OPTIMALITY OF CONSTANTS IN POWER-WEIGHTED BIRMAN–HARDY–RELLICH-TYPE INEQUALITIES WITH LOGARITHMIC REFINEMENTS

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Abstract. The principal aim of this paper is to establish the optimality (i.e., sharpness) of the constants $A(m,\alpha)$ and $B(m,\alpha)$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, of the form

$$A(m,\alpha) = 4^{-m} \prod_{j=1}^{m} (2j-1-\alpha)^2, \quad B(m,\alpha) = 4^{-m} \sum_{k=1}^{m} \prod_{\substack{j=1 \\ j \neq k}}^{m} (2j-1-\alpha)^2,$$

in the power-weighted Birman–Hardy–Rellich-type integral inequalities with logarithmic refinement terms recently proved in [41], namely,

$$\begin{aligned}\int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 &\geqslant A(m,\alpha) \int_0^\rho dx\, x^{\alpha-2m} \big|f(x)\big|^2 \\ &+ B(m,\alpha) \sum_{k=1}^{N} \int_0^\rho dx\, x^{\alpha-2m} \prod_{p=1}^{k} [\ln_p(\gamma/x)]^{-2} \big|f(x)\big|^2, \\ f \in C_0^\infty((0,\rho)),\ m,N \in \mathbb{N},\ &\alpha \in \mathbb{R},\ \rho,\gamma \in (0,\infty),\ \gamma \geqslant e_N \rho,\end{aligned}$$

where sharpness is meant in the sense that $A(m,\alpha)$ as well as the N constants $B(m,\alpha)$ appearing in this inequality are optimal.

Here the iterated logarithms are given by

$$\ln_1(\,\cdot\,) = \ln(\,\cdot\,), \quad \ln_{j+1}(\,\cdot\,) = \ln(\ln_j(\,\cdot\,)), \quad j \in \mathbb{N},$$

and the iterated exponentials are defined via

$$e_0 = 0, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Moreover, we prove the analogous sequence of inequalities on the exterior interval (r,∞) for $f \in C_0^\infty((r,\infty))$, $r \in (0,\infty)$.

Contents

1. Introduction and Notations Employed

Given the notation introduced in (1.4)–(1.8) we will prove in this paper that the constants $A(m,\alpha)$ and the N constants $B(m,\alpha)$ appearing in the power-weighted

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Birman–Hardy–Rellich-type integral inequalities with logarithmic refinement terms,

$$\begin{aligned}&\int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(m,\alpha)\int_0^\rho dx\, x^{\alpha-2m}\big|f(x)\big|^2\\&\quad + B(m,\alpha)\sum_{k=1}^N \int_0^\rho dx\, x^{\alpha-2m}\prod_{p=1}^k [\ln_p(\gamma/x)]^{-2}\big|f(x)\big|^2,\\&f\in C_0^\infty((0,\rho)),\ m,N\in\mathbb{N},\ \alpha\in\mathbb{R},\ \rho,\gamma\in(0,\infty),\ \gamma\geqslant e_N\rho,\end{aligned} \tag{1.1}$$

recently proved in [41], are optimal (i.e., sharp). Moreover, we prove optimality of $A(m,\alpha)$ and the N constants $B(m,\alpha)$ for the analogous sequence of inequalities on the exterior interval (r,∞), that is,

$$\begin{aligned}&\int_r^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(m,\alpha)\int_r^\infty dx\, x^{\alpha-2m}\big|f(x)\big|^2\\&\quad + B(m,\alpha)\sum_{k=1}^N \int_r^\infty dx\, x^{\alpha-2m}\prod_{p=1}^k [\ln_p(x/\Gamma)]^{-2}\big|f(x)\big|^2,\\&f\in C_0^\infty((r,\infty)),\ m,N\in\mathbb{N},\ \alpha\in\mathbb{R},\ r,\Gamma\in(0,\infty),\ r\geqslant e_N\Gamma.\end{aligned} \tag{1.2}$$

Of course, (1.1) (resp., (1.2)) extends to $N=0$, $\rho=\infty$ (resp., to $N=0$, $r=0$) upon disregarding all logarithmic terms (i.e., upon putting $B(m,\alpha)=0$).

In their simplest (i.e., unweighted) form, the Birman–Hardy–Rellich inequalities, as recorded by Birman in 1961, and in English translation in 1966 [19] (see also [45, pp. 83–84]), are given by

$$\begin{aligned}&\int_0^\rho dx\, \big|f^{(m)}(x)\big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}}\int_0^\rho dx\, x^{-2m}|f(x)|^2,\\&\qquad f\in C_0^m((0,\rho)),\ m\in\mathbb{N},\quad 0<\rho\leqslant\infty.\end{aligned} \tag{1.3}$$

The case $m=1$ in (1.3) represents Hardy's celebrated inequality [51], [52, Sect. 9.8] (see also [61, Chs. 1, 3, App.]), the case $m=2$ is due to Rellich [81, Sect. II.7]. The power-weighted extension of (1.3) is then represented by the first line of (1.1) (i.e., by deleting the second line in (1.1) which contains additional logarithmic refinements).

Even though a detailed history of the power-weighted Birman–Hardy–Rellich inequalities was provided in the companion paper [41], we will now repeat the highlights of this history for matters of completeness.

We start with the observation that the inequalities (1.3) and their power weighted generalizations, that is, the first line in (1.1), are known to be strict, that is, equality holds in (1.3), resp., in the first line in (1.1) (in fact, for the entire inequality (1.1)) if and only if $f=0$ on $(0,\rho)$. Moreover, these inequalities are optimal, meaning, the constants $[(2m-1)!!]^2/2^{2m}$ in (1.3), respectively, the constants $A(m,\alpha)$ in (1.1) are sharp, although, this must be qualified and will be revisited below as different authors frequently prove sharpness for different function spaces. In the present one-dimensional context at hand, sharpness of (1.3) (and typically, it's power weighted version, the first line in (1.1)), are often proved in an integral form (rather than the currently presented differential form) where $f^{(m)}$ on the left-hand side is replaced by F and f on the right-hand side by m repeated integrals over F. For pertinent one-dimensional sources, we refer, for instance, to [14, p. 3–5], [22], [24, p. 104–105], [42, 49, 51], [52, p. 240–243], [61, Ch. 3], [62, p. 5–11], [64, 72, 80]. We also note that higher-order Hardy inequalities, including various weight functions, are discussed in [60, Sect. 5], [61, Chs. 2–5], [62, Chs. 1–4], [63], and [79, Sect. 10] (however, Birman's sequence of inequalities (1.3) is not mentioned in these sources). In addition, there are numerous sources which treat multi-dimensional versions of these inequalities on various domains $\Omega \subseteq \mathbb{R}^n$, which, when specialized to radially symmetric functions (e.g., when Ω represents a ball), imply one-dimensional Birman–Hardy–Rellich-type inequalities with power weights under various restrictions on these weights. However, none of the results obtained in this manner imply (1.1), under optimal hypotheses on α and γ. We also mention that a large number of these references treat the L^p-setting, and in some references $x \in (a,b)$ is replaced by $d(x)$, the distance of x to the boundary of (a,b), respectively, Ω, but this represents quite a different situation (especially in the multi-dimensional context) and hence is not further discussed in this paper.

To put the logarithmic refinements in (1.1) (i.e., the second line in (1.1)) into some perspective and to compare with existing results in the literature, we offer the following comments: originally, logarithmic refinements of Hardy's inequality started with oscillation theoretic considerations going back to Hartman [53] (see also [54, p. 324–325]) and have been used in connection with Hardy's inequality in [38, 43], and more recently, in [39, 40]. Since then there has been enormous activity in this context and we mention, for instance, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12], [14, Chs. 3, 5], [16, 17, 18, 21, 23, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36, 37, 39, 44, 46, 47], [48, Chs. 2,6,7], [56, 57, 65, 66, 67, 68, 70, 71, 74, 76, 77], [81, Sect. 2.7], [82, 83, 84, 88, 89, 90, 91]. The vast majority of these references deals with analogous multi-dimensional settings (relevant to our setting in particular in the case of radially symmetric functions), several also with the L^p-context. For $m \geqslant 2$ the inequalities (1.1) and (1.2) proven in [41] were new in the following sense: the weight parameter $\alpha \in \mathbb{R}$ is unrestricted (as opposed to prior results) and at the same time the conditions on the logarithmic parameters γ and Γ are sharp.

The issue of sharpness of the constants $A(m, \alpha)$ and $B(m, \alpha)$ appearing in (1.1) is a rather delicate one and hence we offer the following remarks, the gist of which can be found in [41, Appendix A].

We start by noting that the smaller the underlying function space, the larger the efforts needed to prove optimality. Many of the results cited in the remainder of this remark, under particular restrictions on the weight parameter α, establish sharpness for larger classes of functions f which do not automatically continue to hold in the $C_0^\infty((0,\rho))$-context. It is this simple observation that adds considerable complexity to sharpness proofs for the space $C_0^\infty((0,\rho))$. (The issue of dependence of optimal constants on the underlying function space is nicely illustrated in [30].) By the same token, optimality proofs obtained for C_0^∞ function spaces automatically hold for larger function spaces as long as the inequalities have already been established for the larger function spaces with the same constants $A(m, \alpha), B(m, \alpha)$. This comment applies, in particular, to many papers that prove sharpness results in multi-dimensional situations for larger function spaces such as[1] $C_0^\infty(B(0;\rho))$ or (homogeneous, weighted) Sobolev spaces rather than $C_0^\infty(B(0;\rho)\backslash\{0\})$. Unless $C_0^\infty(B(0;\rho)\backslash\{0\})$ is dense in the appropriate norm, one cannot *a priori* assume that the optimal constants $A(m, \widetilde{\alpha})$ and $B(m, \widetilde{\alpha})$ (with $\widetilde{\alpha}$ appropriately depending on n,

[1]Here $B(0;\rho) \subseteq \mathbb{R}^n$ denotes the open ball in $\mathbb{R}^n$, $n \geqslant 2$, with center at the origin $x = 0$ and radius $\rho > 0$.

e.g., $\widetilde{\alpha} = \alpha + n - 1$) remain the same for $C_0^\infty(B(0;\rho))$ and $C_0^\infty(B(0;\rho)\backslash\{0\})$, say. At least in principle, they could actually increase for the space $C_0^\infty(B(0;\rho)\backslash\{0\})$.

Turning to a review of the existing literature, sharpness of the constant $A(m,0)$, $m \in \mathbb{N}$ (i.e., in the unweighted case, $\alpha = 0$), corresponding to the space $C_0^\infty((0,\infty))$ has been shown by Yafaev [91]. In fact, he also established this result for fractional m (in this context we also refer to appropriate norm bounds in $L^p(\mathbb{R}^n; d^n x)$ of operators of the form $|x|^{-\beta}| - i\nabla|^{-\beta}$, $1 < p < n/\beta$, see [13, Sect. 1.7], [14, 55, 58, 59, 78, 86, 87, Sects. 1.7, 4.2]). Sharpness of $A(2,0)$ (i.e., in the unweighted Rellich case) was shown by Rellich [81, p. 91–101] in connection with the space $C_0^\infty((0,\infty))$; his multi-dimensional results also yield sharpness of $A(2,n-1)$ for $n \in \mathbb{N}$, $n \geqslant 3$, again for $C_0^\infty((0,\infty))$; in this context see also [14, Corollary 6.3.5]. An exhaustive study of optimality of $A(2,\widetilde{\alpha})$ (i.e., Rellich inequalities with power weights) for the space $C_0^\infty(\Omega\backslash\{0\})$ for cones $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, appeared in Caldiroli and Musina [21]. The authors, in particular, describe situations where $A(2,\widetilde{\alpha})$ has to be replaced by other constants and also treat the special case of radially symmetric functions in detail. Additional results for power weighted Rellich inequalities appeared in [74, 75]; further extensions of power weighted Rellich inequalities with sharp constants on $C_0^\infty(\mathbb{R}^n\backslash\{0\})$ were obtained in [69]; for optimal power weighted Hardy, Rellich, and higher-order inequalities on homogeneous groups, see [82, 83]. Many of these references also discuss sharp (power weighted) Hardy inequalities, implying optimality for $A(1,\widetilde{\alpha})$. Moreover, replacing $f(x)$ by $F(x) = \int_0^x dt\, f(t)$ $\big($or $F(x) = \int_x^\infty dt\, f(t)\big)$, optimality of the Hardy constant $A(1,0)$ for larger, L^p-based function spaces, can already be found in [52, Sect. 9.8] (see also [14, Theorem 1.2.1], [61, Ch. 3], [62, p. 5–11], [64, 72, 80], in connection with $A(1,\alpha)$). We mention that Theorems 4.1 and 4.7, which assert optimality of $A(m,\alpha)$ in (1.1) and (1.2), were already proved in [41, Theorem A.1] using a different method.

