RADIAL AND LOGARITHMIC REFINEMENTS OF HARDY'S INEQUALITY

FRITZ GESZTESY, LANCE L. LITTLEJOHN, ISAAC MICHAEL, AND MICHAEL M. H. PANG

Dedicated, with great admiration, to the memory of Michael Solomyak (May 16, 1931 – July 31, 2016).

Abstract. The principal purpose of this note is to derive variants of Hardy's inequality involving radial derivatives and logarithmic refinements.

1. Introduction

To describe the principal aim of this note we start by recalling the classical Hardy inequality

$$\int_{\Omega} |(\nabla f)(x)|^2 \, d^n x \geqslant \frac{(n-2)^2}{4} \int_{\Omega} |x|^{-2} |f(x)|^2 \, d^n x, \tag{1.1}$$

valid for $f \in C_0^{\infty}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ open, $n \in \mathbb{N}$, $n \geqslant 2$ (interpreting the right-hand side of (1.1) as zero if $n = 2$, and hence rendering it trivial in that case). The following extension of Hardy's inequality (in the special case where Ω equals $B_n(x_0; \rho)$, the open ball in $\mathbb{R}^n$ of radius $\rho > 0$ centered at $x_0 \in \mathbb{R}^n$), involving logarithmic refinements, was derived in [7],

$$\begin{aligned} \int_{\Omega} |(\nabla f)(x)|^2 d^n x \geqslant \int_{\Omega} |x - x_0|^{-2} |f(x)|^2 \bigg\{ & \frac{(n-2)^2}{4} \\ & + \frac{1}{4} \sum_{j=1}^{m} \prod_{k=1}^{j} [\ln_k(\gamma/|x-x_0|)]^{-2} \bigg\} d^n x, \end{aligned} \tag{1.2}$$

valid for $f \in C_0^{\infty}(\Omega)$, assuming that $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 2$, is open and bounded with $x_0 \in \Omega$, $m \in \mathbb{N}$, and the logarithmic terms $\ln_k(\gamma/|x - x_0|), k \in \mathbb{N}$, are recursively given by

$$\begin{aligned} &\ln_1(\gamma/|x-x_0|) := \ln(\gamma/|x-x_0|), \quad 0 < |x-x_0| < \gamma, \\ &\ln_{k+1}(\gamma/|x-x_0|) := \ln(\ln_k(\gamma/|x-x_0|)), \quad 0 < |x-x_0| < \gamma/e_{k+1}, \quad k \in \mathbb{N}, \end{aligned} \tag{1.3}$$

for $\gamma > 0$, $x \in \mathbb{R}^n \backslash \{x_0\}$, $n \in \mathbb{N}$, $n \geqslant 2$, with $0 < |x - x_0| < \operatorname{diam}(\Omega) < \gamma/e_m$, where

$$e_1 := 1, \quad e_{k+1} := e^{e_k}, \quad k \in \mathbb{N}. \tag{1.4}$$

We denote $\sum_{j=1}^{0}(\,\cdot\,) := 0$ and $\prod_{k=1}^{0}(\,\cdot\,) := 1$, so when $m = 0$, $x_0 = 0$, (1.2) formally agrees with (1.1).

Date: June 23, 2019.
2010 *Mathematics Subject Classification.* Primary: 35A23, 35J30; Secondary: 47A63, 47F05.
Key words and phrases. Hardy's inequality, factorizations of differential operators, radial and logarithmic refinements.
St. Petersburg Math. J. **30**, 429–436 (2019).

Due to the incredible amount of work on the classical Hardy inequality, we cannot possibly do justice to the existing literature and hence only refer to some of the standard monographs on the subject such as, [3], [15], [16], and [17]. In addition, we note that factorizations in the context of Hardy's inequality in balls with optimal constants and logarithmic correction terms were already studied in [6], [10], based on prior work in [12], [13], and [14], although this appears to have gone unnoticed in the recent literature on this subject. Higher-order logarithmic refinements of the multi-dimensional Hardy–Rellich-type inequality appeared in [1, Theorem 2.1], and a sequence of such multi-dimensional Hardy–Rellich-type inequalities, with additional generalizations, appeared in [19, Theorems 1.8–1.10].

