appeared in [79, 80]; further extensions of power weighted Rellich inequalities with sharp constants on $C_0^\infty(\mathbb{R}^n\backslash\{0\})$ were obtained in [73]; for optimal power weighted Hardy, Rellich, and higher-order inequalities on homogeneous groups, see [87, 88]. Many of these references also discuss sharp (power weighted) Hardy inequalities, implying optimality for $A(1,\widetilde{\alpha})$. Moreover, replacing $f(x)$ by $F(x) = \int_0^x dt\, f(t)$ $\big($or $F(x) = \int_x^\infty dt\, f(t)\big)$, optimality of the Hardy constant $A(1,0)$ for larger, L^p-based function spaces, can already be found in [54, Sect. 9.8] (see also [14, Theorem 1.2.1], [63, Ch. 3], [64, p. 5–11], [67, 76, 85], in connection with $A(1,\alpha)$).

Sharpness results for $A(m,\alpha)$ and $B(m,\alpha)$ together are much less frequently discussed in the literature, even under suitable restrictions on m and α. The results we found in the literature primarily follow upon specializing multi-dimensional results for function spaces such as $C_0^\infty(\Omega\backslash\{0\})$, or $C_0^\infty(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ open, and appropriate restrictions on m, α, and $n \geqslant 2$, for radially symmetric functions to the one-dimensional case at hand (cf. the previous paragraph). In this context we mention that the Hardy case $m = 1$, without a weight function, is studied in [1, 2, 5, 9, 20, 24, 27, 36, 52, 59, 69, 90, 94] (all for $N = 1$), and in [10, 29, 48] (all for $N \in \mathbb{N}$); the case with power weight functions is discussed in [17], [49], [50, Ch. 6] (for $N \in \mathbb{N}$); see also [70]. The Rellich case $m = 2$ with a general power weight on $C_0^\infty(\Omega\backslash\{0\})$ is discussed in [21] (for $N = 1$); the Rellich case $m = 2$, without weight function on $C_0^\infty(\Omega)$, is studied in [27, 28, 30] (all for $N = 1$), the case $N \in \mathbb{N}$ is studied in [4]; the case of additional power weights is treated in [49], [50, Ch. 6], [75]. The general case $m \in \mathbb{N}$ is discussed in [6] (for $N = 1$) and in [15], [49], [50, Ch. 6], [95] (all for $N \in \mathbb{N}$ and including power weights). Employing oscillation theory, sharpness of the unweighted Hardy case $A(1,0) = B(1,0) = 1/4$, with $N \in \mathbb{N}$, was proved in [45].

The special results available on sharpness of $B(m,\alpha)$ are all saddled with enormous complexity, especially, for larger values of $N \in \mathbb{N}$. In fact, a careful proof for general N will rival the length of this paper and hence has not been attempted here as briefly discussed in the following remark. $\diamond$

Remark A.4. The proof of optimality of $A(\ell,\alpha)$ in Theorem A.1 consists of two principal steps:

(i) Identify a function y_0 (see (A.1)) which is not in $C_0^\infty((\rho,\infty))$, but which satisfies (see (A.4)),

$$\frac{\int_\rho^\infty dx\, x^\alpha \big|y_0^{(m)}(x)\big|^2}{\int_\rho^\infty dx\, x^{\alpha 2\ell} \big|y_0^{(m-\ell)}(x)\big|^2} = A(\ell,\alpha). \tag{A.22}$$

(ii) Exhibit a family $\{y_{0,\varepsilon}\}_{\varepsilon>0} \subset C_0^\infty((\rho,\infty))$ of multiplicative mollifications of y_0 (see (A.8)) that approaches y_0 as $\varepsilon \downarrow 0$ and for which (see (A.14))

$$\lim_{\varepsilon\downarrow 0} \frac{\int_\rho^\infty dx\, x^\alpha \big|y_{0,\varepsilon}^{(m)}(x)\big|^2}{\int_\rho^\infty dx\, x^{\alpha 2\ell} dx \big|y_{0,\varepsilon}^{(m-\ell)}(x)\big|^2} = A(\ell,\alpha). \tag{A.23}$$

Unfortunately, due to the ensuing complexity when having to apply the product rule of differentiation again and again, this approach in connection with $A(\ell,\alpha)$ cannot naturally be adapted to a proof of optimality of $B(\ell,\alpha)$. The proof of optimality of $B(\ell,\alpha)$ we are currently working out requires substantial modification to steps (i) and (ii) above. We sketch the new approach in the special case $\ell = m$ in inequality

(3.1). We abbreviate

$$W_{m,\alpha,N}(x) = A(m,\alpha) + B(m,\alpha) \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_p(\gamma/x)]^{-2}. \tag{A.24}$$