Sharpness results for $A(m,\alpha)$ and $B(m,\alpha)$ together are much less frequently discussed in the literature, even under suitable restrictions on m and α. The results we found primarily follow upon specializing multi-dimensional results for function spaces such as $C_0^\infty(\Omega\backslash\{0\})$, or $C_0^\infty(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ open, and appropriate restrictions on m, α, and $n \geqslant 2$, for radially symmetric functions to the one-dimensional case at hand (cf. the previous paragraph). In this context we mention that the Hardy case $m = 1$, without a weight function, is studied in [1, 2, 5, 9, 20, 23, 26, 36, 50, 57, 65, 85, 89] (all for $N = 1$), and in [10, 28, 46] (all for $N \in \mathbb{N}$); the case with power weight functions is discussed in [17], [47], [48, Ch. 6] (for $N \in \mathbb{N}$); see also [66]. The Rellich case $m = 2$ with a general power weight on $C_0^\infty(\Omega\backslash\{0\})$ is discussed in [21] (for $N = 1$); the Rellich case $m = 2$, without weight function on $C_0^\infty(\Omega)$, is studied in [26, 27, 29] (all for $N = 1$), the case $N \in \mathbb{N}$ is studied in [4]; the case of additional power weights is treated in [47], [48, Ch. 6], [71]. The general case $m \in \mathbb{N}$ is discussed in [6] (for $N = 1$) and in [15], [47], [48, Ch. 6], [90] (all for $N \in \mathbb{N}$ and including power weights, but with additional restrictions). Employing oscillation theory, sharpness of the unweighted Hardy case $A(1,0) = B(1,0) = 1/4$, with $N \in \mathbb{N}$, was proved in [43].

As will become clear in the course of this paper, the special results available on sharpness of the N constants $B(m,\alpha)$ are all saddled with considerable complexity, especially, for larger values of $N \in \mathbb{N}$. For this reason only sharpness of the constants $A(m,\alpha)$ was derived in [41, Appendix A] and sharpness of $A(m,\alpha)$ and $B(m,\alpha)$

was postponed to this paper which therefore should be viewed as a companion of [41].

In Section 2 (a very massive one) we establish all the preliminary results, culminating in Lemmas 2.13 and 2.14, required in the remainder of this paper. The methods used in this section are adaptations of those in [15, Sect. 3]. The basic approximation procedure is introduced in Section 3, with Corollaries 3.12 and 3.13 summarizing the principal results. Our final Section 4 then proves optimality of the $N+1$ constants $A(m,\alpha)$ and $B(m,\alpha)$ for the interval $(0,\rho)$ in Theorems 4.1 and 4.2 and for the interval (r,∞) in Theorems 4.7 and 4.8 based on Lemmas 2.13 and 2.14 and Corollaries 3.12 and 3.13. We also mention that Theorems 4.2 and 4.8 still hold if the repeated log-terms $\ln_p(\,\cdot\,)$ (see (1.5) below) are replaced by the type of repeated log-terms used, for example, in [15, 16, 17, 90].[2]

We conclude this introduction by establishing the principal notation used in this paper: for $j \in \mathbb{N}_0$ (with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) we define e_j by

$$e_0 = 0, \quad e_1 = 1, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}. \tag{1.4}$$

For $N \in \mathbb{N}$, $\gamma, \rho \in (0,\infty)$, with $\gamma \geqslant \rho e_N$, and $1 \leqslant j \leqslant N$, we define $\ln_j(\gamma/x)$, for $0 < x < \rho$, by

$$\ln_1(\gamma/x) = \ln(\gamma/x), \quad \ln_{j+1}(\gamma/x) = \ln(\ln_j(\gamma/x)), \quad 1 \leqslant j \leqslant N-1. \tag{1.5}$$

For the rest of this paper we shall assume that $N \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\gamma, \rho \in (0,\infty)$, with $\gamma \geqslant \rho e_{N+1}$. We shall write

$$A(m,\alpha) = 4^{-m} \prod_{j=1}^{m} (2j-1-\alpha)^2, \tag{1.6}$$

$$B(m,\alpha) = 4^{-m} \sum_{k=1}^{m} \prod_{j=1, j\neq k}^{m} (2j-1-\alpha)^2. \tag{1.7}$$

Note that if $\alpha \in \mathbb{R}\backslash\{2j-1\}_{1\leqslant j\leqslant m}$, one has

$$B(m,\alpha) = A(m,\alpha) \sum_{j=1}^{m} (2j-1-\alpha)^{-2}. \tag{1.8}$$

We assume $\psi \in C^\infty(\mathbb{R})$ satisfies the following properties:

$$(i) \quad \psi \text{ is non-increasing}, \tag{1.9}$$

$$(ii) \quad \psi(x) = \begin{cases} 1, & x \leqslant 8\rho/10, \\ 0, & x \geqslant 9\rho/10. \end{cases} \tag{1.10}$$

For $g \in C^\infty((0,\rho))$ we shall write

$$\begin{aligned} J_N[g] = {} & \int_0^\rho dx\, x^\alpha \big|g^{(m)}(x)\big|^2 - A(m,\alpha) \int_0^\rho dx\, x^{\alpha-2m} |g(x)|^2 \\ & - B(m,\alpha) \sum_{k=1}^{N} \int_0^\rho dx\, x^{\alpha-2m} |g(x)|^2 \prod_{j=1}^{k} [\ln_j(\gamma/x)]^{-2}, \end{aligned} \tag{1.11}$$

[2] Detailed proofs of Theorems 4.2 and 4.8 for the type of log-terms used in [15, 16, 17, 90] are available from the authors upon request.

provided that

$$\int_0^\rho dx\, x^\alpha \big|g^{(m)}(x)\big|^2 < \infty, \quad \int_0^\rho dx\, x^{\alpha-2m} |g(x)|^2 < \infty. \tag{1.12}$$

For $j = 0, 1, \dots, N$ and $\beta \in \mathbb{R}$ we introduce

$$\begin{aligned} &\sigma_0(\beta) = (2m - 1 - \alpha + \beta)/2, \\ &\sigma_j(\beta) = -(1 - \beta)/2, \quad j = 1, \dots, N. \end{aligned} \tag{1.13}$$

For $0 \leqslant j \leqslant k \leqslant N$ and $\underline{\varepsilon} = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N)$, where $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N > 0$, we shall write

$$\begin{aligned} \Gamma_{j,k}(\underline{\varepsilon}) &= \Gamma_{j,k}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N), \\ &= \int_0^\rho dx\, x^{-1+\varepsilon_0} [\ln_1(\gamma/x)]^{-1-\varepsilon_1} \cdots [\ln_j(\gamma/x)]^{-1-\varepsilon_j} \\ &\quad \times [\ln_{j+1}(\gamma/x)]^{-\varepsilon_{j+1}} \cdots [\ln_k(\gamma/x)]^{-\varepsilon_k} \\ &\quad \times [\ln_{k+1}(\gamma/x)]^{1-\varepsilon_{k+1}} \cdots [\ln_N(\gamma/x)]^{1-\varepsilon_N} [\psi(x)]^2. \end{aligned} \tag{1.14}$$

In particular, if $N \in \mathbb{N}$,

$$\begin{aligned} \Gamma_{0,0}(\underline{\varepsilon}) &= \int_0^\rho dx\, x^{-1+\varepsilon_0} \prod_{k=1}^N [\ln_k(\gamma/x)]^{1-\varepsilon_k} [\psi(x)]^2, \\ \Gamma_{0,k}(\underline{\varepsilon}) &= \int_0^\rho dx\, x^{-1+\varepsilon_0} \prod_{\ell=1}^k [\ln_\ell(\gamma/x)]^{-\varepsilon_\ell} \prod_{p=k+1}^N [\ln_p(\gamma/x)]^{1-\varepsilon_p} [\psi(x)]^2, \\ &\qquad k = 1, \dots, N, \\ \Gamma_{k,k}(\underline{\varepsilon}) &= \int_0^\rho dx\, x^{-1+\varepsilon_0} \prod_{\ell=1}^k [\ln_\ell(\gamma/x)]^{-1-\varepsilon_\ell} \prod_{p=k+1}^N [\ln_p(\gamma/x)]^{1-\varepsilon_p} [\psi(x)]^2, \\ &\qquad k = 1, \dots, N, \\ \Gamma_{N,N}(\underline{\varepsilon}) &= \int_0^\rho dx\, x^{-1+\varepsilon_0} \prod_{\ell=1}^N [\ln_\ell(\gamma/x)]^{-1-\varepsilon_\ell} [\psi(x)]^2. \end{aligned} \tag{1.15}$$

For $k \in \mathbb{N}$ we shall write P_k for the polynomial

$$P_k(\sigma) = \sigma(\sigma - 1) \cdots (\sigma - k + 1), \quad \sigma \in \mathbb{R}. \tag{1.16}$$

For $\underline{\beta} = (\beta_0, \beta_1, \dots, \beta_N)$, where $\beta_0, \beta_1, \dots, \beta_N \in \mathbb{R}$, we introduce

$$\begin{aligned} v_{\underline{\beta}}(x) &= v_{\beta_0, \beta_1, \dots, \beta_N}(x) \\ &= \begin{cases} x^{\sigma_0(\beta_0)}, & 0 < x < \rho,\ N = 0, \\ x^{\sigma_0(\beta_0)} \prod_{\ell=1}^N [\ln_\ell(\gamma/x)]^{-\sigma_\ell(\beta_\ell)}, & 0 < x < \rho,\ N \in \mathbb{N}, \end{cases} \end{aligned} \tag{1.17}$$

and

$$f_{\underline{\beta}}(x) = f_{\beta_0, \beta_1, \dots, \beta_N}(x) = v_{\underline{\beta}}(x) \psi(x), \quad 0 < x < \rho. \tag{1.18}$$

If $N \in \mathbb{N}$ and $\underline{\varepsilon}_1 = (\varepsilon_1, \dots, \varepsilon_N)$, where $\varepsilon_1, \dots, \varepsilon_N > 0$, we define $h_{\ell, \underline{\varepsilon}_1} : (0, \rho) \to \mathbb{R}$, $\ell \in \mathbb{N}$, iteratively by

$$\begin{aligned} h_{1,\underline{\varepsilon}_1}(x) &= h_{1,\varepsilon_1,\dots,\varepsilon_N}(x) = \sum_{k=1}^N \sigma_k(\varepsilon_k) \prod_{j=1}^k [\ln_j(\gamma/x)]^{-1}, \\ h_{\ell+1,\underline{\varepsilon}_1}(x) &= x h'_{\ell,\underline{\varepsilon}_1}(x), \quad \ell \in \mathbb{N}. \end{aligned} \tag{1.19}$$

Note that, since $\gamma/x > \gamma/\rho \geqslant e_{N+1}$, one infers that

$$[\ln_j(\gamma/x)]^{-1} \leqslant 1, \quad 0 < x < \rho,\ j = 1, \dots, N. \tag{1.20}$$

For $0 \leqslant j \leqslant k \leqslant N$ and $\beta_0, \beta_1, \dots, \beta_N \in \mathbb{R}$, we define $a_{j,k}(\underline{\beta}) = a_{j,k}(\beta_0, \beta_1, \dots, \beta_N)$ by

$$\begin{aligned}
a_{0,0}(\underline{\beta}) &= \big[P_m(\sigma_0(\beta_0))\big]^2 - A(m,\alpha),\\
a_{N,N}(\underline{\beta}) &= \sigma_N(\beta_N)\Big\{P_m(\sigma_0(\beta_0))P_m''(\sigma_0(\beta_0))[\sigma_N(\beta_N)+1]\\
&\qquad + \big[P_m'(\sigma_0(\beta_0))\big]^2\sigma_N(\beta_N)\Big\},\\
a_{j,j}(\underline{\beta}) &= \sigma_j(\beta_j)\Big\{P_m(\sigma_0(\beta_0))P_m''(\sigma_0(\beta_0))[\sigma_j(\beta_j)+1]\\
&\qquad + \big[P_m'(\sigma_0(\beta_0))\big]^2\sigma_j(\beta_j)\Big\} - B(m,\alpha), \quad 1 \leqslant j \leqslant N-1,\\
a_{0,j}(\underline{\beta}) &= 2\sigma_j(\beta_j)P_m(\sigma_0(\beta_0))P_m'(\sigma_0(\beta_0)), \quad 1 \leqslant j \leqslant N,\\
a_{j,k}(\underline{\beta}) &= \sigma_k(\beta_k)\Big\{P_m(\sigma_0(\beta_0))P_m''(\sigma_0(\beta_0))[2\sigma_j(\beta_j)+1]\\
&\qquad + 2\big[P_m'(\sigma_0(\beta_0))\big]^2\sigma_j(\beta_j)\Big\}, \quad 1 \leqslant j < k \leqslant N.
\end{aligned} \tag{1.21}$$

If $N \in \mathbb{N}$, $\beta_0, \beta_1, \dots, \beta_N \in \mathbb{R}$, and $1 \leqslant j \leqslant k \leqslant N$, then we define $b_{j,k}(\underline{\beta}) = b_{j,k}(\beta_0, \beta_1, \dots, \beta_N)$ by

$$\begin{aligned}
b_{j,j}(\underline{\beta}) &= \frac{1}{4}\Big[P_m(\sigma_0(\beta_0))P_m''(\sigma_0(\beta_0)) + \big[P_m'(\sigma_0(\beta_0))\big]^2\Big](\beta_j - \beta_j^2)\\
&\quad + a_{j,j}(\underline{\beta}), \quad 1 \leqslant j \leqslant N,\\
b_{j,k}(\underline{\beta}) &= a_{j,k}(\underline{\beta}) - \frac{1}{4}\Big[P_m(\sigma_0(\beta_0))P_m''(\sigma_0(\beta_0)) + \big[P_m'(\sigma_0(\beta_0))\big]^2\Big]\\
&\quad \times (1-2\beta_j)(1-\beta_k), \quad 1 \leqslant j < k \leqslant N.
\end{aligned} \tag{1.22}$$

For the rest of this paper we shall assume that $M \in (0,\infty)$ is fixed and that $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, constants denoted by c_j, $j \in \mathbb{N}$, will depend on $N \in \mathbb{N} \cup \{0\}$, $\gamma, \rho \in (0,\infty)$ with $\gamma \geqslant \rho e_{N+1}$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $M \in (0,\infty)$, and $\psi \in C^\infty(\mathbb{R})$, but will be independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$.