The principal goal in this paper is to offer an improvement of (1.2) by replacing the gradient with the radial derivative ∂_r, given by

$$\partial_r := |x|^{-1} x \cdot \nabla, \quad x \in \mathbb{R}^n \backslash \{0\}, \ r = |x|, \ n \in \mathbb{N}, \ n \geqslant 2. \tag{1.5}$$

Obviously,

$$|(\nabla f)(x)| \geqslant |(\partial_r f)(x)|, \quad x \in \mathbb{R}^n \backslash \{0\}, \ f \in C_0^\infty(\mathbb{R}^n). \tag{1.6}$$

With (1.6) in mind, we will show that (1.1), (1.2) still hold when ∇ is replaced by ∂_r. More precisely, we will prove

$$\int_\Omega |(\partial_r f)(x)|^2 \, d^n x \geqslant \frac{(n-2)^2}{4} \int_\Omega |x|^{-2} |f(x)|^2 \, d^n x, \tag{1.7}$$

valid for $f \in C_0^\infty(\Omega)$, $n \in \mathbb{N}$, $n \geqslant 2$ (again, interpreting the right-hand side of (1.7) as zero in the case $n = 2$), and

$$\begin{split} \int_\Omega |(\partial_r f)(x)|^2 \, d^n x \geqslant \int_\Omega |x - x_0|^{-2} |f(x)|^2 \bigg\{ \frac{(n-2)^2}{4} \\ + \frac{1}{4} \sum_{j=1}^m \prod_{k=1}^j [\ln_k(\gamma/|x - x_0|)]^{-2} \bigg\} \, d^n x, \end{split} \tag{1.8}$$

valid for $f \in C_0^\infty(\Omega)$, assuming that $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 2$, is open and bounded with $x_0 \in \Omega$, and $\gamma > 0$ satisfies $0 < |x - x_0| < \operatorname{diam}(\Omega) < \gamma/e_m$, $m \in \mathbb{N}$.

While (1.7) is well-known, see, for instance, [2, p. 19], [3, Theorem 1.2.5] (in the case $p = 2$, $\varepsilon = 0$), [5], [7], and [18], inequality (1.8) is the principal result of this note.

2. Refinements of Hardy's Inequality

In this section we present our radial and logarithmic refinements of Hardy's inequality.

We start with some preliminary results. Introducing the differential operators, T_0 on $C_0^\infty(\Omega)$ if $0 \notin \Omega$, respectively, $C_0^\infty(\Omega \backslash \{0\})$ if $0 \in \Omega$, $\Omega \subseteq \mathbb{R}^n$ open, and T_m, $m \in \mathbb{N}$, on $C_0^\infty(B_n(0;\rho) \backslash \{0\})$, $n \geqslant 2$, as follows,

$$T_0 := \partial_r + [(n-2)/2]|x|^{-1}, \quad m = 0, \tag{2.1}$$

$$T_m := \partial_r + [(n-2)/2]|x|^{-1} + (1/2)|x|^{-1} \sum_{j=1}^m \prod_{k=1}^j [\ln_k(\gamma/|x|)]^{-1}, \quad m \in \mathbb{N}, \tag{2.2}$$

their formal adjoints (with respect to $L^2(\Omega) := L^2(\Omega; d^n x)$), denoted by T_0^+ and defined on $C_0^\infty(\Omega)$, respectively, on $C_0^\infty(\Omega\backslash\{0\})$, and T_m^+, $m \in \mathbb{N}$, defined on $C_0^\infty(B_n(0;\rho)\backslash\{0\})$, are then given by (cf. (1.3), (1.4))

$$T_0^+ = -\partial_r - (n/2)|x|^{-1}, \quad m = 0, \tag{2.3}$$

$$T_m^+ = -\partial_r - (n/2)|x|^{-1} + (1/2)|x|^{-1} \sum_{j=1}^{m} \prod_{k=1}^{j} [\ln_k(\gamma/|x|)]^{-1}, \quad m \in \mathbb{N}. \tag{2.4}$$