Instead of identifying one explicit functon y_0 which satisfies (A.22), we use a modification of the proof of [15, Theorem 2] to identify a family $\{f_{0,\varepsilon} \colon (\rho,\infty) \to \mathbb{C}\}_{\varepsilon>0}$ of functions which are not in $C_0^\infty((\rho,\infty))$ but for which

$$\lim_{\varepsilon\downarrow 0} \frac{\int_\rho^\infty dx\, x^\alpha \big|f_{0,\varepsilon}^{(m)}(x)\big|^2 - \int_\rho^\infty dx\, x^{\alpha-2m} W_{m,\alpha,N}(x) |f_{0,\varepsilon}(x)|^2}{\int_\rho^\infty dx\, x^{\alpha-2m} \prod_{p=1}^{N} [\ln_p(\gamma/x)]^{-2} |f_{0,\varepsilon}(x)|^2} = B(m,\alpha). \tag{A.25}$$

Instead of a family of multiplicative mollifications as in (A.8), for each $\varepsilon > 0$ we employ a family $\{f_{0,\varepsilon,\nu}\}_{\nu>0} \subset C_0^\infty((\rho,\infty))$ of mollifications of $f_{0,\varepsilon}$ using convolution with an approximate identity which has the properties:

$$\lim_{\nu\downarrow 0} \int_\rho^\infty dx\, x^\alpha \big|f_{0,\varepsilon,\nu}^{(m)}(x)\big|^2 = \int_\rho^\infty dx\, x^\alpha |f_{0,\varepsilon}(x)|^2, \tag{A.26}$$

and for $k = 0, 1, \cdots, N$,

$$\begin{aligned} &\lim_{\nu\downarrow 0} \int_\rho^\infty dx\, x^{\alpha-2m} \prod_{p=1}^{k} [\ln_p(\gamma/x)]^{-2} |f_{0,\varepsilon,\nu}(x)|^2 \\ &\quad = \int_\rho^\infty dx\, x^{\alpha-2m} \prod_{p=1}^{k} [\ln_p(\gamma/x)]^{-2} |f_{0,\varepsilon}(x)|^2. \end{aligned} \tag{A.27}$$

Thus, roughly speaking, one gets

$$\lim_{\varepsilon,\nu\downarrow 0} \frac{\int_\rho^\infty dx\, x^\alpha \big|f_{0,\varepsilon,\nu}^{(m)}(x)\big|^2 - \int_\rho^\infty dx\, x^{\alpha-2m} W_{m,\alpha,N}(x) |f_{0,\varepsilon}(x)|^2}{\int_\rho^\infty dx\, x^{\alpha-2m} \prod_{p=1}^{N} [\ln_p(\gamma/x)]^{-2} |f_{0,\varepsilon,\nu}(x)|^2} = B(m,\alpha). \tag{A.28}$$

This approach is that much longer than the proof of Theorem A.1, that we felt we had no choice but to write a separate paper [43] for the proof of optimality of $B(\ell,\alpha)$. ◇

Appendix B. The Interval Case (ρ,∞) in Theorem 3.5

We recall the space

$$\begin{aligned} H_{m,\alpha}([\rho,\infty)) = \big\{ f : [\rho,\infty) \to \mathbb{C} \,\big|\, &\text{for all } R > \rho, f^{(k)} \in AC([\rho,R]),\ 0 \leqslant k \leqslant m-1; \\ &f^{(k)}(\rho) = 0,\ 0 \leqslant k \leqslant m-1;\ f^{(m)} \in L^2\big((\rho,\infty); x^\alpha dx\big) \big\}, \end{aligned} \tag{B.1}$$

and introduce the bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle_{m,\alpha}$ on $H_{m,\alpha}([\rho,\infty))$ by

$$\langle f, g\rangle_{m,\alpha} = \int_\rho^\infty x^\alpha dx\, \overline{f^{(m)}(x)} g^{(m)}(x), \quad f, g \in H_{m,\alpha}([\rho,\infty)). \tag{B.2}$$

Proposition B.1. *The bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle_{m,\alpha}$ is an inner product on the space $H_{m,\alpha}([\rho,\infty))$, in fact, $(H_{m,\alpha}([\rho,\infty)), \langle\,\cdot\,,\,\cdot\,\rangle_{m,\alpha})$ is a Hilbert space.*

Proof. Assuming $\langle f, f\rangle_{m,\alpha} = 0$, $f \in H_{m,\alpha}([\rho,\infty))$, one obtains

$$\int_\rho^\infty x^\alpha dx\, \big|f^{(m)}(x)\big|^2 = 0, \tag{B.3}$$

and hence $f^{(m)} = 0$ a.e. on (ρ, ∞). Thus, employing $f^{(m-1)}(\rho) = 0$, one concludes that

$$f^{(m-1)}(x) = \int_\rho^x dt\, f^{(m)}(t) = 0, \quad x \geqslant \rho. \tag{B.4}$$

Similarly, as $f^{(m-2)}(\rho) = 0$,

$$f^{(m-2)}(x) = \int_\rho^x dt\, f^{(m-1)}(t) = 0, \quad x \geqslant \rho, \tag{B.5}$$

and hence inductively,

$$f^{(k)}(x) = 0, \quad 0 \leqslant k \leqslant m-1,\ x \geqslant \rho. \tag{B.6}$$

Thus, $\langle \cdot\,, \cdot \rangle_{m,\alpha}$ is an inner product on $H_{m,\alpha}([\rho,\infty))$.