2. Preliminary Results

We mention again that the methods used in this section are adapted from [15, Sect. 3].

Lemma 2.1. *Let $j \in \{1, \dots, N+1\}$ and $\beta \in \mathbb{R}$. Then, for all $0 < x < \rho$,*

$$\frac{d}{dx}[\ln_j(\gamma/x)]^{-\beta} = \beta x^{-1}[\ln_1(\gamma/x)]^{-1} \cdots [\ln_{j-1}(\gamma/x)]^{-1}[\ln_j(\gamma/x)]^{-1-\beta}. \tag{2.1}$$

Proof. For $j = 1$ (2.1) clearly holds. Suppose that (2.1) holds for $j \in \{1, \dots, N\}$. Then

$$\begin{aligned}
\frac{d}{dx}[\ln_{j+1}(\gamma/x)]^{-\beta} &= \frac{d}{dx}[\ln(\ln_j(\gamma/x))]^{-\beta}\\
&= -\beta[\ln_{j+1}(\gamma/x)]^{-1-\beta}[\ln_j(\gamma/x)]^{-1}\frac{d}{dx}[\ln_j(\gamma/x)]^{-(-1)}\\
&= -\beta[\ln_{j+1}(\gamma/x)]^{-1-\beta}[\ln_j(\gamma/x)]^{-1}(-1)x^{-1}\prod_{k=1}^{j-1}[\ln_k(\gamma/x)]^{-1}
\end{aligned}$$

$$= \beta x^{-1} \prod_{k=1}^{j} [\ln_k(\gamma/x)]^{-1} [\ln_{j+1}(\gamma/x)]^{-1-\beta}. \tag{2.2}$$

The result now follows by induction. □

Lemma 2.2.

(i) $\big[P_m(\sigma_0(0))\big]^2 = A(m,\alpha)$.

(ii) $\frac{1}{4}\Big\{\big[P'_m(\sigma_0(0))\big]^2 - P_m(\sigma_0(0))P''_m(\sigma_0(0))\Big\} = B(m,\alpha)$.

Proof. Since (i) is clear, we only need to prove (ii). Since both sides of (ii) are continuous in α, we may assume that $\alpha \in \mathbb{R}\backslash\{1,3,\dots,2m-1\}$. For $\sigma \in \mathbb{R}\backslash\{0,1,\dots,m-1\}$ one gets

$$\begin{aligned} P'_m(\sigma) &= (\sigma-1)(\sigma-2)\cdots(\sigma-m+1) \\ &\quad + \sigma(\sigma-2)\cdots(\sigma-m+1) + \cdots + \sigma(\sigma-1)\cdots(\sigma-m+2) \\ &= \sigma^{-1}P_m(\sigma) + (\sigma-1)^{-1}P_m(\sigma) + \cdots + (\sigma-m+1)^{-1}P_m(\sigma), \end{aligned} \tag{2.3}$$

hence

$$P'_m(\sigma)\big[P_m(\sigma)\big]^{-1} = \sum_{j=0}^{m-1} (\sigma-j)^{-1}, \tag{2.4}$$

thus, differentiating both sides,

$$P_m(\sigma)P''_m(\sigma) - [P'_m(\sigma)]^2 = -[P_m(\sigma)]^2 \sum_{j=0}^{m-1} (\sigma-j)^{-2}. \tag{2.5}$$

Put $\sigma = (2m-1-\alpha)/2$. Then $\sigma \in \mathbb{R}\backslash\{0,1,\dots,m-1\}$ if and only if $\alpha \in \mathbb{R}\backslash\{1,3,\dots,2m-1\}$. So, by (2.5), part (i), and (1.8), for $\alpha \in \mathbb{R}\backslash\{1,3,\dots,2m-1\}$, one obtains

$$\begin{aligned} &[P'_m((2m-1-\alpha)/2)]^2 - P_m((2m-1-\alpha)/2)P''_m((2m-1-\alpha)/2) \\ &\quad = [P_m((2m-1-\alpha)/2)]^2 \sum_{j=0}^{m-1} \left(\frac{2m-1-\alpha}{2} - j\right)^{-2}, \end{aligned} \tag{2.6}$$

that is,

$$\begin{aligned} &[P'_m(\sigma_0(0))]^2 - P_m(\sigma_0(0))P''_m(\sigma_0(0)) \\ &\quad = 4\big[P_m(\sigma_0(0))\big]^2 \sum_{j=0}^{m-1} (2(m-j)-1-\alpha)^{-2} \\ &\quad = 4A(m,\alpha) \sum_{j=1}^{m} (2j-1-\alpha)^{-2} \\ &\quad = 4B(m,\alpha). \end{aligned} \tag{2.7}$$

□

Remark 2.3. Let $h_{\ell,\underline{\varepsilon}_1} : (0,\rho) \to \mathbb{R}, \ell \in \mathbb{N}$, be as in (1.19). For all $\ell \in \mathbb{N}$ with $\ell \geqslant 3$, there exists $c_1(\ell) > 0$ such that for all $\varepsilon_1,\dots,\varepsilon_N \in (0,M)$ one has

$$|h_{\ell,\underline{\varepsilon}_1}(x)| \leqslant c_1(\ell)[\ln(\gamma/x)]^{-3}, \quad 0 < x < \rho. \tag{2.8}$$

◇

Lemma 2.4. *Suppose $N \in \mathbb{N}$. Let $v_{\underline{\varepsilon}} = v_{\varepsilon_0,\varepsilon_1,\ldots\varepsilon_N} : (0,\rho) \to (0,\infty)$ be defined as in (1.17). Then, for $\tau \in \mathbb{N}$,*

$$\begin{aligned} v_{\underline{\varepsilon}}^{(\tau)}(x) &= x^{\sigma_0(\varepsilon_0)-\tau} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j(\varepsilon_j)} \Big\{ P_\tau(\sigma_0(\varepsilon_0)) \\ &\quad + P_\tau'(\sigma_0(\varepsilon_0)) h_{1,\underline{\varepsilon}_1}(x) + (1/2) P_\tau''(\sigma_0(\varepsilon_0)) [h_{1,\underline{\varepsilon}_1}(x)]^2 + (1/2) P_\tau''(\sigma_0(\varepsilon_0)) h_{2,\underline{\varepsilon}_1}(x) \\ &\quad + E_{\tau,\underline{\varepsilon}}(x) \Big\}, \quad 0 < x < \rho, \end{aligned} \tag{2.9}$$

where $E_{\tau,\underline{\varepsilon}}(x)$ is of the form

$$\begin{aligned} E_{\tau,\underline{\varepsilon}}(x) &= E_{\tau,\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_N}(x) \\ &= \sum_{j=1}^{Q(\tau)} p_{\tau,j} [h_{1,\underline{\varepsilon}_1}(x)]^{w_{\tau,j,1}} \cdots [h_{\tau,\underline{\varepsilon}_1}(x)]^{w_{\tau,j,\tau}}, \quad 0 < x < \rho, \end{aligned} \tag{2.10}$$

for some $Q(\tau) \in \mathbb{N}, w_{\tau,j,k} \in \mathbb{N} \cup \{0\}$ for all $j \in \{1,\ldots,Q(\tau)\}$ and $k \in \{1,\ldots,\tau\}$, $p_{\tau,j} \in \mathbb{R}$ for all $j \in \{1,\ldots,Q(\tau)\}$. Moreover, there exists $c_2 = c_2(\tau) > 0$, independent of $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_N$, such that

$$\big| p_{\tau,j} [h_{1,\underline{\varepsilon}_1}(x)]^{w_{\tau,j,1}} \cdots [h_{\tau,\underline{\varepsilon}_1}(x)]^{w_{\tau,j,\tau}} \big| \leqslant c_2 [\ln(\gamma/x)]^{-3}, \quad 0 < x < \rho, \tag{2.11}$$

for all $j \in \{1,\ldots,Q(\tau)\}$. Hence

$$\big| E_{\tau,\underline{\varepsilon}}(x) \big| \leqslant c_2 Q(\tau) [\ln(\gamma/x)]^{-3}, \quad 0 < x < \rho. \tag{2.12}$$

Proof. We prove this result by induction on $\tau \in \mathbb{N}$. For brevity we shall write $\sigma_j = \sigma_j(\varepsilon_j), j = 0,1,\ldots,N$, in this proof. For $\tau = 1$ we have, by Lemma 2.1,

$$v_{\underline{\varepsilon}}'(x) = x^{\sigma_0-1} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \big(\sigma_0 + h_{1,\underline{\varepsilon}_1}(x) \big), \quad 0 < x < \rho. \tag{2.13}$$

For $\tau = 2$ we have

$$\begin{aligned} v_{\underline{\varepsilon}}''(x) &= x^{\sigma_0-2} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \big(\sigma_0 - 1 + h_{1,\underline{\varepsilon}_1}(x) \big) \big(\sigma_0 + h_{1,\underline{\varepsilon}_1}(x) \big) \\ &\quad + x^{\sigma_0-1} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \big(x^{-1} h_{2,\underline{\varepsilon}_1}(x) \big) \\ &= x^{\sigma_0-2} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \big\{ \sigma_0(\sigma_0-1) + (2\sigma_0-1) h_{1,\underline{\varepsilon}_1}(x) \\ &\quad + [h_{1,\underline{\varepsilon}_1}(x)]^2 + h_{2,\underline{\varepsilon}_1}(x) \big\}. \end{aligned} \tag{2.14}$$

For $\tau = 3$ we have

$$\begin{aligned} v_{\underline{\varepsilon}}'''(x) &= x^{\sigma_0-3} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \big(\sigma_0 - 2 + h_{1,\underline{\varepsilon}_1}(x) \big) \big\{ \sigma_0(\sigma_0-1) \\ &\quad + (2\sigma_0-1) h_{1,\underline{\varepsilon}_1}(x) + [h_{1,\underline{\varepsilon}_1}(x)]^2 + h_{2,\underline{\varepsilon}_1}(x) \big\} \\ &\quad + x^{\sigma_0-3} \prod_{j=1}^{N} [\ln_j(\gamma/x)]^{-\sigma_j} \big\{ (2\sigma_0-1) h_{2,\underline{\varepsilon}_1}(x) + 2 h_{1,\underline{\varepsilon}_1}(x) h_{2,\underline{\varepsilon}_1}(x) \\ &\quad + h_{3,\underline{\varepsilon}_1}(x) \big\} \end{aligned}$$

$$= x^{\sigma_0-3}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-\sigma_j}\Big\{P_3(\sigma_0)+P_3'(\sigma_0)h_{1,\underline{\varepsilon}_1}(x) + (1/2)P_3''(\sigma_0)[h_{1,\underline{\varepsilon}_1}(x)]^2+(1/2)P_3''(\sigma_0)h_{2,\underline{\varepsilon}_1}(x)+E_{3,\underline{\varepsilon}}(x)\Big\}, \tag{2.15}$$

where

$$E_{3,\underline{\varepsilon}}(x) = [h_{1,\underline{\varepsilon}_1}(x)]^3 + 3h_{1,\underline{\varepsilon}_1}(x)h_{2,\underline{\varepsilon}_1}(x) + h_{3,\underline{\varepsilon}_1}(x), \tag{2.16}$$

hence the result holds for $\tau = 3$ by Remark 2.3 and (1.20). Next, we assume that the lemma holds for $\tau \in \mathbb{N}$. Differentiating (2.9) yields

$$\begin{aligned}
v_{\underline{\varepsilon}}^{(\tau+1)}(x) &= x^{\sigma_0-\tau-1}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-\sigma_j}\big(\sigma_0-\tau+h_{1,\underline{\varepsilon}_1}(x)\big)\Big\{P_\tau(\sigma_0) \\
&\quad + P_\tau'(\sigma_0)h_{1,\underline{\varepsilon}_1}(x) + (1/2)P_\tau''(\sigma_0)[h_{1,\underline{\varepsilon}_1}(x)]^2 + (1/2)P_\tau''(\sigma_0)h_{2,\underline{\varepsilon}_1}(x) + E_{\tau,\underline{\varepsilon}}(x)\Big\} \\
&\quad + x^{\sigma_0-\tau-1}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-\sigma_j}\Big\{P_\tau'(\sigma_0)h_{2,\underline{\varepsilon}_1}(x) \\
&\quad + P_\tau''(x)h_{1,\underline{\varepsilon}_1}(x)h_{2,\underline{\varepsilon}_1}(x) + (1/2)P_\tau''(\sigma_0)h_{3,\underline{\varepsilon}_1}(x) + xE_{\tau,\underline{\varepsilon}}'(x)\Big\} \\
&= x^{\sigma_0-(\tau+1)}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-\sigma_j}\Big\{P_\tau(\sigma_0)(\sigma_0-\tau) + \big[P_\tau(\sigma_0) \\
&\quad + P_\tau'(\sigma_0)(\sigma_0-\tau)\big]h_{1,\underline{\varepsilon}_1}(x) + \big[(1/2)P_\tau''(\sigma_0)(\sigma_0-\tau)+P_\tau'(\sigma_0)\big][h_{1,\underline{\varepsilon}_1}(x)]^2 \\
&\quad + \big[(1/2)P_\tau''(\sigma_0)(\sigma_0-\tau)+P_\tau'(\sigma_0)\big]h_{2,\underline{\varepsilon}_1}(x) + E_{\tau+1,\underline{\varepsilon}}(x)\Big\} \\
&= x^{\sigma_0-(\tau+1)}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-\sigma_j}\Big\{P_{\tau+1}(\sigma_0) + P_{\tau+1}'(\sigma_0)h_{1,\underline{\varepsilon}_1}(x) \\
&\quad + (1/2)P_{\tau+1}''(\sigma_0)[h_{1,\underline{\varepsilon}_1}(x)]^2 + (1/2)P_{\tau+1}''(\sigma_0)h_{2,\underline{\varepsilon}_1}(x) + E_{\tau+1,\underline{\varepsilon}}(x)\Big\},
\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}
E_{\tau+1,\underline{\varepsilon}}(x) &= (1/2)P_\tau''(\sigma_0)[h_{1,\underline{\varepsilon}_1}(x)]^3 + (3/2)P_\tau''(\sigma_0)h_{1,\underline{\varepsilon}_1}(x)h_{2,\underline{\varepsilon}_1}(x) \\
&\quad + (\sigma_0-\tau)E_{\tau,\underline{\varepsilon}}(x) + h_{1,\underline{\varepsilon}_1}(x)E_{\tau,\underline{\varepsilon}}(x) + (1/2)P_\tau''(\sigma_0)h_{3,\underline{\varepsilon}_1}(x) + xE_{\tau,\underline{\varepsilon}}'(x).
\end{aligned} \tag{2.18}$$