Remark 2.1. In the following we will employ a standard convention when repeated use of differential expressions is involved: Given differential expressions S_j, $j = 1, 2$, their product $S_1 S_2$ is used in the usual (operator) sense, that is,

$$(S_1 S_2 f)(x) = (S_1(S_2 f))(x), \tag{2.5}$$

for f in the underlying function space, and analogously for products of three and more differential expressions. ◇

Next, we note that one obtains inductively,

$$\begin{aligned}
&\partial_r |x|^{-1} \prod_{k=1}^{m} [\ln_k(\gamma/|x|)]^{-1} - |x|^{-1} \prod_{k=1}^{m} [\ln_k(\gamma/|x|)]^{-1} \partial_r \\
&\quad = -|x|^{-2} \prod_{k=1}^{m} [\ln_k(\gamma/|x|)]^{-1} + |x|^{-2} \prod_{k=1}^{m} [\ln_k(\gamma/|x|)]^{-1} \sum_{j=1}^{m-1} \prod_{k=1}^{j} [\ln_k(\gamma/|x|)]^{-1} \\
&\qquad + |x|^{-2} \prod_{k=1}^{m} [\ln_k(\gamma/|x|)]^{-2}, \quad m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},
\end{aligned} \tag{2.6}$$

where again $\sum_{j=1}^{0} (\,\cdot\,) := 0$, $\prod_{k=1}^{0} (\,\cdot\,) = 1$.

Using (2.6), one can prove the following lemma, which will be useful in establishing Theorem 2.4.

Lemma 2.2. *Let $n \in \mathbb{N}, n \geqslant 2$ and $m \in \mathbb{N}_0$. Then*

$$\begin{aligned}
&T_m^+ |x|^{-1} \prod_{k=1}^{m+1} [\ln_k(\gamma/|x|)]^{-1} + |x|^{-1} \prod_{k=1}^{m+1} [\ln_k(\gamma/|x|)]^{-1} T_m \\
&\quad = -|x|^{-2} \prod_{k=1}^{m+1} [\ln_k(\gamma/|x|)]^{-2}.
\end{aligned} \tag{2.7}$$

Proof. First, one notes,

$$\begin{aligned}
T_m^+ |x|^{-1} \prod_{k=1}^{m+1} [\ln_k(\gamma/|x|)]^{-1} &= -\partial_r |x|^{-1} \prod_{k=1}^{m+1} [\ln_k(\gamma/|x|)]^{-1} \\
&\quad - (n/2)|x|^{-2} \prod_{k=1}^{m+1} [\ln_k(\gamma/|x|)]^{-1} \\
&\quad + (1/2)|x|^{-2} \prod_{k=1}^{m+1} [\ln_k(\gamma/|x|)]^{-1} \sum_{j=1}^{m} \prod_{k=1}^{j} [\ln_k(\gamma/|x|)]^{-1},
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned} |x|^{-1}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1}T_m &= |x|^{-1}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1}\partial_r \\ &\quad + [(n-2)/2]|x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1} \\ &\quad + (1/2)|x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}[\ln_k(\gamma/|x|)]^{-1}. \end{aligned} \tag{2.9}$$