To prove completeness of $(H_{m,\alpha}([\rho,\infty)), \langle \cdot\,, \cdot \rangle_{m,\alpha})$, one assumes that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $H_{m,\alpha}([\rho,\infty))$. Then $\big\{f_n^{(m)}\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2\big((\rho,\infty); x^\alpha dx\big)$. Hence, there exists $g \in L^2\big((\rho,\infty); x^\alpha dx\big)$ such that

$$\lim_{n\to\infty} \big\|f_n^{(m)} - g\big\|_{L^2((\rho,\infty);x^\alpha dx)} = 0. \tag{B.7}$$

Introducing $f : [\rho, \infty) \to \mathbb{C}$ by

$$f(x) = \int_\rho^x \int_\rho^{t_1} \cdots \int_\rho^{t_{m-1}} dt_1 \cdots dt_{m-1} du\, g(u), \quad x \geqslant \rho, \tag{B.8}$$

then $f^{(k)} \in AC([\rho, R])$ for all $R > \rho$ and $f^{(k)}(\rho) = 0$, $0 \leqslant k \leqslant m-1$, and $f^{(m)} = g \in L^2\big((\rho,\infty); x^\alpha dx\big)$, and hence $f \in H_{m,\alpha}([\rho,\infty))$. In addition,

$$\begin{aligned} |||f_n - f|||_{H_{m,\alpha}([\rho,\infty))} &= \big\|f_n^{(m)} - f^{(m)}\big\|_{L^2((\rho,\infty);x^\alpha dx)} \\ &= \big\|f_n^{(m)} - g\big\|_{L^2((\rho,\infty);x^\alpha dx)} \underset{n\to\infty}{\longrightarrow} 0, \end{aligned} \tag{B.9}$$

completing the proof. □

We recall that the norm in Hilbert space $H_{m,\alpha}([\rho,\infty))$ is denoted by $|||\cdot|||_{m,\alpha}$. The fact that $C_0^\infty((\rho,\infty)) \subset H_{m,\alpha}([\rho,\infty))$ then leads to the introduction of the homogeneous weighted Sobolev space

$$\dot{H}_0^m\big((\rho,\infty); x^\alpha dx\big) = \overline{C_0^\infty((\rho,\infty))}^{|||\cdot|||_{m,\alpha}}, \tag{B.10}$$

that is, the closure of $C_0^\infty((\rho,\infty))$ in $H_{m,\alpha}([\rho,\infty))$. Proposition B.1 then yields the following result.

Corollary B.2. *Assume that $f \in \dot{H}_0^m\big((\rho,\infty); x^\alpha dx\big)$, $\alpha \in \mathbb{R}$. Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C_0^\infty((\rho,\infty))$ such that for $0 \leqslant k \leqslant m$,*

$$\lim_{n\to\infty} f_n^{(k)}(x) = f^{(k)}(x) \text{ for a.e. } x > \rho. \tag{B.11}$$

Proof. Since $f \in \dot{H}_0^m\big((\rho,\infty); x^\alpha dx\big)$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C_0^\infty((\rho,\infty))$ such that

$$\lim_{n\to\infty} \big\|f_n^{(m)} - f^{(m)}\big\|_{L^2((\rho,\infty);x^\alpha dx)} = 0. \tag{B.12}$$

By taking a subsequence, if necessary, one can assume that

$$\lim_{n\to\infty} f_n^{(m)}(x) = f^{(m)}(x) \text{ for a.e. } x > \rho. \tag{B.13}$$

Since $f^{(k)}(\rho) = 0$, $0 \leqslant k \leqslant m-1$,

$$\begin{aligned}
\big|f_n^{(m-1)}(x) - f^{(m-1)}(x)\big| &= \bigg|\int_\rho^x dt\, f_n^{(m)}(t) - \int_\rho^x dt\, f^{(m)}(t)\bigg| \\
&\leqslant \int_\rho^x dt\, \big|f_n^{(m)}(t) - f^{(m)}(t)\big| \\
&\leqslant \bigg[\int_\rho^x dt\, \big|f_n^{(m)}(t) - f^{(m)}(t)\big|^2\bigg]^{1/2} (x-\rho)^{1/2} \\
&\leqslant \max\big(\rho^{-\alpha/2}, x^{-\alpha/2}\big) \bigg[\int_\rho^x dt\, t^\alpha \big|f_n^{(m)}(t) - f^{(m)}(t)\big|^2\bigg]^{1/2} (x-\rho)^{1/2} \\
&\underset{n\to\infty}{\longrightarrow} 0, \quad x \geqslant \rho. \qquad \text{(B.14)}
\end{aligned}$$