Thus, by (1.19), $E_{\tau+1,\underline{\varepsilon}}(x)$ can be written in the form

$$E_{\tau+1,\underline{\varepsilon}}(x) = \sum_{j=1}^{Q(\tau+1)} p_{\tau+1,j}[h_{1,\underline{\varepsilon}_1}(x)]^{w_{\tau+1,j,1}}\cdots[h_{\tau+1,\underline{\varepsilon}_1}(x)]^{w_{\tau+1,j,\tau+1}} \tag{2.19}$$

for some $Q(\tau+1) \in \mathbb{N}, w_{\tau+1,j,k} \in \mathbb{N}\cup\{0\}$ for $j \in \{1,\dots,Q(\tau+1)\}$ and $k \in \{1,\dots,\tau+1\}$, $p_{\tau+1,j} \in \mathbb{R}$ for $j \in \{1,\dots,Q(\tau+1)\}$. By (2.18), (1.19), (1.20), and Remark 2.3, there exists $\widetilde{c}_2 > 0$, independent of $\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N \in (0,M)$, such that, for all $0 < x < \rho$,

$$\big|p_{\tau+1,j}[h_{1,\underline{\varepsilon}_1}(x)]^{w_{\tau+1,j,1}}\cdots[h_{\tau+1,\underline{\varepsilon}_1}(x)]^{w_{\tau+1,j,\tau+1}}\big| \leqslant \widetilde{c}_2[\ln(\gamma/x)]^{-3}. \tag{2.20}$$

Hence the lemma holds for $\tau+1$. $\square$

Lemma 2.5. *Suppose $N \in \mathbb{N}$. Let $v_{\underline{\varepsilon}} = v_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N} : (0,\rho) \to (0,\infty)$ be defined as in* (1.17)*, $f_{\underline{\varepsilon}} = f_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N} : (0,\rho) \to [0,\infty)$ be defined as in* (1.18)*, and, for*

$0 \leqslant j \leqslant k \leqslant N$, $a_{j,k}(\underline{\varepsilon}) = a_{j,k}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N)$ *be defined as in* (1.21). *Let* $G_{1,\underline{\varepsilon}} = G_1(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N) \in \mathbb{R}$ *be defined by*[3]

$$\int_0^\rho dx\, x^\alpha \big|f_{\underline{\varepsilon}}^{(m)}(x)\big|^2 = \int_0^\rho dx\, x^\alpha \big|v_{\underline{\varepsilon}}^{(m)}(x)\big|^2 [\psi(x)]^2 + G_{1,\underline{\varepsilon}}. \tag{2.21}$$

Then there exists $c_3 > 0$, *independent of* $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N$, *such that*

$$\big|G_{1,\underline{\varepsilon}}\big| \leqslant c_3, \tag{2.22}$$

and

$$\begin{aligned} J_{N-1}[f_{\underline{\varepsilon}}] &= G_{1,\underline{\varepsilon}} + \sum_{0\leqslant j\leqslant k\leqslant N} a_{j,k}(\underline{\varepsilon})\Gamma_{j,k}(\underline{\varepsilon}) \\ &\quad + \int_0^\rho dx\, x^{2(\sigma_0(\varepsilon_0)-m)+\alpha} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j(\varepsilon_j)} G_{2,\underline{\varepsilon}}(x)[\psi(x)]^2, \end{aligned} \tag{2.23}$$

where $G_{2,\underline{\varepsilon}} = G_{2,\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N} : (0,\rho) \to \mathbb{R}$ *satisfies*

$$\big|G_{2,\underline{\varepsilon}}(x)\big| \leqslant c_3[\ln(\gamma/x)]^{-3}, \quad 0 < x < \rho. \tag{2.24}$$

Proof. We shall write $\sigma_j = \sigma_j(\varepsilon_j)$, $j = 0, 1, \dots, N$, in this proof. By Lemma 2.4 we have

$$\begin{aligned} \big|v_{\underline{\varepsilon}}^{(m)}(x)\big|^2[\psi(x)]^2 &= x^{2(\sigma_0-m)} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j} \Big[P_m(\sigma_0) \\ &\quad + P_m'(\sigma_0)h_{1,\underline{\varepsilon}_1}(x) + \tfrac{1}{2}P_m''(\sigma_0)[h_{1,\underline{\varepsilon}_1}(x)]^2 + \tfrac{1}{2}P_m''(\sigma_0)h_{2,\underline{\varepsilon}_1}(x) + E_{m,\underline{\varepsilon}}(x)\Big]^2[\psi(x)]^2 \\ &= x^{2(\sigma_0-m)} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j} \bigg\{ \big[P_m(\sigma_0)\big]^2 + 2P_m(\sigma_0)P_m'(\sigma_0)h_{1,\underline{\varepsilon}_1}(x) \\ &\quad + \Big[P_m(\sigma_0)P_m''(\sigma_0) + \big[P_m'(\sigma_0)\big]^2\Big][h_{1,\underline{\varepsilon}_1}(x)]^2 + P_m(\sigma_0)P_m''(\sigma_0)h_{2,\underline{\varepsilon}_1}(x) \\ &\quad + G_{2,\underline{\varepsilon}}(x)\bigg\}[\psi(x)]^2, \end{aligned} \tag{2.25}$$

where, by Lemma 2.4, $G_{2,\underline{\varepsilon}} = G_{2,\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N} : (0,\rho) \to \mathbb{R}$ satisfies

$$\big|G_{2,\underline{\varepsilon}}(x)\big| \leqslant c_4[\ln(\gamma/x)]^{-3}, \quad 0 < x < \rho, \tag{2.26}$$

for some $c_4 > 0$ independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$. Direct computation shows

$$\int_0^\rho dx\, x^{2(\sigma_0-m)+\alpha} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j} h_{1,\underline{\varepsilon}_1}(x)[\psi(x)]^2 = \sum_{j=1}^N \sigma_j \Gamma_{0,j}(\underline{\varepsilon}), \tag{2.27}$$

$$\begin{aligned} &\int_0^\rho dx\, x^{2(\sigma_0-m)+\alpha} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j} [h_{1,\underline{\varepsilon}_1}(x)]^2[\psi(x)]^2 \\ &\quad = \sum_{j=1}^N \sigma_j^2 \Gamma_{j,j}(\underline{\varepsilon}) + 2\sum_{1\leqslant j<k\leqslant N} \sigma_j\sigma_k\Gamma_{j,k}(\underline{\varepsilon}), \end{aligned} \tag{2.28}$$

---

[3]One notes that, since $\varepsilon_0 > 0$, (1.10) and Lemma 2.4 imply that the integrals in (2.21) are finite and hence $G_{1,\underline{\varepsilon}}$ is well-defined.

$$\int_0^\rho dx\, x^{2(\sigma_0-m)+\alpha} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j} h_{2,\underline{\varepsilon}_1}(x)[\psi(x)]^2$$
$$= \sum_{j=1}^N \sigma_j \Gamma_{j,j}(\underline{\varepsilon}) + \sum_{1\leqslant j<k\leqslant N} \sigma_k \Gamma_{j,k}(\underline{\varepsilon}). \tag{2.29}$$

Combining (2.25) and (2.27)-(2.29) yields

$$\begin{aligned}
\int_0^\rho dx\, x^\alpha \big|v_{\underline{\varepsilon}}^{(m)}(x)\big|^2 [\psi(x)]^2 &= \big[P_m(\sigma_0)\big]^2 \Gamma_{0,0}(\underline{\varepsilon}) \\
&+ \sum_{j=1}^N 2P_m(\sigma_0)P_m'(\sigma_0)\sigma_j \Gamma_{0,j}(\underline{\varepsilon}) \\
&+ \sum_{j=1}^N \Big\{ \Big[P_m(\sigma_0)P_m''(\sigma_0) + \big[P_m'(\sigma_0)\big]^2\Big]\sigma_j^2 + P_m(\sigma_0)P_m''(\sigma_0)\sigma_j \Big\} \Gamma_{j,j}(\underline{\varepsilon}) \\
&+ \sum_{1\leqslant j<k\leqslant N} \Big\{ 2\Big[P_m(\sigma_0)P_m''(\sigma_0) + \big[P_m'(\sigma_0)\big]^2\Big]\sigma_j\sigma_k + P_m(\sigma_0)P_m''(\sigma_0)\sigma_k \Big\} \Gamma_{j,k}(\underline{\varepsilon}) \\
&+ \int_0^\rho dx\, x^{2(\sigma_0-m)+\alpha} \prod_{j=1}^N [\ln_j(\gamma/x)]^{-2\sigma_j} G_{2,\underline{\varepsilon}}(x)[\psi(x)]^2.
\end{aligned} \tag{2.30}$$

Equation (2.23) now follows from (1.11), (2.21), and (2.30). Since

$$f_{\underline{\varepsilon}}^{(m)}(x) = \sum_{j=0}^m \binom{m}{j} v_{\underline{\varepsilon}}^{(m-j)}(x)\psi^{(j)}(x), \tag{2.31}$$

we have, by (1.10),

$$\begin{aligned}
\big|G_{1,\underline{\varepsilon}}\big| &= \bigg| \int_0^\rho dx\, x^\alpha \big|f_{\underline{\varepsilon}}^{(m)}(x)\big|^2 - \int_0^\rho dx\, x^\alpha \big|v_{\underline{\varepsilon}}^{(m)}(x)\big|^2 [\psi(x)]^2 \bigg| \\
&= \bigg| \int_0^\rho dx\, x^\alpha \bigg\{ 2v_{\underline{\varepsilon}}^{(m)}(x)\psi(x) \sum_{j=1}^m \binom{m}{j} v_{\underline{\varepsilon}}^{(m-j)}(x)\psi^{(j)}(x) \\
&\quad + \bigg(\sum_{j=1}^m \binom{m}{j} v_{\underline{\varepsilon}}^{(m-j)}(x)\psi^{(j)}(x) \bigg)^2 \bigg\} \bigg| \\
&\leqslant 2\sum_{j=1}^m \binom{m}{j} \int_{(0.8)\rho}^{(0.9)\rho} dx\, x^\alpha \big|v_{\underline{\varepsilon}}^{(m)}(x) v_{\underline{\varepsilon}}^{(m-j)}(x)\big| \big|\psi(x)\big| \big|\psi^{(j)}(x)\big| \\
&\quad + \sum_{j,k=1}^m \binom{m}{j}\binom{m}{k} \int_{(0.8)\rho}^{(0.9)\rho} dx x^\alpha \, \big|v_{\underline{\varepsilon}}^{(m-j)}(x) v_{\underline{\varepsilon}}^{(m-k)}(x)\big| \big|\psi^{(j)}(x)\psi^{(k)}(x)\big|.
\end{aligned} \tag{2.32}$$

Hence Lemma 2.4 implies that there exists $c_5 > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that $|G_{1,\underline{\varepsilon}}| \leqslant c_5$. Thus Lemma 2.5 is proved upon putting $c_3 = \max\{c_4, c_5\}$. □

Lemma 2.6. *Let $k \in \{0, 1, \dots, N\}$ and $\beta_0, \beta_1, \dots, \beta_k \geqslant 0$. Then*

$$\int_0^\rho dx\, x^{-1+\beta_0} [\ln_1(\gamma/x)]^{-1-\beta_1} \cdots [\ln_k(\gamma/x)]^{-1-\beta_k} < \infty \tag{2.33}$$

if and only if

$$\begin{cases} \quad \beta_0 > 0, \\ \text{or } \beta_0 = 0 \text{ and } \beta_1 > 0, \\ \text{or } \beta_0 = \beta_1 = 0 \text{ and } \beta_2 > 0, \\ \vdots \\ \text{or } \beta_0 = \beta_1 = \cdots = \beta_{k-1} = 0 \text{ and } \beta_k > 0. \end{cases} \tag{2.34}$$

Proof. This follows from Lemma 2.1 and (1.20). □

Lemma 2.7. *Let $\beta \in (-\infty, 1)$. Then there exists $c_6 = c_6(\beta) > 0$, independent of $\varepsilon_0 \in (0, M)$, such that*

$$\int_0^\rho d\, x^{-1+\varepsilon_0} [\ln_1(\gamma/x)]^{-\beta} [\psi(x)]^2 \leqslant c_6 \varepsilon_0^{-1+\beta}. \tag{2.35}$$