Thus, applying (2.6) yields

$$\begin{aligned} &T_m^+|x|^{-1}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1} + |x|^{-1}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1}T_m \\ &\quad = \bigg(-\partial_r|x|^{-1}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1} + |x|^{-1}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1}\partial_r\bigg) \\ &\qquad - |x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1} + |x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}[\ln_k(\gamma/|x|)]^{-1}. \\ &\quad = \bigg(|x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1} - |x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}[\ln_k(\gamma/|x|)]^{-1} \\ &\qquad - |x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-2}\bigg) - |x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1} \\ &\qquad + |x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1}\sum_{j=1}^{m}\prod_{k=1}^{j}[\ln_k(\gamma/|x|)]^{-1} \\ &\quad = -|x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-2}. \end{aligned} \tag{2.10}$$

□

Lemma 2.3. *Let $n \in \mathbb{N}$, $n \geqslant 2$, and $m \in \mathbb{N}_0$. Then*

$$\begin{aligned} T_m^+T_m &= -\partial_r^2 - (n-1)|x|^{-1}\partial_r - [(n-2)/2]^2|x|^{-2} \\ &\quad - (1/4)|x|^{-2}\sum_{j=1}^{m}\prod_{k=1}^{j}[\ln_k(\gamma/|x|)]^{-2}. \end{aligned} \tag{2.11}$$

Proof. We use induction on $m \in \mathbb{N}$. For $m = 0$, one observes

$$\begin{aligned} T_0^+T_0 &= \big(-\partial_r - (n/2)|x|^{-1}\big)\big(\partial_r + [(n-2)/2]|x|^{-1}\big) \\ &= -\partial_r^2 - (n-1)|x|^{-1}\partial_r - [(n-2)/2]^2|x|^{-2}. \end{aligned} \tag{2.12}$$

For $m = 1$, a direct computation, employing (2.6), yields,

$$\begin{aligned} T_1^+T_1 &= \Big(T_0^+ + (1/2)|x|^{-1}[\ln(\gamma/|x|)]^{-1}\Big)\Big(T_0 + (1/2)|x|^{-1}[\ln(\gamma/|x|)]^{-1}\Big) \\ &= T_0^+T_0 - (1/2)|x|^{-2}[\ln(\gamma/|x|)]^{-2} + (1/4)|x|^{-2}[\ln(\gamma/|x|)]^{-2} \end{aligned} \tag{2.13}$$

$$= -\partial_r^2 - (n-1)|x|^{-1}\partial_r + [(n-2)/2]^2|x|^{-2} - (1/4)|x|^{-2}[\ln(\gamma/|x|)]^{-2}.$$

Assuming (2.11) holds for $m \in \mathbb{N}$, an application of Lemma 2.2 then yields for $m+1$,

$$\begin{aligned}
T_{m+1}^+ T_{m+1} &= \bigg(T_m^+ + (1/2)|x|^{-1}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1}\bigg) \\
&\quad \times \bigg(T_m + (1/2)|x|^{-1}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1}\bigg) \\
&= T_m^+ T_m + (1/2)\bigg(T_m^+|x|^{-1}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1} \\
&\quad + |x|^{-1}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-1}T_m\bigg) + (1/4)|x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-2} \\
&= T_m^+ T_m - (1/2)|x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-2} + (1/4)|x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-2} \\
&= T_m^+ T_m - (1/4)|x|^{-2}\prod_{k=1}^{m+1}[\ln_k(\gamma/|x|)]^{-2} \\
&= -\partial_r^2 - (n-1)|x|^{-1}\partial_r - [(n-2)/2]^2|x|^{-2} \\
&\quad - (1/4)|x|^{-2}\sum_{j=1}^{m+1}\prod_{k=1}^{j}[\ln_k(\gamma/|x|)]^{-2}. \qquad (2.14)
\end{aligned}$$

□

Given these preliminaries, we can now show the following result.