Next, fix $R > \rho$. Then for all $n \in \mathbb{N}$ sufficiently large, and for all $x \in [\rho, R]$, there exists $C(\rho, \alpha, R) \in (0, \infty)$ such that

$$\begin{aligned}
\big|f_n^{(m-1)}(x)\big| &\leqslant \int_\rho^x dt\, \big|f_n^{(m)}(t)\big| \\
&\leqslant \bigg[\int_\rho^x dt\, \big|f_n^{(m)}(t)\big|^2\bigg]^{1/2} (x-\rho)^{1/2} \\
&\leqslant \max\big(\rho^{-\alpha/2}, x^{-\alpha/2}\big) \bigg[\int_\rho^x dt\, t^\alpha \big|\big[f_n^{(m)}(t) - f^{(m)}(t)\big] + f^{(m)}(t)\big|^2\bigg]^{1/2} \\
&\quad \times (x-\rho)^{1/2} \\
&\leqslant \max\big(\rho^{-\alpha/2}, x^{-\alpha/2}\big) \bigg[2\int_\rho^x dt\, t^\alpha \big|f_n^{(m)}(t) - f^{(m)}(t)\big|^2 \\
&\qquad + 2\int_\rho^x dt\, t^\alpha \big|f^{(m)}(t)\big|^2\bigg]^{1/2} (x-\rho)^{1/2} \\
&\leqslant \max\big(\rho^{-\alpha/2}, x^{-\alpha/2}\big) \bigg[o(1) + 2\int_\rho^x dt\, t^\alpha \big|f^{(m)}(t)\big|^2\bigg]^{1/2} (x-\rho)^{1/2} \\
&\leqslant C(\rho, \alpha, R) \big\|f^{(m)}\big\|_{L^2((\rho,\infty); x^\alpha dx)}, \quad x \in [\rho, R]. \qquad \text{(B.15)}
\end{aligned}$$

Thus, (B.14), (B.15), and an application of Lebesgue's dominated convergence theorem implies

$$\lim_{n\to\infty} \Big\| f_n^{(m-1)}\big|_{[\rho,R]} - f^{(m-1)}\big|_{[\rho,R]} \Big\|_{L^1((\rho,R); dt)} = 0. \qquad \text{(B.16)}$$

Next, one infers that

$$\begin{aligned}
\big|f_n^{(m-2)}(x) - f^{(m-2)}(x)\big| &= \bigg|\int_\rho^x dt\, \big[f_n^{(m-1)}(t) - f^{(m-1)}(t)\big]\bigg| \\
&\leqslant \int_\rho^x dt\, \big|f_n^{(m-1)}(t) - f^{(m-1)}(t)\big| \\
&\leqslant \int_\rho^R dt\, \big|f_n^{(m-1)}(t) - f^{(m-1)}(t)\big| \\
&= \Big\| f_n^{(m-1)}\big|_{[\rho,R]} - f^{(m-1)}\big|_{[\rho,R]} \Big\|_{L^1((\rho,R); dt)}
\end{aligned}$$

$$\underset{n\to\infty}{\longrightarrow} 0, \quad x \in [\rho, R], \tag{B.17}$$

by (B.16). Similarly, for all $n \in \mathbb{N}$ sufficiently large, and for all $x \in [\rho, R]$, one has

$$\begin{aligned}
\big|f_n^{(m-2)}(x)\big| &\leqslant \int_\rho^x dt\, \big|f_n^{(m-1)}(x)\big| \leqslant \int_\rho^R dt\, \big|f_n^{(m-1)}(x)\big| \\
&= \Big\| \big[f_n^{(m-1)} - f^{(m-1)}\big] + f^{(m-1)}\big|_{\rho,R}\Big\|_{L^1((\rho,R);dt)} \\
&= o(1) + \Big\| f^{(m-1)}\big|_{\rho,R}\Big\|_{L^1((\rho,R);dt)} \\
&\leqslant 2\Big\| f^{(m-1)}\big|_{\rho,R}\Big\|_{L^1((\rho,R);dt)}, \quad x \in [\rho, R].
\end{aligned} \tag{B.18}$$

Thus, (B.17), (B.18), and an application of Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty} \Big\| f_n^{(m-2)}\big|_{[\rho,R]} - f^{(m-2)}\big|_{[\rho,R]}\Big\|_{L^1((\rho,R);dt)} = 0. \tag{B.19}$$

Iterating these arguments proves that for all $0 \leqslant k \leqslant m-1$,

$$\lim_{n\to\infty} f_n^{(k)}(x) = f^{(k)}(x) \ \text{ for a.e. } x \in [\rho, R]. \tag{B.20}$$

Since $R > \rho$ was arbitrary, this concludes the proof of Corollary B.2. □

Acknowledgments. We gratefully acknowledge discussions with Marius Mitrea.