Proof. Writing $\tau = \varepsilon_0^{-1}[\ln(\gamma/\rho)]^{-1} > 0$, and using the change of variables

$$\begin{aligned} s &= \varepsilon_0^{-1}[\ln(\gamma/x)]^{-1} \quad \Big(\text{i.e., } x = \gamma e^{\frac{-1}{\varepsilon_0 s}}\Big), \\ ds &= \varepsilon_0^{-1} x^{-1}[\ln(\gamma/x)]^{-2} dx \quad \Big(\text{i.e., } dx = \gamma \varepsilon_0^{-1} s^{-2} e^{\frac{-1}{\varepsilon_0 s}} ds\Big), \end{aligned} \tag{2.36}$$

one obtains

$$\begin{aligned} &\int_0^\rho dx\, x^{-1+\varepsilon_0} [\ln(\gamma/x)]^{-\beta} [\psi(x)]^2 \leqslant \int_0^\rho dx\, x^{-1+\varepsilon_0} [\ln(\gamma/x)]^{-\beta} \\ &\quad = \gamma^{\varepsilon_0} \varepsilon_0^{-1+\beta} \int_0^\tau ds\, s^{-2+\beta} e^{\frac{-1}{s}} \leqslant \bigg(\gamma^{\varepsilon_0} \int_0^\infty ds\, s^{-2+\beta} e^{\frac{-1}{s}}\bigg) \varepsilon_0^{-1+\beta}. \end{aligned} \tag{2.37}$$

□

Lemma 2.8. *Suppose $N \geqslant 2$. Let $\beta \in (-\infty, 1)$ and $1 \leqslant j \leqslant N-1$. Then there exists $c_7 = c_7(\beta) > 0$, independent of $\varepsilon_j \in (0, M)$, such that*

$$\begin{aligned} &\int_0^\rho dx\, x^{-1} \prod_{k=1}^{j-1} [\ln_k(\gamma/x)]^{-1} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\ln_{j+1}(\gamma/x)]^{-\beta} [\psi(x)]^2 \\ &\quad \leqslant c_7 \varepsilon_j^{-1+\beta}. \end{aligned} \tag{2.38}$$

Proof. Writing $\tau = \varepsilon_j^{-1}[\ln_{j+1}(\gamma/\rho)]^{-1} > 0$, and using the change of variables

$$s = \varepsilon_j^{-1}[\ln_{j+1}(\gamma/x)]^{-1}, \tag{2.39}$$

so that, by Lemma 2.1,

$$ds = \varepsilon_j^{-1} x^{-1} [\ln_1(\gamma/x)]^{-1} \cdots [\ln_j(\gamma/x)]^{-1} [\ln_{j+1}(\gamma/x)]^{-2} dx, \tag{2.40}$$

one gets

$$\begin{aligned} &\int_0^\rho dx\, x^{-1} \prod_{k=1}^{j-1} [\ln_k(\gamma/x)]^{-1} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\ln_{j+1}(\gamma/x)]^{-\beta} [\psi(x)]^2 \\ &\quad \leqslant \varepsilon_j \int_0^\tau ds\, [\ln_j(\gamma/x)]^{-\varepsilon_j} [\ln_{j+1}(\gamma/x)]^{2-\beta}. \end{aligned} \tag{2.41}$$

By (2.39) one has

$$(\varepsilon_j s)^{-1} = \ln(\ln_j(\gamma/x)) \quad \Big(\text{i.e., } \ln_j(\gamma/x) = e^{\frac{1}{\varepsilon_j s}}\Big). \tag{2.42}$$

Hence

$$\begin{aligned}\int_0^\rho dx\, x^{-1} \prod_{k=1}^{j-1} [\ln_k(\gamma/x)]^{-1} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\ln_{j+1}(\gamma/x)]^{-\beta} [\psi(x)]^2 \\ \leqslant \int_0^\tau ds\, \varepsilon_j e^{\frac{-1}{s}} (\varepsilon_j s)^{-2+\beta} \leqslant \bigg(\int_0^\infty ds\, e^{\frac{-1}{s}} s^{-2+\beta}\bigg) \varepsilon_j^{-1+\beta}.\end{aligned} \tag{2.43}$$

□

Next, we need to introduce some more notation: For $\tau \in \{0, 1, \dots, N-1\}$ and $\tau < j \leqslant k \leqslant N$ we write

$$\begin{aligned}\big(\Gamma_\tau(\underline{\varepsilon})\big)_{j,k} = \int_0^\rho dx \bigg\{ x^{-1} \prod_{\ell=1}^{\tau} [\ln_\ell(\gamma/x)]^{-1} \prod_{\ell=\tau+1}^{j} [\ln_\ell(\gamma/x)]^{-1-\varepsilon_\ell} \\ \times \prod_{\ell=j+1}^{k} [\ln_\ell(\gamma/x)]^{-\varepsilon_\ell} \prod_{\ell=k+1}^{N} [\ln_\ell(\gamma/x)]^{1-\varepsilon_\ell} [\psi(x)]^2 \bigg\}.\end{aligned} \tag{2.44}$$

By Lemma 2.6, $\big(\Gamma_\tau(\underline{\varepsilon})\big)_{j,k}$ is well-defined for $\tau \in \{0, 1, \dots, N-1\}$ and $\tau < j \leqslant k \leqslant N$ as the integral on the right-hand side of (2.44) is finite.

Lemma 2.9. *(i) There exists $c_8 > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that*

$$\varepsilon_0 \Gamma_{0,0}(\underline{\varepsilon}) = \sum_{j=1}^{N} (1-\varepsilon_j) \Gamma_{0,j}(\underline{\varepsilon}) + G_{3,\underline{\varepsilon}}, \tag{2.45}$$

and for $j = 1, \dots, N$,

$$\varepsilon_0 \Gamma_{0,j}(\underline{\varepsilon}) = -\sum_{k=1}^{j} \varepsilon_k \Gamma_{k,j}(\underline{\varepsilon}) + \sum_{k=j+1}^{N} (1-\varepsilon_k) \Gamma_{j,k}(\underline{\varepsilon}) + G_{4,j,\underline{\varepsilon}}, \tag{2.46}$$

where

$$\big|G_{3,\underline{\varepsilon}}\big| \leqslant c_8, \quad \big|G_{4,j,\underline{\varepsilon}}\big| \leqslant c_8. \tag{2.47}$$

(ii) Suppose $N \geqslant 2$. Let $1 \leqslant j \leqslant N-1$. Then there exists $c_9 = c_9(j) > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that

$$\varepsilon_j \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{j,j} = \sum_{k=j+1}^{N} (1-\varepsilon_k) \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{j,k} + G_{5,j,\underline{\varepsilon}_j}, \tag{2.48}$$

where $\underline{\varepsilon}_j = (\varepsilon_j, \dots, \varepsilon_N)$, and, for $j+1 \leqslant k \leqslant N$,

$$\varepsilon_j \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{j,k} = -\sum_{\ell=j+1}^{k} \varepsilon_\ell \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{\ell,k} + \sum_{\ell=k+1}^{N} (1-\varepsilon_\ell) \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{k,\ell} + G_{6,j,k,\underline{\varepsilon}_j}, \tag{2.49}$$

and where

$$\big|G_{5,j,\underline{\varepsilon}_j}\big| \leqslant c_9, \quad \big|G_{6,j,k,\underline{\varepsilon}_j}\big| \leqslant c_9. \tag{2.50}$$

(iii) There exists $c_{10} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that

$$\begin{aligned}
&\varepsilon_0^2 \Gamma_{0,0}(\underline{\varepsilon}) - 2\varepsilon_0 \sum_{j=1}^{N} (1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon}) \\
&\quad = \sum_{j=1}^{N} (\varepsilon_j - \varepsilon_j^2)\Gamma_{j,j}(\underline{\varepsilon}) - \sum_{1\leqslant j<k\leqslant N} (1-2\varepsilon_j)(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) + G_{7,\underline{\varepsilon}},
\end{aligned} \tag{2.51}$$

where

$$\left|G_{7,\underline{\varepsilon}}\right| \leqslant c_{10}. \tag{2.52}$$

Proof. (i) We observe

$$\begin{aligned}
&\frac{d}{dx}\Big(x^{\varepsilon_0}[\ln_1(\gamma/x)]^{1-\varepsilon_1}\cdots[\ln_N(\gamma/x)]^{1-\varepsilon_N}[\psi(x)]^2\Big) \\
&\qquad - 2x^{\varepsilon_0}[\ln_1(\gamma/x)]^{1-\varepsilon_1}\cdots[\ln_N(\gamma/x)]^{1-\varepsilon_N}\psi(x)\psi'(x) \\
&\quad = \varepsilon_0 x^{-1+\varepsilon_0}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{1-\varepsilon_j}[\psi(x)]^2. \\
&\qquad - (1-\varepsilon_1)x^{-1+\varepsilon_0}[\ln_1(\gamma/x)]^{-\varepsilon_1}\prod_{j=2}^{N}[\ln_j(\gamma/x)]^{1-\varepsilon_j}[\psi(x)]^2 \\
&\qquad \vdots \\
&\qquad - (1-\varepsilon_N)x^{-1+\varepsilon_0}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-\varepsilon_j}[\psi(x)]^2,
\end{aligned} \tag{2.53}$$

integrating both sides yields

$$G_{3,\underline{\varepsilon}} = \varepsilon_0\Gamma_{0,0}(\underline{\varepsilon}) - \sum_{j=1}^{N}(1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon}). \tag{2.54}$$

Similarly, for $j \in \{1, \dots, N\}$,

$$\begin{aligned}
&\frac{d}{dx}\Bigg(x^{\varepsilon_0}\prod_{k=1}^{j}[\ln_k(\gamma/x)]^{-\varepsilon_k}\prod_{k=j+1}^{N}[\ln_k(\gamma/x)]^{1-\varepsilon_k}[\psi(x)]^2\Bigg) \\
&\qquad - 2x^{\varepsilon_0}\prod_{k=1}^{j}[\ln_k(\gamma/x)]^{-\varepsilon_k}\prod_{k=j+1}^{N}[\ln_k(\gamma/x)]^{1-\varepsilon_k}\psi(x)\psi'(x) \\
&\quad = \varepsilon_0 x^{-1+\varepsilon_0}\prod_{k=1}^{j}[\ln_k(\gamma/x)]^{-\varepsilon_k}\prod_{k=j+1}^{N}[\ln_k(\gamma/x)]^{1-\varepsilon_k}[\psi(x)]^2 \\
&\qquad + \varepsilon_1 x^{-1+\varepsilon_0}[\ln_1(\gamma/x)]^{-1-\varepsilon_1}\prod_{k=2}^{j}[\ln_k(\gamma/x)]^{-\varepsilon_k}\prod_{k=j+1}^{N}[\ln_k(\gamma/x)]^{1-\varepsilon_k}[\psi(x)]^2 \\
&\qquad \vdots \\
&\qquad + \varepsilon_j x^{-1+\varepsilon_0}\prod_{k=1}^{j}[\ln_k(\gamma/x)]^{-1-\varepsilon_k}\prod_{k=j+1}^{N}[\ln_k(\gamma/x)]^{1-\varepsilon_k}[\psi(x)]^2
\end{aligned}$$

$$
\begin{aligned}
& - (1-\varepsilon_{j+1}) x^{-1+\varepsilon_0} \prod_{k=1}^{j} [\ln_k(\gamma/x)]^{-1-\varepsilon_k} [\ln_{j+1}(\gamma/x)]^{-\varepsilon_{j+1}} \\
& \quad \times \prod_{k=j+2}^{N} [\ln_k(\gamma/x)]^{1-\varepsilon_k} [\psi(x)]^2 \\
& \vdots \\
& - (1-\varepsilon_N) x^{-1+\varepsilon_0} \prod_{k=1}^{j} [\ln_k(\gamma/x)]^{-1-\varepsilon_k} \prod_{k=j+1}^{N} [\ln_k(\gamma/x)]^{-\varepsilon_k} [\psi(x)]^2, \qquad (2.55)
\end{aligned}
$$

integrating both sides yields

$$
G_{4,j,\underline{\varepsilon}} = \varepsilon_0 \Gamma_{0,j}(\underline{\varepsilon}) + \sum_{k=1}^{j} \varepsilon_k \Gamma_{k,j}(\underline{\varepsilon}) - \sum_{k=j+1}^{N} (1-\varepsilon_k) \Gamma_{j,k}(\underline{\varepsilon}). \qquad (2.56)
$$

By (1.10), there exists $c_8 > 0$, independent of $\varepsilon_0, \varepsilon_1, \cdots \varepsilon_N \in (0, M)$, such that

$$
\big|G_{3,\underline{\varepsilon}}\big| \leqslant c_8, \quad \big|G_{4,j,\underline{\varepsilon}}\big| \leqslant c_8. \qquad (2.57)
$$