Theorem 2.4. *Let $\Omega \subseteq \mathbb{R}^n$ open, $n \in \mathbb{N}$, $n \geqslant 2$.*
(i) Then, for all $f \in C_0^\infty(\Omega)$,

$$\int_\Omega |(\nabla f)(x)|^2\, d^n x \geqslant \int_\Omega |(\partial_r f)(x)|^2\, d^n x \geqslant \frac{(n-2)^2}{4}\int_\Omega |x|^{-2}|f(x)|^2\, d^n x. \qquad (2.15)$$

(ii) Let $m \in \mathbb{N}$, and suppose in addition that $\Omega \subset \mathbb{R}^n$ is bounded with $x_0 \in \Omega$. Assume $\gamma > 0$ is such that $0 < \operatorname{diam}(\Omega) < \gamma/e_m$, where e_m is given as in (1.4), and let $\ln_k(\gamma/|x-x_0|), k \in \mathbb{N}$, be as in (1.3), (1.4). Then, for all $f \in C_0^\infty(\Omega)$,

$$\begin{aligned}
&\int_\Omega |(\nabla f)(x)|^2 d^n x \geqslant \int_\Omega |(\partial_r f)(x)|^2 d^n x \\
&\quad \geqslant \int_\Omega |x-x_0|^{-2}|f(x)|^2\bigg\{\frac{(n-2)^2}{4} + \frac{1}{4}\sum_{j=1}^{m}\prod_{k=1}^{j}[\ln_k(\gamma/|x-x_0|)]^{-2}\bigg\}d^n x.
\end{aligned} \qquad (2.16)$$

Proof. It suffices to focus on item *(ii)* only. In a first step we establish the latter in the special case $\Omega = B_n(0;\rho)$, $x_0 = 0$, with $\rho, \gamma > 0$ and $\rho < \gamma/e_m$. Thus, we will prove for all $f \in C_0^\infty(B_n(0;\rho))$,

$$\begin{aligned} \int_{B_n(0;\rho)} |(\nabla f)(x)|^2 \, d^n x &\geqslant \int_{B_n(0;\rho)} |(\partial_r f)(x)|^2 \, d^n x \\ &\geqslant \int_{B_n(0;\rho)} |x|^{-2} |f(x)|^2 \bigg\{ \frac{(n-2)^2}{4} + \frac{1}{4} \sum_{j=1}^m \prod_{k=1}^j [\ln_k(\gamma/|x|)]^{-2} \bigg\} \, d^n x. \end{aligned} \tag{2.17}$$

Define T_m and T_m^+ as in (2.1)–(2.4), respectively. For simplicity we will work with $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$ for $m \in \mathbb{N}$. However, all integrals extend to $f \in C_0^\infty(B_n(0;\rho))$.

By Lemma 2.3, one has

$$\begin{aligned} 0 &\leqslant \int_{B_n(0;\rho)} |(T_m f)(x)|^2 \, d^n x = \int_{B_n(0;\rho)} \overline{f(x)} (T_m^+ T_m f)(x) \, d^n x \\ &= - \int_{B_n(0;\rho)} \overline{f(x)} (\partial_r^2 f)(x) \, d^n x - (n-1) \int_{B_n(0;\rho)} |x|^{-1} \overline{f(x)} (\partial_r f)(x) \, d^n x \\ &\quad - [(n-2)/2]^2 \int_{B_n(0;\rho)} |x|^{-2} |f(x)|^2 \, d^n x \\ &\quad - (1/4) \sum_{j=1}^m \int_{B_n(0;\rho)} |x|^{-2} |f(x)|^2 \prod_{k=1}^j [\ln_k(\gamma/|x|)]^{-2} \, d^n x. \end{aligned} \tag{2.18}$$

Considering the identity,

$$\begin{aligned} \int_{B_n(0;\rho)} \overline{f(x)} (\partial_r^2 f)(x) \, d^n x &= - \int_{B_n(0;\rho)} |(\partial_r f)(x)|^2 \, d^n x \\ &\quad - (n-1) \int_{B_n(0;\rho)} |x|^{-1} \overline{f(x)} (\partial_r f)(x) \, d^n x, \quad f \in C_0^\infty(B_n(0;\rho)), \end{aligned} \tag{2.19}$$