References

[1] Adimurthi, N. Chaudhuri, and M. Ramaswami, *An improved Hardy-Sobolev inequality and its application*, Proc. Amer. Math. Soc. **130**, 489–505 (2001).

[2] Adimurthi and M. J. Esteban, *An improved Hardy-Sobolev inequality in $W^{1,p}$ and its application to Schrödinger operators*, Nonlin. Diff. Eq. Appl. **12**, 243–263 (2005).

[3] Adimurthi, S. Filippas, and A. Tertikas, *On the best constant of Hardy–Sobolev inequalities*, Nonlinear Anal. **70**, 2826–2833 (2009).

[4] Adimurthi, M. Grossi, and S. Santra, *Optimal Hardy–Rellich inequalities, maximum principle and related eigenvalue problem*, J. Funct. Anal. **240**, 36—83 (2006).

[5] Adimurthi and K. Sandeep, *Existence and non-existence of the first eigenvalue of the perturbed Hardy-Sobolev operator*, Proc. Roy. Soc. Edinburgh **A 132**, 1021–1043 (2002).

[6] Adimurthi and S. Santra, *Generalized Hardy–Rellich inequalities in critical dimension and its applications*, Comm. Contemp. Math. **11**, 367—394 (2009).

[7] Adimurthi and A. Sekar, *Role of the fundamental solution in Hardy–Sobolev-type inequalities*, Proc. Roy. Soc. Edinburgh Sect. **A 136**, 1111–1130 (2006).

[8] W. Allegretto, *Nonoscillation theory of elliptic equations of order* $2n$, Pac. J. Math. **64**, 1–16 (1976).

[9] A. Alvino, R. Volpicelli, and B. Volzone, *On Hardy inequalities with a remainder term*, Ric. Mat. **59**, 265–280 (2010).

[10] H. Ando and T. Horiuchi, *Missing terms in the weighted Hardy–Sobolev inequalities and its application*, Kyoto J. of Math. **52**, 759—796 (2012).

[11] W. Arendt, G. Ruiz Goldstein, and J. A. Goldstein, *Outgrowths of Hardy's inequality*, in *Recent Advances in Differential Equations and Mathematical Physics*, N. Chernov, Y. Karpeshina, I. W. Knowles, R. T. Lewis, and R. Weikard (eds.), Contemp. Math. **412**, Amer. Math. Soc., Providence, RI, 2006, pp. 51–68.

[12] F. G. Avkhadiev, *The generalized Davies problem for polyharmonic operators*, Sib. Math. J. **58**, 932–942 (2017).

[13] A. Balinsky and W. D. Evans, *Spectral Analysis of Relativistic Operators*, Imperial College Press, London, 2011.

[14] A. A. Balinsky, W. D. Evans, and R. T. Lewis, *The Analysis and Geometry of Hardy's Inequality*, Universitext, Springer, Cham, 2015.

[15] G. Barbatis, *Best constants for higher-order Rellich inequalities in $L^p(\Omega)$*, Math Z. **255**, 877–896 (2007).

[16] G. Barbatis, S. Fillipas, and A. Tertikas, *Series expansion for L^p Hardy inequalities*, Indiana Univ. Math. J. **52**, 171–190 (2003).

[17] G. Barbatis, S. Filippas, and A. Tertikas, *Sharp Hardy and Hardy–Sobolev inequalities with point singularities on the boundary*, J. Math. Pures Appl. **117**, , 146–184 (2018).

[18] D. M. Bennett, *An extension of Rellich's inequality*, Proc. Amer. Math. Soc. **106**, 987–993 (1989).

[19] M. S. Birman, *The spectrum of singular boundary problems*, Mat. Sb. (N.S.) **55** (97), 125–174 1961 (Russian). Engl. transl. in Amer. Math. Soc. Transl., Ser. 2, **53**, 23–80 (1966).

[20] H. Brezis and M. Marcus, *Hardy's inequalities revisited*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25**, 217–237 (1997).

[21] P. Caldiroli and R. Musina, *Rellich inequalities with weights*, Calc. Var. **45**, 147–164 (2012).