(ii) One has

$$
\begin{aligned}
& \frac{d}{dx}\bigg([\ln_j(\gamma/x)]^{-\varepsilon_j} \prod_{k=j+1}^{N} [\ln_k(\gamma/x)]^{1-\varepsilon_k} [\psi(x)]^2 \bigg) \\
& \quad - 2 [\ln_j(\gamma/x)]^{-\varepsilon_j} \prod_{k=j+1}^{N} [\ln_k(\gamma/x)]^{1-\varepsilon_k} \psi(x)\psi'(x) \\
& = \varepsilon_j x^{-1} \prod_{k=1}^{j-1} [\ln_k(\gamma/x)]^{-1} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} \prod_{k=j+1}^{N} [\ln_k(\gamma/x)]^{1-\varepsilon_k} [\psi(x)]^2 \\
& \quad - (1-\varepsilon_{j+1}) x^{-1} \prod_{k=1}^{j-1} [\ln_k(\gamma/x)]^{-1} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\ln_{j+1}(\gamma/x)]^{-\varepsilon_{j+1}} \\
& \qquad \times \prod_{k=j+2}^{N} [\ln_k(\gamma/x)]^{1-\varepsilon_k} [\psi(x)]^2 \\
& \quad \vdots \\
& \quad - (1-\varepsilon_N) x^{-1} \prod_{k=1}^{j-1} [\ln_k(\gamma/x)]^{-1} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} \\
& \qquad \times \prod_{k=j+1}^{N} [\ln_k(\gamma/x)]^{-\varepsilon_k} [\psi(x)]^2, \qquad (2.58)
\end{aligned}
$$

integrating both sides in (2.58) yields

$$
G_{5,j,\underline{\varepsilon}_j} = \varepsilon_j \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{j,j} - \sum_{k=j+1}^{N} (1-\varepsilon_k) \big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{j,k}. \qquad (2.59)
$$

Similarly one obtains, for $j+1 \leqslant k \leqslant N$,

$$\begin{aligned}
&\frac{d}{dx}\Big([\ln_j(\gamma/x)]^{-\varepsilon_j}\cdots[\ln_k(\gamma/x)]^{-\varepsilon_k}[\ln_{k+1}(\gamma/x)]^{1-\varepsilon_{k+1}}\cdots \\
&\qquad \times [\ln_N(\gamma/x)]^{1-\varepsilon_N}[\psi(x)]^2\Big) - 2[\ln_j(\gamma/x)]^{-\varepsilon_j}\cdots[\ln_k(\gamma/x)]^{-\varepsilon_k} \\
&\qquad \times [\ln_{k+1}(\gamma/x)]^{1-\varepsilon_{k+1}}\cdots[\ln_N(\gamma/x)]^{1-\varepsilon_N}\psi(x)\psi'(x) \\
&= \varepsilon_j x^{-1}\prod_{\ell=1}^{j-1}[\ln_\ell(\gamma/x)]^{-1}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}\prod_{\ell=j+1}^{k}[\ln_\ell(\gamma/x)]^{-\varepsilon_\ell} \\
&\qquad \times \prod_{\ell=k+1}^{N}[\ln_\ell(\gamma/x)]^{1-\varepsilon_\ell}[\psi(x)]^2 \\
&\quad + \\
&\quad \vdots \\
&\quad + \varepsilon_k x^{-1}\prod_{\ell=1}^{j-1}[\ln_\ell(\gamma/x)]^{-1}\prod_{\ell=j}^{k}[\ln_\ell(\gamma/x)]^{-1-\varepsilon_\ell} \\
&\qquad \times \prod_{\ell=k+1}^{N}[\ln_\ell(\gamma/x)]^{1-\varepsilon_\ell}[\psi(x)]^2 \\
&\quad - (1-\varepsilon_{k+1})x^{-1}\prod_{\ell=1}^{j-1}[\ln_\ell(\gamma/x)]^{-1}\prod_{\ell=j}^{k}[\ln_\ell(\gamma/x)]^{-1-\varepsilon_\ell} \\
&\qquad \times [\ln_{k+1}(\gamma/x)]^{-\varepsilon_{k+1}}\prod_{\ell=k+2}^{N}[\ln_\ell(\gamma/x)]^{1-\varepsilon_\ell}[\psi(x)]^2 \\
&\quad - \\
&\quad \vdots \\
&\quad - (1-\varepsilon_N)x^{-1}\prod_{\ell=1}^{j-1}[\ln_\ell(\gamma/x)]^{-1}\prod_{\ell=j}^{k}[\ln_\ell(\gamma/x)]^{-1-\varepsilon_\ell} \\
&\qquad \times \prod_{\ell=k+1}^{N}[\ln_\ell(\gamma/x)]^{-\varepsilon_\ell}[\psi(x)]^2, \qquad (2.60)
\end{aligned}$$

integrating both sides in (2.60) yields

$$G_{6,j,k,\underline{\varepsilon}_j} = \sum_{\ell=j}^{k}\varepsilon_\ell\big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{\ell,k} - \sum_{\ell=k+1}^{N}(1-\varepsilon_\ell)\big(\Gamma_{j-1}(\underline{\varepsilon})\big)_{k,\ell}. \qquad (2.61)$$

By (1.10), there exists $c_9 > 0$, independent of $\varepsilon_j, \dots, \varepsilon_N \in (0, M)$, such that

$$\big|G_{5,j,\underline{\varepsilon}_j}\big| \leqslant c_9, \quad \big|G_{6,j,k,\underline{\varepsilon}_j}\big| \leqslant c_9, \qquad (2.62)$$

for $1 \leqslant j \leqslant N-1$ and $j+1 \leqslant k \leqslant N$.

(iii) By *(i)* we have

$$\begin{aligned}
&\varepsilon_0^2\Gamma_{0,0}(\underline{\varepsilon}) - 2\varepsilon_0 \sum_{j=1}^{N}(1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon}) = -\varepsilon_0 \sum_{j=1}^{N}(1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon}) + \varepsilon_0 G_{3,\underline{\varepsilon}} \\
&\quad = -\sum_{j=1}^{N}(1-\varepsilon_j)\bigg\{-\sum_{k=1}^{j}\varepsilon_k\Gamma_{k,j}(\underline{\varepsilon}) + \sum_{k=j+1}^{N}(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) + G_{4,j,\underline{\varepsilon}}\bigg\} \\
&\qquad + \varepsilon_0 G_{3,\underline{\varepsilon}} \\
&\quad = \sum_{j=1}^{N}\sum_{k=1}^{j}(1-\varepsilon_j)\varepsilon_k\Gamma_{k,j}(\underline{\varepsilon}) - \sum_{j=1}^{N}\sum_{k=j+1}^{N}(1-\varepsilon_j)(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) \\
&\qquad + G_{7,\underline{\varepsilon}},
\end{aligned} \tag{2.63}$$

where there exists $c_{10} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that

$$\big|G_{7,\underline{\varepsilon}}\big| \leqslant c_{10}. \tag{2.64}$$

Thus

$$\begin{aligned}
&\varepsilon_0^2\Gamma_{0,0}(\underline{\varepsilon}) - 2\varepsilon_0 \sum_{j=1}^{N}(1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon}) \\
&\quad = \sum_{j=1}^{N}(\varepsilon_j - \varepsilon_j^2)\Gamma_{j,j}(\underline{\varepsilon}) + \sum_{1\leqslant j<k\leqslant N}(1-\varepsilon_k)\varepsilon_j\Gamma_{j,k}(\underline{\varepsilon}) \\
&\qquad + \sum_{1\leqslant j<k\leqslant N}\varepsilon_j(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) - \sum_{1\leqslant j<k\leqslant N}(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) + G_{7,\underline{\varepsilon}} \\
&\quad = \sum_{j=1}^{N}(\varepsilon_j - \varepsilon_j^2)\Gamma_{j,j}(\underline{\varepsilon}) - \sum_{1\leqslant j<k\leqslant N}(1-2\varepsilon_j)(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) + G_{7,\underline{\varepsilon}}.
\end{aligned} \tag{2.65}$$

□

Lemma 2.10. *Suppose $N \in \mathbb{N}$. Then there exists a constant $c_{11} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, with the following property: Given any fixed $\varepsilon_1, \dots, \varepsilon_N \in (0, M)$, there exists a decreasing sequence $\{\varepsilon_{0,\ell}\}_{\ell=1}^{\infty} \subseteq (0, M)$ and $L_0 \in \mathbb{R}$ such that $\varepsilon_{0,\ell} \downarrow 0$ as $\ell \uparrow \infty$, $|L_0| \leqslant c_{11}$, and, writing $f_{\underline{\varepsilon}} = f_{\varepsilon_{0,\ell},\varepsilon_1,\dots,\varepsilon_N}$ as defined in* (1.18),

$$\lim_{\ell\uparrow\infty} J_{N-1}[f_{\underline{\varepsilon}}] = \sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_N)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k} + L_0. \tag{2.66}$$

Proof. We first note that by Lemma 2.7, there exists $c_{12} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that for all $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$ we have

$$\begin{aligned}
\Gamma_{0,0}(\underline{\varepsilon}) &= \int_0^\rho dx\, x^{-1+\varepsilon_0}[\ln_1(\gamma/x)]^{1-\varepsilon_1}\cdots[\ln_N(\gamma/x)]^{1-\varepsilon_N}[\psi(x)]^2 \\
&\leqslant \int_0^\rho dx\, x^{-1+\varepsilon_0}[\ln_1(\gamma/x)]^{3/2}\bigg\{[\ln_1(\gamma/x)]^{-1/2}\prod_{k=2}^{N}[\ln_k(\gamma/x)]\bigg\}[\psi(x)]^2 \\
&\leqslant c_{12}\varepsilon_0^{-5/2}.
\end{aligned} \tag{2.67}$$

For $j = 1, \dots, N$, by Lemma 2.7, there exists $c_{13} = c_{13}(j) > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that for all $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$ we have

$$\begin{aligned}\Gamma_{0,j}(\underline{\varepsilon}) &= \int_0^\rho dx\, x^{-1+\varepsilon_0} \prod_{k=1}^{j} [\ln_k(\gamma/x)]^{-\varepsilon_k} \prod_{k=j+1}^{N} [\ln_k(\gamma/x)]^{1-\varepsilon_k} [\psi(x)]^2 \\ &\leqslant \int_0^\rho dx\, x^{-1+\varepsilon_0} [\ln_1(\gamma/x)]^{1/2} \bigg\{ [\ln_1(\gamma/x)]^{-1/2} \prod_{k=j+1}^{N} [\ln_k(\gamma/x)] \bigg\} [\psi(x)]^2 \\ &\leqslant c_{13} \varepsilon_0^{-3/2}. \end{aligned} \tag{2.68}$$

Since we are fixing $\varepsilon_1, \cdots \varepsilon_n \in (0, M)$, for $0 \leqslant j \leqslant k \leqslant N$, we shall consider $a_{j,k}(\underline{\varepsilon}) = a_{j,k}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N)$ as functions of $\varepsilon_0 \in (0, M)$ only. Then

$$\begin{aligned} & a_{0,0}(\varepsilon_0) = \big[P_m(\sigma_0(\varepsilon_0))\big]^2 - A(m, \alpha), \\ & a'_{0,0}(\varepsilon_0) = P_m(\sigma_0(\varepsilon_0)) P'_m(\sigma_0(\varepsilon_0)), \\ & a''_{0,0}(\varepsilon_0) = \frac{1}{2} \Big\{ P_m(\sigma_0(\varepsilon_0)) P''_m(\sigma_0(\varepsilon_0)) + \big[P'_m(\sigma_0(\varepsilon_0))\big]^2 \Big\}, \\ & a^{(k)}_{0,0}(\varepsilon_0) = 2^{-k} \bigg\{ \frac{d^k}{d\sigma^k} \Big(\big[P_m(\sigma)\big]^2 \Big) \bigg|_{\sigma = \sigma_0(\varepsilon_0)} \bigg\}, \quad k = 3, \dots, 2m. \end{aligned} \tag{2.69}$$

Similarly one has, for $j = 1, \dots, N$, and $k = 2, \dots, 2m-1$,

$$\begin{aligned} & a_{0,j}(\varepsilon_0) = 2\sigma_j(\varepsilon_j) P_m(\sigma_0(\varepsilon_0)) P'_m(\sigma_0(\varepsilon_0)), \\ & a'_{0,j}(\varepsilon_0) = \sigma_j(\varepsilon_j) \Big\{ \big[P'_m(\sigma_0(\varepsilon_0))\big]^2 + P_m(\sigma_0(\varepsilon_0)) P''_m(\sigma_0(\varepsilon_0)) \Big\}, \\ & a^{(k)}_{0,j}(\varepsilon_0) = 2^{-(k-1)} \sigma_j(\varepsilon_j) \bigg\{ \frac{d^k}{d\sigma^k} \Big(P_m(\sigma) P'_m(\sigma) \Big) \bigg|_{\sigma = \sigma_0(\varepsilon_0)} \bigg\}. \end{aligned} \tag{2.70}$$

Thus, by Lemma 2.2,

$$\begin{aligned} a_{0,0}(\varepsilon_0) &= a_{0,0}(0) + a'_{0,0}(0)\varepsilon_0 + \frac{1}{2} a''_{0,0}(0) \varepsilon_0^2 + \varepsilon_0^3 \bigg(\sum_{k=3}^{2m} (k!)^{-1} a^{(k)}_{0,0}(0) \varepsilon_0^{k-3} \bigg) \\ &= P_m(\sigma_0(0)) P'_m(\sigma_0(0)) \varepsilon_0 + \frac{1}{4} \Big\{ P_m(\sigma_0(0)) P''_m(\sigma_0(0)) + \big[P'_m(\sigma_0(0))\big]^2 \Big\} \varepsilon_0^2 \\ &\quad + \bigg(\sum_{k=3}^{2m} (k!)^{-1} 2^{-k} \bigg\{ \frac{d^k}{d\sigma^k} \Big(\big[P_m(\sigma)\big]^2 \Big) \bigg|_{\sigma = \sigma_0(0)} \bigg\} \varepsilon_0^{k-3} \bigg) \varepsilon_0^3. \end{aligned} \tag{2.71}$$