(2.18) becomes

$$\begin{aligned} 0 &\leqslant \int_{B_n(0;\rho)} |(T_m f)(x)|^2 \, d^n x \\ &= \int_{B_n(0;\rho)} |(\partial_r f)(x)|^2 \, d^n x - [(n-2)/2]^2 \int_{B_n(0;\rho)} |x|^{-2} |f(x)|^2 \, d^n x \\ &\quad - (1/4) \sum_{j=1}^m \int_{B_n(0;\rho)} |x|^{-2} |f(x)|^2 \prod_{k=1}^j [\ln_k(\gamma/|x|)]^{-2} \, d^n x, \end{aligned} \tag{2.20}$$

implying,

$$\begin{aligned} \int_{B_n(0;\rho)} |(\partial_r f)(x)|^2 d^n x \geqslant \int_{B_n(0;\rho)} |x|^{-2} |f(x)|^2 \bigg\{ \frac{(n-2)^2}{4} \\ + \frac{1}{4} \sum_{j=1}^m \prod_{k=1}^j [\ln_k(\gamma/|x|)]^{-2} \bigg\} d^n x. \end{aligned} \tag{2.21}$$

Next, let $\Omega = B_n(x_0;\rho) \subset \mathbb{R}^n$. The proof of (2.16) is entirely analogous to that of (2.17), upon replacing T_m by

$$T_{m,x_0} := \partial_r + [(n-2)/2]|x-x_0|^{-1} + (1/2)|x-x_0|^{-1} \sum_{j=1}^m \prod_{k=1}^j [\ln_k(\gamma/|x-x_0|)]^{-1}, \tag{2.22}$$

and similarly, replacing T_m^+ by

$$T_{m,x_0}^+ = -\partial_r - (n/2)|x-x_0|^{-1} + (1/2)|x-x_0|^{-1} \sum_{j=1}^{m} \prod_{k=1}^{j} [\ln_k(\gamma/|x-x_0|)]^{-1}. \quad (2.23)$$

It then follows that

$$\begin{aligned} T_{m,x_0}^+ T_{m,x_0} &= -\partial_r^2 - (n-1)|x-x_0|^{-1}\partial_r - [(n-2)/2]^2|x-x_0|^{-2} \\ &\quad - (1/4)|x-x_0|^{-2} \sum_{j=1}^{m} \prod_{k=1}^{j} [\ln_k(\gamma/|x-x_0|)]^{-2}, \end{aligned} \quad (2.24)$$

and continuing as in the proof of (2.17) yields (2.16) for $\Omega = B_n(x_0;\rho)$.

For an arbitrary, bounded domain $\Omega \subset \mathbb{R}^n$ with some fixed $x_0 \in \Omega$, one picks some $\rho > 0$ such that $0 < \mathrm{diam}(\Omega) < \rho < \gamma/e_m$. Since $C_0^\infty(\Omega) \subseteq C_0^\infty(B_n(x_0;\rho))$ (extending functions in $C_0^\infty(\Omega)$ by zero outside Ω), inequality (2.16) follows. $\square$

Remark 2.5. (*i*) Upon referring to the spherically symmetric case and oscillation theory for the second-order differential expression

$$-\frac{d^2}{dr^2} - \frac{1}{4r^2} - \frac{1}{4r^2} \sum_{j=1}^{m} \prod_{k=1}^{j} [\ln_k(\gamma/r)]^{-2}, \quad (2.25)$$

with $r > 0$ for $m = 0$ and $0 < r < \gamma/e_m$ for $m \in \mathbb{N}$, discussed in [11], one verifies that the constants $(n-2)^2/4$ and $1/4$ in (2.16) are optimal.