[22] R. S. Chisholm, W. N. Everitt, and L. L. Littlejohn, *An integral operator inequality with applications*, J. of Inequal. & Applications **3**, 245–266 (1999).

[23] C. Y. Chuah, F. Gesztesy, L. L. Littlejohn, T. Mei, I. Michael, and M. M. H. Pang, *On weighted Hardy-type inequalities*, Math. Ineqal. & Appl. **23**, 625–646 (2020).

[24] C. Cowan, *Optimal Hardy inequalities for general elliptic operators with improvements*, Commun. Pure Appl. Anal. **9**, 109–140 (2010).

[25] E. B. Davies, *Spectral Theory and Differential Operators*, Cambridge Studies in Advanced Mathematics, Vol. 42, Cambridge University Press, Cambridge, UK, 1995.

[26] E. B. Davies and A. M. Hinz, *Explicit constants for Rellich inequalities in $L_p(\Omega)$*, Math. Z. **227**, 511–523 (1998).

[27] A. Detalla, T. Horiuchi, and H. Ando, *Missing terms in Hardy–Sobolev inequalities and its application*, Far East J. Math. Sci. **14**, 333–359 (2004).

[28] A. Detalla, T. Horiuchi, and H. Ando, *Missing terms in Hardy–Sobolev inequalities*, Proc. Japan Acad. **A 80**, 160–165 (2004).

[29] A. Detalla, T. Horiuchi, and H. Ando, *Sharp remainder terms of Hardy–Sobolev inequalities*, Math. J. Ibaraki Univ. **37**, 39–52 (2005).

[30] A. Detalla, T. Horiuchi, and H. Ando, *Sharp remainder terms of the Rellich inequality and its application*, Bull. Malays. Math. Sci. Soc. **35** (2A), 519–528 (2012).

[31] Yu. A. Dubinskii, *Hardy inequalities with exceptional parameter values and applications*, Doklady Math. **80**, 558–562 (2009).

[32] Yu. A. Dubinskii, *A Hardy-type inequality and its applications*, Proc. Steklov Inst. Math. **269**, 106–126 (2010).

[33] Yu. A. Dubinskii, *Bilateral scales of Hardy inequalities and their applications to some problems in mathematical physics*, J. Math. Sci. **201**, 751–795 (2014).

[34] N. T. Duy, N. Lam, and N. A. Triet, *Hardy–Rellich identities with Bessel pairs*, Arch. Math. **113**, 95–112 (2019).

[35] N. T. Duy, N. Lam, and N. A. Triet, *Improved Hardy and Hardy–Rellich type inequalities with Bessel pairs via factorization*, J. Spectral Th. (to appear).

[36] S. Fillipas, and A. Tertikas, *Optimizing improved Hardy inequalities*, J. Funct. Anal. **192**, 186–233 (2002); Corrigendum: J. Funct. Anal. **255**, 2095 (2008); see also [78].

[37] G. B. Folland, *Real Analysis. Modern Techniques and Their Applications*, 2nd ed., Wiley-Interscience, New York, 1999.

[38] F. Gazzola, H.-C. Grunau, and E. Mitidieri, *Hardy inequalities with optimal constants and remainder terms*, Trans. Amer. Math. Soc. **356**, 2149–2168 (2003).

[39] F. Gesztesy, *On non-degenerate ground states for Schrödinger operators*, Rep. Math. Phys. **20**, 93–109 (1984).

[40] F. Gesztesy and L. L. Littlejohn, *Factorizations and Hardy–Rellich-type inequalities*, in *Partial Differential Equations, Mathematical Physics, and Stochastic Analysis. A Volume in Honor of Helge Holden's 60th Birthday*, EMS Congress Reports, F. Gesztesy, H. Hanche-Olsen, E. Jakobsen, Y. Lyubarskii, N. Risebro, and K. Seip (eds.), 207–226 (2018).

[41] F. Gesztesy, L. L. Littlejohn, I. Michael, and M. M. H. Pang, *Radial and logarithmic refinements of Hardy's inequality*, St. Petersburg Math. J. **30**, 429–436 (2019).

[42] F. Gesztesy, I. Michael, and M. M. H. Pang, *A sequence of multi-dimensional weighted Birman–Hardy–Rellich inequalities with radial and logarithmic refinements*, in preparation.