Put

$$G_8(\varepsilon_0) = \sum_{k=3}^{2m} (k!)^{-1} 2^{-k} \bigg\{ \frac{d^k}{d\sigma^k} \Big(\big[P_m(\sigma)\big]^2 \Big) \bigg|_{\sigma = \sigma_0(0)} \bigg\} \varepsilon_0^{k-3}, \tag{2.72}$$

then there exists $c_{14} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that

$$\big| G_8(\varepsilon_0) \big| \leqslant c_{14}, \quad \varepsilon_0 \in (0, M). \tag{2.73}$$

Similarly, for $j = 1, \dots, N$,

$$\begin{aligned} a_{0,j}(\varepsilon_0) &= a_{0,j}(0) + a'_{0,j}(0)\varepsilon_0 + \sum_{k=2}^{2m-1} (k!)^{-1} a^{(k)}_{0,j}(0) \varepsilon_0^k \\ &= 2\sigma_j(\varepsilon_j) P_m(\sigma_0(0)) P'_m(\sigma_0(0)) \end{aligned}$$

$$+ \sigma_j(\varepsilon_j)\Big\{\big[P_m'(\sigma_0(0))\big]^2 + P_m(\sigma_0(0))P_m''(\sigma_0(0))\Big\}\varepsilon_0$$
$$+ \Bigg(\sum_{k=2}^{2m-1}(k!)^{-1}2^{-(k-1)}\sigma_j(\varepsilon_j)\bigg\{\frac{d^k}{d\sigma^k}\Big(P_m(\sigma)P_m'(\sigma)\Big)\bigg|_{\sigma=\sigma_0(0)}\bigg\}\varepsilon_0^{k-2}\Bigg)\varepsilon_0^2. \quad (2.74)$$

For $j = 1, \dots, N$, put

$$G_{9,j}(\varepsilon_0,\varepsilon_j) = \sum_{k=2}^{2m-1}(k!)^{-1}2^{-(k-1)}\sigma_j(\varepsilon_j)\bigg\{\frac{d^k}{d\sigma^k}\Big(P_m(\sigma)P_m'(\sigma)\Big)\bigg|_{\sigma=\sigma_0(0)}\bigg\}\varepsilon_0^{k-2}, \quad (2.75)$$

then there exists $c_{15} = c_{15}(j) > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that

$$\big|G_{9,j}(\varepsilon_0,\varepsilon_j)\big| \leqslant c_{15}, \quad j = 1,\dots,N,\ \varepsilon_0,\varepsilon_j \in (0,M). \quad (2.76)$$

Hence, applying Lemma 2.9,

$$\begin{aligned}
&a_{0,0}(\underline{\varepsilon})\Gamma_{0,0}(\underline{\varepsilon}) + \sum_{j=1}^N a_{0,j}(\underline{\varepsilon})\Gamma_{0,j}(\underline{\varepsilon})\\
&\quad = P_m(\sigma_0(0))P_m'(\sigma_0(0))\varepsilon_0\Gamma_{0,0}(\underline{\varepsilon})\\
&\qquad + \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\}\varepsilon_0^2\Gamma_{0,0}(\underline{\varepsilon})\\
&\qquad + G_8(\varepsilon_0)\varepsilon_0^3\Gamma_{0,0}(\underline{\varepsilon}) + \sum_{j=1}^N\Big\{2\sigma_j(\varepsilon_j)P_m(\sigma_0(0))P_m'(\sigma_0(0))\Gamma_{0,j}(\underline{\varepsilon})\\
&\qquad + \sigma_j(\varepsilon_j)\Big(\big[P_m'(\sigma_0(0))\big]^2 + P_m(\sigma_0(0))P_m''(\sigma_0(0))\Big)\varepsilon_0\Gamma_{0,j}(\underline{\varepsilon})\\
&\qquad + G_{9,j}(\varepsilon_0,\varepsilon_j)\varepsilon_0^2\Gamma_{0,j}(\underline{\varepsilon})\Big\}\\
&\quad = P_m(\sigma_0(0))P_m'(\sigma_0(0))\bigg\{\varepsilon_0\Gamma_{0,0}(\underline{\varepsilon}) - \sum_{j=1}^N(1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon})\bigg\}\\
&\qquad + \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\}\bigg\{\varepsilon_0^2\Gamma_{0,0}(\underline{\varepsilon})\\
&\qquad - 2\varepsilon_0\sum_{j=1}^N(1-\varepsilon_j)\Gamma_{0,j}(\underline{\varepsilon})\bigg\} + G_8(\varepsilon_0)\varepsilon_0^3\Gamma_{0,0}(\underline{\varepsilon}) + \sum_{j=1}^N G_{9,j}(\varepsilon_0,\varepsilon_j)\varepsilon_0^2\Gamma_{0,j}(\underline{\varepsilon})\\
&\quad = P_m(\sigma_0(0))P_m'(\sigma_0(0))G_{3,\underline{\varepsilon}}\\
&\qquad + \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\}\bigg\{\sum_{j=1}^N(\varepsilon_j-\varepsilon_j^2)\Gamma_{j,j}(\underline{\varepsilon})\\
&\qquad - \sum_{1\leqslant j<k\leqslant N}(1-2\varepsilon_j)(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon}) + G_{7,\underline{\varepsilon}}\bigg\}\\
&\qquad + G_8(\varepsilon_0)\varepsilon_0^3\Gamma_{0,0}(\underline{\varepsilon}) + \sum_{j=1}^N G_{9,j}(\varepsilon_0,\varepsilon_j)\varepsilon_0^2\Gamma_{0,j}(\underline{\varepsilon}). \qquad (2.77)
\end{aligned}$$

Put

$$G_{10,\underline{\varepsilon}} = P_m(\sigma_0(0))P_m'(\sigma_0(0))G_{3,\underline{\varepsilon}}$$

$$+ \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\}G_{7,\underline{\varepsilon}} \qquad (2.78)$$
$$+ G_8(\varepsilon_0)\varepsilon_0^3\Gamma_{0,0}(\underline{\varepsilon}) + \sum_{j=1}^N G_{9,j}(\varepsilon_0,\varepsilon_j)\varepsilon_0^2\Gamma_{0,j}(\underline{\varepsilon}).$$

Then by Lemma 2.9, (2.67), (2.68), (2.73), and (2.76), there exists $c_{16} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that

$$\big|G_{10,\underline{\varepsilon}}\big| \leqslant c_{16}, \quad \varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M). \qquad (2.79)$$

Let $\{\varepsilon_{0,\ell}\}_{\ell=1}^\infty$ be any decreasing sequence in $(0, M)$ with $\lim_{\ell\uparrow\infty} \varepsilon_{0,\ell} = 0$. Applying Lemma 2.5, (2.77), and (2.78), we have, with $\varepsilon_0 = \varepsilon_{0,\ell}$,

$$\begin{aligned}
J_{N-1}[f_{\underline{\varepsilon}}] &= G_{1,\underline{\varepsilon}} + \int_0^\rho dx\, x^{-1+\varepsilon_{0,\ell}} \prod_{j=1}^N [\ln_j(\gamma/x)]^{1-\varepsilon_j} G_{2,\underline{\varepsilon}}(x)[\psi(x)]^2 \\
&\quad + a_{0,0}(\underline{\varepsilon})\Gamma_{0,0}(\underline{\varepsilon}) + \sum_{j=1}^N a_{0,j}(\underline{\varepsilon})\Gamma_{0,j}(\underline{\varepsilon}) + \sum_{1\leqslant j\leqslant k\leqslant N} a_{j,k}(\underline{\varepsilon})\Gamma_{j,k}(\underline{\varepsilon}) \\
&= G_{1,\underline{\varepsilon}} + \int_0^\rho dx\, x^{-1+\varepsilon_{0,\ell}} \prod_{j=1}^N [\ln_j(\gamma/x)]^{1-\varepsilon_j} G_{2,\underline{\varepsilon}}(x)[\psi(x)]^2 \\
&\quad + G_{10,\underline{\varepsilon}} + \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\}\Big\{\sum_{j=1}^N (\varepsilon_j - \varepsilon_j^2)\Gamma_{j,j}(\underline{\varepsilon}) \\
&\quad - \sum_{1\leqslant j<k\leqslant N} (1-2\varepsilon_j)(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon})\Big\} + \sum_{1\leqslant j\leqslant k\leqslant N} a_{j,k}(\underline{\varepsilon})\Gamma_{j,k}(\underline{\varepsilon}) \\
&= \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\}\Big\{\sum_{j=1}^N (\varepsilon_j - \varepsilon_j^2)\Gamma_{j,j}(\underline{\varepsilon}) \\
&\quad - \sum_{1\leqslant j<k\leqslant N} (1-2\varepsilon_j)(1-\varepsilon_k)\Gamma_{j,k}(\underline{\varepsilon})\Big\} + G_{11,\underline{\varepsilon}} + \sum_{1\leqslant j\leqslant k\leqslant N} a_{j,k}(\underline{\varepsilon})\Gamma_{j,k}(\underline{\varepsilon}),
\end{aligned} \qquad (2.80)$$

where

$$\begin{aligned}
G_{11,\underline{\varepsilon}} &= G_1(\varepsilon_{0,\ell}, \varepsilon_1, \dots, \varepsilon_N) \\
&\quad + \int_0^\rho dx\, x^{-1+\varepsilon_{0,\ell}} \prod_{j=1}^N [\ln_j(\gamma/x)]^{1-\varepsilon_j} G_{2,\underline{\varepsilon}}(x)[\psi(x)]^2 \\
&\quad + G_{10}(\varepsilon_{0,\ell}, \varepsilon_1, \dots, \varepsilon_N).
\end{aligned} \qquad (2.81)$$

By (2.24) and Lemma 2.1 there exist $c_{17}, c_{18} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that

$$\begin{aligned}
&\bigg|\int_0^\rho dx\, x^{-1+\varepsilon_0} \prod_{j=1}^N [\ln_j(\gamma/x)]^{1-\varepsilon_j} G_{2,\underline{\varepsilon}}(x)[\psi(x)]^2\bigg| \\
&\quad \leqslant \int_0^\rho dx\, c_3 x^{-1}[\ln_1(\gamma/x)]^{-3/2}\Big\{[\ln_1(\gamma/x)]^{-1/2} \prod_{j=2}^N [\ln_j(\gamma/x)]\Big\}[\psi(x)]^2
\end{aligned}$$

$$\leqslant c_{17} \int_0^\rho dx\, x^{-1} [\ln_1(\gamma/x)]^{-3/2} [\psi(x)]^2 = c_{18} < \infty. \tag{2.82}$$

This, together with (2.22) and (2.79), implies that there exists $c_{11} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, such that

$$\big|G_{11,\underline{\varepsilon}}\big| \leqslant c_{11}, \quad \varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M). \tag{2.83}$$

By compactness of $[-c_{11}, c_{11}]$, there exist a subsequence $\{\varepsilon_{0,\ell_p}\}_{p=1}^\infty$ and $L_0 \in [-c_{11}, c_{11}]$, such that

$$\lim_{p\uparrow\infty} G_{11}(\varepsilon_{0,\ell_p}, \varepsilon_1, \dots, \varepsilon_N) = L_0. \tag{2.84}$$

We shall regard this subsequence as $\{\varepsilon_{0,\ell}\}_{\ell=1}^\infty$. For $1 \leqslant j \leqslant k \leqslant N$ we have, by monotone convergence,

$$\lim_{\ell\uparrow\infty} \Gamma_{j,k}(\varepsilon_{0,\ell}, \varepsilon_1, \dots, \varepsilon_N) = \big(\Gamma_0(\underline{\varepsilon})\big)_{j,k}(\varepsilon_1, \dots, \varepsilon_N). \tag{2.85}$$

The lemma now follows from taking the limit $\ell \uparrow \infty$ in (2.80) and using (2.81) and (2.83)–(2.85). □

Lemma 2.11. *Suppose $N \geqslant 2$. Then there exists a constant $c_{19} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, M)$, with the following property: Let $p \in \{1, \dots, N-1\}$ and let $\varepsilon_{p+1}, \dots, \varepsilon_N \in (0, M)$ be fixed. Then there exist $L_p \in \mathbb{R}$, with $|L_p| \leqslant c_{19}$, and a decreasing sequence $\{\varepsilon_{p,\ell}\}_{\ell=1}^\infty \subseteq (0, M)$ with $\varepsilon_{p,\ell} \downarrow 0$ as $\ell \uparrow \infty$, such that*

$$\begin{aligned} &\lim_{\ell\uparrow\infty} \sum_{p\leqslant j\leqslant k\leqslant N} b_{j,k}(0, \dots, 0, \varepsilon_{p,\ell}, \varepsilon_{p+1}, \dots, \varepsilon_N)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{j,k} \\ &\quad = \sum_{p+1\leqslant j\leqslant k\leqslant N} b_{j,k}(0, \dots, 0, \varepsilon_{p+1}, \dots, \varepsilon_N)\big(\Gamma_p(\underline{\varepsilon})\big)_{j,k} + L_p. \end{aligned} \tag{2.86}$$