(*ii*) We note that our proof of (2.17), most likely, is not the shortest possible one, but brevity was not the point we had in mind. Instead, as demonstrated in [7] (see also [18]), the value of our strategy of proof, relying on factorizations as in (2.11), lies in the wide applicability of this approach to higher-order inequalities, such as the well-known Rellich inequality and beyond. This will be more systematically explored elsewhere [8]. $\diamond$

We conclude with some applications of (2.15), (2.16) to Schrödinger operators with strongly singular potentials: Let $J \subseteq \mathbb{N}$ be an index set, and $\{x_j\}_{j\in J} \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 2$, be a set of points such that for some $\varepsilon_0 > 0$,

$$\inf_{\substack{j,j'\in J \\ j\neq j'}} |x_j - x_{j'}| \geqslant \varepsilon_0. \quad (2.26)$$

In addition, let $m \in \mathbb{N}$, $\xi_j, \eta_j \in \mathbb{R}$, $j \in J$, and $\delta, \gamma, \xi, \eta \in (0,\infty)$ with

$$|\xi_j| \leqslant \xi < (n-2)^2/4, \quad |\eta_j| \leqslant \eta < 1/4,\ j \in J, \quad 0 < \varepsilon_0 < 4\gamma/e_m, \quad n \geqslant 3. \quad (2.27)$$

Next, we introduce the potential

$$W(x) = \sum_{j\in J} e^{-\delta|x-x_j|} \bigg[\frac{\xi_j}{|x-x_j|^2} + \eta_j \chi_{B_n(x_j;\varepsilon_0/4)}(x) \sum_{\ell=1}^{m} \prod_{k=1}^{\ell} [\ln_k(\gamma/|x-x_j|)]^{-2} \bigg],$$
$$x \in \mathbb{R}^n \backslash \{x_j\}_{j\in J},\ n \geqslant 3, \quad (2.28)$$

with χ_M the characteristic function of $M \subset \mathbb{R}^n$.

Then an application of (2.16) (actually, (2.17) with $\rho = \varepsilon_0/4$) combined with [9, Theorem 3.2] shows that W (and hence, any scalar potential V satisfying $|V| \leqslant |W| + W_0$ a.e. on $\mathbb{R}^n$, with $0 \leqslant W_0 \in L^\infty(\mathbb{R}^n)$) is form bounded with respect to $H_0 = -\Delta$, $\mathrm{dom}(H_0) = H^2(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, $n \geqslant 3$, with form bound strictly less than one (cf. also [4, p. 28–29], and the example in [9, p. 1033–1034]). In this context we recall that dom $\big(H_0^{1/2}\big) = H^1(\mathbb{R}^n)$, and that $C_0^\infty(\mathbb{R}^n)$ is a form core for H_0.

Finally, replacing (2.28) by

$$W(x) = \sum_{j\in J} e^{-\delta|x-x_j|}\eta_j \chi_{B_n(x_j;\varepsilon_0/4)}(x) \sum_{\ell=1}^{m} \prod_{k=1}^{\ell} [\ln_k(\gamma/|x-x_j|)]^{-2}, \quad x \in \mathbb{R}^2\backslash\{x_j\}_{j\in J}, \tag{2.29}$$

with $\delta,\gamma,\eta \in (0,\infty)$ and $|\eta_j| \leqslant \eta < 1/4$, $j \in J$, $0 < \varepsilon_0 < 4\gamma/e_m$, these form boundedness considerations with respect to $H_0 = -\Delta$, $\mathrm{dom}(H_0) = H^2(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$, with form bound strictly less than one, extend to $n = 2$.

Acknowledgments. We are indebted to Gerald Teschl for inquiring whether (1.8) holds, which initiated the present note. In addition, we are grateful to the anonymous referee for a very careful reading of our manuscript.