[43] F. Gesztesy, I. Michael, and M. M. H. Pang, *Optimality of constants in weighted Birman–Hardy–Rellich inequalities with logarithmic refinements*, in preparation.
[44] F. Gesztesy, L. L. Littlejohn, I. Michael, and R. Wellman, *On Birman's sequence of Hardy–Rellich-type inequalities*, J. Diff. Eq. **264**, 2761–2801 (2018).
[45] F. Gesztesy and M. Ünal, *Perturbative oscillation criteria and Hardy-type inequalities*, Math. Nachr. **189**, 121–144 (1998).
[46] K. T. Gkikas and G. Psaradakis, *Optimal non-homogeneous improvements for the series expansion of Hardy's inequality*, arXiv:1805.10935.
[47] I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, Israel Program for Scientific Translations, Jerusalem, 1965, Daniel Davey & Co., Inc., New York, 1966.
[48] N. Ghoussoub and A. Moradifam, *On the best possible remaining term in the Hardy inequality*, Proc. Nat. Acad. Sci. **105**, No. 37, 13746–13751 (2008).
[49] N. Ghoussoub and A. Moradifam, *Bessel pairs and optimal Hardy and Hardy–Rellich inequalities*, Math. Ann. **349**, 1–57 (2011).
[50] N. Ghoussoub and A. Moradifam, *Functional Inequalities: New Perspectives and New Applications*, Math. Surveys Monographs, Vol. 187, Amer. Math. Soc., Providence, RI, 2013.
[51] G. R. Goldstein, J. A. Goldstein, R. M. Mininni, and S. Romanelli, *Scaling and variants of Hardy's inequality*, Proc. Amer. Math. Soc. **147**, 1165–1172 (2019).
[52] G. Grillo, *Hardy and Rellich-type inequalities for metrics defined by vector fields*, Potential Anal. **18**, 187–217 (2003).
[53] G. H. Hardy, *Notes on some points in the integral calculus, LX. An inequality between integrals*, Messenger Math. **54**, 150–156 (1925).
[54] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, reprinted, 1988.
[55] P. Hartman, *On the linear logarithmic-exponential differential equation of the second-order*, Amer. J. Math. **70**, 764–779 (1948).
[56] P. Hartman, *Ordinary Differential Equations*, 2nd ed., SIAM, Philadelphia, 2002.
[57] I. W. Herbst, *Spectral theory of the operator* $(p^2+m^2)^{1/2} - Ze^2/r$, Commun. Math. Phys. **53**, 285–294(1977).
[58] A. M. Hinz, *Topics from spectral theory of differential operators*, in *Spectral Theory of Schrödinger Operators*, R. del Rio and C. Villegas-Blas (eds.), Contemp. Math. **340**, Amer. Math. Soc., Providence, R.I., (2004), pp. 1–50.
[59] N. Ioku and M. Ishiwata, *A scale invariant form of a critical Hardy inequality*, Int. Math. Res. Notes **2015**, No. 18, 8830–8846 (2015).
[60] H. Kalf, U.-W. Schmincke, J. Walter, and R. Wüst, *On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials*, in *Spectral Theory and Differential Equations*, W. N. Everitt (ed.), Lecture Notes in Math., Vol. 448, Springer, Berlin, 1975, pp. 182–226.
[61] V. F. Kovalenko, M. A. Perelmuter, and Ya. A. Semenov, *Schrödinger operators with* $L_w^{1/2}(\mathbb{R}^\ell) - potentials$, J. Math. Phys. **22**, 1033–1044 (1981).
[62] A. Kufner, *Weighted Sobolev Spaces*, A Wiley-Interscience Publication, John Wiley & Sons, 1985.
[63] A. Kufner, L. Maligranda, and L.-E. Persson, *The Hardy Inequality: About its History and Some Related Results*, Vydavatelský Servis, Pilsen, 2007.
[64] A. Kufner, L.-E. Persson, and N. Samko, *Weighted Inequalities of Hardy Type*, 2nd ed., World Scientific, Singapore, 2017.
[65] A. Kufner and A. Wannebo, *Some remarks on the Hardy inequality for higher order derivatives*, Int. Ser. Num. Math. **103**, 33–48 (1992).
[66] S. T. Kuroda, *An Introduction to Scattering Theory*, Aarhus University Lecture Notes Series, No. 51, 1978.
[67] E. Landau, *A note on a theorem concerning series of positive terms: extract from a letter of Prof. E. Landau to Prof. I. Schur*, J. London Math. Soc. **1**, 38–39 (1926).
[68] G. Leoni, *A First Course in Sobolev Spaces*, 2nd ed., Graduate Studies in Math., Vol. 181, Amer. Math. Soc., Providence, RI, 2017.
[69] S. Machihara, T. Ozawa, and H. Wadade, *Hardy type inequalities on balls*, Tohoku Math. J. **65**, 321–330 (2013).

[70] S. Machihara, T. Ozawa, and H. Wadade, *Scaling invariant Hardy inequalities of multiple logarithmic type on the whole space*, J. Inequal. Appls. **2015:281**, pp. 1–13.
[71] S. Machihara, T. Ozawa, and H. Wadade, *Remarks on the Hardy type inequalities with remainder terms in the framework of equalities*, arXiv:1611.03580.
[72] S. Machihara, T. Ozawa, and H. Wadade, *Remarks on the Rellich inequality*, Math. Z. **286**, 1367–1373 (2017).
[73] G. Metafune, M. Sobajima, and C. Spina, *Weighted Calderón–Zygmund and Rellich inequalities in L^p*, Math. Ann. **361**, 313–366 (2015).
[74] E. Mitidieri, *A simple approach to Hardy inequalities*, Math. Notes **67**, 479–486 (2000).
[75] A. Moradifam, *Optimal weighted Hardy–Rellich inequalities on $H^2 \cap H_0^1$*, J. London Math. Soc. **85**, 22–40 (2012).
[76] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math. **44**, 31–38 (1972).
[77] E. Müeller-Pfeiffer, *Spectral Theory of Ordinary Differential Operators*, Ellis Horwood Limited, West Sussex, 1981.
[78] R. Musina, *A note on the paper "Optimizing improved Hardy inequalities" by S. Filippas and A. Tertikas*, J. Funct. Anal. **256**, 2741–2745 (2009).
[79] R. Musina, *Weighted Sobolev spaces of radially symmetric functions*, Ann. Mat. **193**, 1629–1659 (2014).
[80] R. Musina, *Optimal Rellich–Sobolev constants and their extremals*, Diff. Integral Eq. **27**, 570–600 (2014).
[81] Q. A. Ngô and V. N. Nguyen, *A supercritical Sobolev type inequality in higher order Sobolev spaces and related higher order elliptic problems*, arXiv:1905.01864.
[82] E. S. Noussair and N. Yoshida, *Nonoscillation criteria for elliptic equations of order $2m$*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **59** (1975), no. 1–2, 57–64 (1976).
[83] N. Okazawa, H. Tamura, and T. Yokota, *Square Laplacian perturbed by inverse fourth-power potential. I Self-adjointness (real case)*, Proc. Roy. Soc. Edinburgh **141A**, 409–416 (2011).
[84] B. Opic and A. Kufner, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series, Vol. 219. Longman Scientific & Technical, Harlow, 1990.
[85] L.-E. Persson and S. G. Samko, *A note on the best constants in some Hardy inequalities*, J. Math. Inequalities **9**, 437–447 (2015).
[86] F. Rellich, *Perturbation Theory of Eigenvalue Problems*, Gordon and Breach, 1969.
[87] M. Ruzhansky and D. Suragan, *Hardy and Rellich inequalities, and sharp remainders on homogeneous groups*, Adv. Math. **317**, 799–822 (2017).
[88] M. Ruzhansky and N. Yessirkegenov, *Factorizations and Hardy–Rellich inequalities on stratified groups*, J. Spectral Th. (to appear), arXiv:1706.05108.
[89] M. Sano, *Extremal functions of generalized critical Hardy inequalities*, J. Diff. Eq. **267**, 2594–2615 (2019).
[90] M. Sano and F. Takahashi, *Sublinear eigenvalue problems with singular weights related to the critical Hardy inequality*, Electronic J. Diff. Eq. **2016**, No. 212, pp. 1–12.
[91] U.-W. Schmincke, *Essential selfadjointness of a Schrödinger operator with strongly singular potential*, Math. Z. **124**, 47–50, (1972).
[92] B. Simon, *Hardy and Rellich inequalities in non-integral dimension*, J. Operator Th. **9**, 143–146 (1983).
[93] M. Solomyak, *A remark on the Hardy inequalities*, Integral Eq. Operator Th. **19**, 120–124 (1994).
[94] F. Takahashi, *A simple proof of Hardy's inequality in a limiting case*, Arch. Math. **104**, 77–82 (2015).
[95] A. Tertikas and N. B. Zographopoulos, *Best constants in the Hardy–Rellich inequalities and related improvements*, Adv. Math. **209**, 407–459 (2007).
[96] D. Yafaev, *Sharp constants in the Hardy–Rellich inequalities*, J. Funct. Anal. **168**, 121–144 (1999).

Department of Mathematics, Baylor University, Sid Richardson Bldg., 1410 S. 4th Street, Waco, TX 76706, USA
E-mail address: Fritz_Gesztesy@baylor.edu
URL: http://www.baylor.edu/math/index.php?id=935340

Department of Mathematics, Baylor University, Sid Richardson Bldg., 1410 S. 4th Street, Waco, TX 76706, USA
E-mail address: Lance_Littlejohn@baylor.edu
URL: http://www.baylor.edu/math/index.php?id=53980

Department of Mathematics, Louisiana State University, 303 Lockett Hall, Baton Rouge, LA 70803, USA
E-mail address: imichael@lsu.edu
URL: http://blogs.baylor.edu/isaac_michael/

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: pangm@missouri.edu
URL: https://www.math.missouri.edu/people/pang