References

[1] Adimurthi, M. Grossi, and S. Santra, *Optimal Hardy–Rellich inequalities, maximum principle and related eigenvalue problem*, J. Funct. Anal. **240**, 36—83 (2006).
[2] A. Balinsky and W. D. Evans, *Spectral Analysis of Relativistic Operators*, Imperial College Press, London, 2011.
[3] A. A. Balinsky, W. D. Evans, and R. T. Lewis, *The Analysis and Geometry of Hardy's Inequality*, Universitext, Springer, 2015.
[4] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry*, Texts and Monographs in Physics, Springer, Berlin, 1987.
[5] W. G. Faris, *Weak Lebesgue spaces and quantum mechanical binding*, Duke Math. J. **43**, 365–373 (1976).
[6] F. Gesztesy, *On non-degenerate ground states for Schrödinger operators*, Rep. Math. Phys. **20**, 93–109 (1984).
[7] F. Gesztesy and L. L. Littlejohn, *Factorizations and Hardy–Rellich-type inequalities*, in *Non-Linear Partial Differential Equations, Mathematical Physics, and Stochastic Analysis. The Helge Holden Anniversary Volume*, F. Gesztesy, H. Hanche-Olsen, E. R. Jakobsen, Yu. Lyubarskii, N. H. Risebro, and K. Seip (eds.), EMS Congress Reports, EMS, ETH–Zürich, 2018, pp. 207–226.
[8] F. Gesztesy, L. L. Littlejohn, I. Michael, and M. Pang, in preparation.
[9] F. Gesztesy, M. Mitrea, I. Nenciu, and G. Teschl, *Decoupling of deficiency indices and applications to Schrödinger-type operators with possibly strongly singular potentials*, Adv. Math. **301**, 1022–1061 (2016).
[10] F. Gesztesy and L. Pittner, *A generalization of the virial theorem for strongly singular potentials*, Rep. Math. Phys. **18**, 149–162 (1980).
[11] F. Gesztesy and M. Ünal, *Perturbative oscillation criteria and Hardy-type inequalities*, Math. Nachr. **189**, 121–144 (1998).
[12] H. Kalf, *On the characterization of the Friedrichs extension of ordinary or elliptic differential operators with a strongly singular potential*, J. Funct. Anal. **10**, 230–250 (1972).
[13] H. Kalf, U.-W. Schmincke, J. Walter, and R. Wüst, *On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials*, in *Spectral Theory and Differential Equations*, W. N. Everitt (ed.), Lecture Notes in Math., Vol. 448, Springer, Berlin, 1975, pp. 182–226.
[14] H. Kalf and J. Walter, *Strongly singular potentials and essential self-adjointness of singular elliptic operators in $C_0^\infty(\mathbb{R}^n\backslash\{0\})$*, J. Funct. Anal. **10**, 114–130 (1972).
[15] A. Kufner, L. Maligranda, and L.-E. Persson, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelský Servis, Pilsen, 2007.

[16] A. Kufner, L.-E. Persson, and N. Samko, *Weighted Inequalities of Hardy Type*, 2nd ed., World Scientific, Singapore, 2017.
[17] B. Opic and A. Kufner, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series, Vol. 219, Longman Scientific & Technical, Harlow, 1990.
[18] M. Ruzhansky and N. Yessirkegenov, *Factorizations and Hardy–Rellich inequalities on stratified groups*, arXiv:1706.05108.
[19] A. Tertikas and N. B. Zographopoulos, *Best constants in the Hardy–Rellich inequalities and related improvements*, Adv. Math. **209**, 407–459 (2007).

Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798-7328, USA
Email address: Fritz_Gesztesy@baylor.edu
URL: http://www.baylor.edu/math/index.php?id=935340

Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798-7328, USA
Email address: Lance_Littlejohn@baylor.edu
URL: http://www.baylor.edu/math/index.php?id=53980

Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798-7328, USA
Email address: Isaac_Michael@baylor.edu
URL: http://blogs.baylor.edu/isaac_michael/

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
Email address: pangm@missouri.edu
URL: https://www.math.missouri.edu/people/pang