Proof. By Lemma 2.2 one obtains

$$\begin{aligned} &b_{p,p}(0, \dots, 0, \varepsilon_p, \varepsilon_{p+1}, \dots, \varepsilon_N) \\ &\quad = \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\}\big(\varepsilon_p - \varepsilon_p^2\big). \\ &\qquad - \frac{1}{2}(1-\varepsilon_p)\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0))\frac{1}{2}(1+\varepsilon_p) - \big[P_m'(\sigma_0(0))\big]^2\frac{1}{2}(1-\varepsilon_p)\Big\} \\ &\qquad - B(m,\alpha) \\ &\quad = \frac{1}{4}\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) - \big[P_m'(\sigma_0(0))\big]^2\Big\}\varepsilon_p = -B(m,\alpha)\varepsilon_p, \end{aligned} \tag{2.87}$$

and, for $j = p+1, \dots, N$, one gets

$$\begin{aligned} &b_{p,j}(0, \dots, 0, \varepsilon_p, \varepsilon_{p+1}, \dots, \varepsilon_N) \\ &\quad = \sigma_j(\varepsilon_j)\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0))\varepsilon_p - \big[P_m'(\sigma_0(0))\big]^2(1-\varepsilon_p)\Big\} \\ &\qquad + \frac{1}{2}\sigma_j(\varepsilon_j)\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) + \big[P_m'(\sigma_0(0))\big]^2\Big\}(1-2\varepsilon_p) \\ &\quad = \frac{1}{2}\sigma_j(\varepsilon_j)\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0)) - \big[P_m'(\sigma_0(0))\big]^2\Big\} = B(m,\alpha)(1-\varepsilon_j). \end{aligned} \tag{2.88}$$

Thus, by Lemma 2.9,

$$\Big| b_{p,p}(0, \dots, 0, \varepsilon_p, \varepsilon_{p+1}, \dots, \varepsilon_N)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{p,p}$$

$$
\begin{aligned}
&\quad + \sum_{j=p+1}^{N} b_{p,j}(0,\dots,0,\varepsilon_p,\varepsilon_{p+1},\dots,\varepsilon_N)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{p,j}\Bigg| \\
&= \Bigg| - B(m,\alpha)\Big\{\varepsilon_p\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{p,p} - \sum_{j=p+1}^{N}(1-\varepsilon_j)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{p,j}\Big\}\Bigg| \\
&\leqslant c_{19},
\end{aligned} \tag{2.89}
$$

where $c_{19} = B(m,\alpha)\max\{c_9(1),\dots,c_9(N-1)\} > 0$ is once again independent of $\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N \in (0,M)$. Hence by compactness of $[-c_{19},c_{19}]$ there exist a decreasing subsequence $\{\varepsilon_{p,\ell}\}_{\ell=1}^{\infty}$ of $\{\frac{1}{\ell}\}_{\ell=1}^{\infty}$ and $L_p \in [-c_{19},c_{19}]$ such that

$$
\begin{aligned}
L_p &= \lim_{\ell\uparrow\infty} b_{p,p}(0,\dots,0,\varepsilon_{p,\ell},\varepsilon_{p+1},\dots,\varepsilon_N)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{p,p} \\
&\qquad + \sum_{j=p+1}^{N} b_{p,j}(0,\dots,0,\varepsilon_{p,\ell},\varepsilon_{p+1},\dots,\varepsilon_N)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{p,j}.
\end{aligned} \tag{2.90}
$$

By monotone convergence

$$
\begin{aligned}
&\lim_{\ell\uparrow\infty} \sum_{p+1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{p,\ell},\varepsilon_{p+1},\dots,\varepsilon_N)\big(\Gamma_{p-1}(\underline{\varepsilon})\big)_{j,k} \\
&\quad = \sum_{p+1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{p+1},\dots,\varepsilon_N)\big(\Gamma_{p}(\underline{\varepsilon})\big)_{j,k}.
\end{aligned} \tag{2.91}
$$

The lemma now follows from (2.90), (2.91). □

Lemma 2.12. *We have*

$$
\lim_{\varepsilon_N\downarrow 0} b_{N,N}(0,\dots,0,\varepsilon_N) = B(m,\alpha). \tag{2.92}
$$

Proof. We have, by Lemma 2.2

$$
\begin{aligned}
&\lim_{\varepsilon_N\downarrow 0} b_{N,N}(0,\dots,0,\varepsilon_N) = \lim_{\varepsilon_N\downarrow 0} a_{N,N}(0,\dots,0,\varepsilon_N) \\
&\quad = \lim_{\varepsilon_N\downarrow 0} -\frac{1}{4}(1-\varepsilon_N)\Big\{P_m(\sigma_0(0))P_m''(\sigma_0(0))(1+\varepsilon_N) - \big[P_m'(\sigma_0(0))\big]^2(1-\varepsilon_N)\Big\} \\
&\quad = \frac{1}{4}\Big\{\big[P_m'(\sigma_0(0))\big]^2 - P_m(\sigma_0(0))P_m''(\sigma_0(0))\Big\} = B(m,\alpha).
\end{aligned} \tag{2.93}
$$

□

Lemma 2.13. *Suppose $N \in \mathbb{N}$. Then given any $\eta > 0$, there exist $\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N \in (0,M)$ such that if $f_{\underline{\varepsilon}} = f_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N}$ is as defined in* (1.18)*, one has*

$$
\left| J_{N-1}[f_{\underline{\varepsilon}}]\left[\int_0^{\rho} dx\, x^{\alpha-2m}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-2}\big|f_{\underline{\varepsilon}}(x)\big|^2\right]^{-1} - B(m,\alpha)\right| \leqslant \eta. \tag{2.94}
$$

Proof. Let $c_{20} = \max\{c_{11},c_{19}\} > 0$, independent of $\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N \in (0,M)$, where c_{11} and c_{19} are as in Lemmas 2.10 and 2.11. By Lemma 2.6 and monotone convergence one infers

$$
\lim_{\varepsilon_N\downarrow 0}\int_0^{\rho} dx\, x^{-1}\prod_{j=1}^{N-1}[\ln_j(\gamma/x)]^{-1}[\ln_N(\gamma/x)]^{-1-\varepsilon_N}[\psi(x)]^2 = \infty. \tag{2.95}
$$

Thus, we can choose $\varepsilon_N \in (0,M)$ sufficiently small such that

$$\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-1} [\ln_j(\gamma/x)]^{-1} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 > 1, \tag{2.96}$$

and

$$c_{20} \bigg[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-1} [\ln_j(\gamma/x)]^{-1} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 \bigg]^{-1} < \eta, \tag{2.97}$$

and, by Lemma 2.12,

$$\big| b_{N,N}(0,\dots,0,\varepsilon_N) - B(m,\alpha) \big| < \eta. \tag{2.98}$$

Thus, for any $R_{N-1} \in [-c_{20}, c_{20}]$, one has

$$\begin{aligned}
&\bigg| \big\{ b_{N,N}(0,\dots,0,\varepsilon_N) \big(\Gamma_{N-1}(\underline{\varepsilon})\big)_{N,N} + R_{N-1} \big\} \bigg[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-1} [\ln_j(\gamma/x)]^{-1} \\
&\qquad \times [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 \bigg]^{-1} - B(m,\alpha) \bigg| \\
&\quad \leqslant \big| b_{N,N}(0,\dots,0,\varepsilon_N) - B(m,\alpha) \big| \\
&\qquad + c_{20} \bigg| \bigg[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-1} [\ln_j(\gamma/x)]^{-1} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 \bigg]^{-1} \bigg| \\
&\quad < 2\eta.
\end{aligned} \tag{2.99}$$

Suppose first that $N \geqslant 2$. Then, by Lemma 2.11, there exist $L_{N-1} \in [-c_{19}, c_{19}]$ and a decreasing sequence $\{\varepsilon_{N-1,\ell}\}_{\ell=1}^\infty \subseteq (0,M)$, with $\lim_{\ell\uparrow\infty} \varepsilon_{N-1,\ell} = 0$, such that

$$\begin{aligned}
&\lim_{\ell\uparrow\infty} \sum_{N-1 \leqslant j \leqslant k \leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1,\ell},\varepsilon_N) \big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k} \\
&\quad = b_{N,N}(0,\dots,0,\varepsilon_N) \big(\Gamma_{N-1}(\underline{\varepsilon})\big)_{N,N} + L_{N-1}.
\end{aligned} \tag{2.100}$$

By (2.96) and monotone convergence, and replacing $\{\varepsilon_{N-1,\ell}\}_{\ell=1}^\infty$ by a subsequence if necessary, one can assume that

$$\begin{aligned}
&\int_0^\rho dx \bigg\{ x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} [\ln_{N-1}(\gamma/x)]^{-1-\varepsilon_{N-1,\ell}} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} \\
&\quad \times [\psi(x)[^2 \bigg\} > 1, \quad \ell \in \mathbb{N}.
\end{aligned} \tag{2.101}$$

Combining (2.97), (2.100), (2.101), and (2.99) with $R_{N-1} = L_{N-1}$, and using monotone convergence, there exists $\varepsilon_{N-1} \in (0,M)$ satisfying

$$\begin{aligned}
&\bigg| \bigg\{ \sum_{N-1 \leqslant j \leqslant k \leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N) \big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k} \bigg\} \\
&\quad \times \bigg(\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} [\ln_{N-1}(\gamma/x)]^{-1-\varepsilon_{N-1}} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} \\
&\quad \times [\psi(x)]^2 \bigg)^{-1} - B(m,\alpha) \bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \bigg| \bigg\{ \sum_{N-1 \leqslant j \leqslant k \leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k} \\
&\quad - \Big[b_{N,N}(0,\dots,0,\varepsilon_N)\big(\Gamma_{N-1}(\underline{\varepsilon})\big)_{N,N} + L_{N-1} \Big] \bigg\} \\
&\quad\quad \times \bigg[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} \\
&\quad\quad \times [\ln_{N-1}(\gamma/x)]^{-1-\varepsilon_{N-1}} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 \bigg]^{-1} \bigg| \\
&\quad + \bigg| \Big[b_{N,N}(0,\dots,0,\varepsilon_N)\big(\Gamma_{N-1}(\underline{\varepsilon})\big)_{N,N} + L_{N-1} \Big] \\
&\quad\quad \times \bigg[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} \\
&\quad\quad \times [\ln_{N-1}(\gamma/x)]^{-1-\varepsilon_{N-1}} [\ln_N(\gamma/x)]^{-1-\varepsilon_N} [\psi(x)]^2 \bigg]^{-1} - B(m,\alpha) \bigg| \\
&< \eta + 2\eta = 3\eta,
\end{aligned} \tag{2.102}
$$

and

$$
c_{20} \bigg[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} \prod_{j=N-1}^{N} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\psi(x)]^2 \bigg]^{-1} < \eta, \tag{2.103}
$$

as well as

$$
\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} \prod_{j=N-1}^{N} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\psi(x)]^2 > 1. \tag{2.104}
$$

One notes that by (2.102), (2.103), for all $R_{N-2} \in [-c_{20}, c_{20}]$,

$$
\begin{aligned}
&\bigg| \bigg\{ \sum_{N-1 \leqslant j \leqslant k \leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k} + R_{N-2} \bigg\} \\
&\quad\quad \times \bigg[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} \prod_{j=N-1}^{N} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\psi(x)]^2 \bigg]^{-1} \\
&\quad - B(m,\alpha) \bigg| \\
&\leqslant \bigg| \bigg\{ \sum_{N-1 \leqslant j \leqslant k \leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k} \bigg\} \\
&\quad\quad \times \bigg[\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} \prod_{j=N-1}^{N} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\psi(x)]^2 \bigg]^{-1} \\
&\quad - B(m,\alpha) \bigg| + c_{20} \bigg(\int_0^\rho dx\, x^{-1} \prod_{j=1}^{N-2} [\ln_j(\gamma/x)]^{-1} \prod_{j=N-1}^{N} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} \\
&\quad\quad \times [\psi(x)]^2 \bigg)^{-1}
\end{aligned}
$$

$$< 3\eta + \eta = 4\eta. \tag{2.105}$$

So we have chosen $\varepsilon_{N-1}, \varepsilon_N \in (0, M)$. If $N - 1 \geqslant 2$, then, by Lemma 2.11, there exist $L_{N-2} \in [-c_{19}, c_{19}]$ and a decreasing sequence $\{\varepsilon_{N-2,\ell}\}_{\ell=1}^{\infty} \subseteq (0, M)$ with $\lim_{\ell\uparrow\infty} \varepsilon_{N-2,\ell} = 0$ such that

$$\begin{aligned}
&\lim_{\ell\uparrow\infty} \sum_{N-2\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-2,\ell},\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-3}(\underline{\varepsilon})\big)_{j,k} \\
&\quad = \sum_{N-1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k} + L_{N-2}.
\end{aligned} \tag{2.106}$$

By (2.104) and monotone convergence, and replacing $\{\varepsilon_{N-2,\ell}\}_{\ell=1}^{\infty}$ by a subsequence, if necessary, one can assume that

$$\begin{aligned}
&\int_0^{\rho} dx\, x^{-1} \prod_{j=1}^{N-3} [\ln_j(\gamma/x)]^{-1} [\ln_{N-2}(\gamma/x)]^{-1-\varepsilon_{N-2,\ell}} \\
&\quad \times \prod_{j=N-1}^{N} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\psi(x)]^2 > 1, \quad \ell \in \mathbb{N}.
\end{aligned} \tag{2.107}$$

Combining (2.103), (2.106), (2.107), and (2.105) with $R_{N-2} = L_{N-2}$, and monotone convergence, there exists $\varepsilon_{N-2} \in (0, M)$ satisfying

$$\begin{aligned}
&\bigg|\bigg\{ \sum_{N-2\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-2},\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-3}(\underline{\varepsilon})\big)_{j,k} \bigg\} \\
&\qquad \times \bigg[\int_0^{\rho} dx\, x^{-1} \prod_{j=1}^{N-3} [\ln_j(\gamma/x)]^{-1} \prod_{j=N-2}^{N} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\psi(x)]^2 \bigg]^{-1} \\
&\quad - B(m,\alpha) \bigg| \\
&\leqslant \bigg|\bigg\{ \sum_{N-2\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-2},\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-3}(\underline{\varepsilon})\big)_{j,k} \\
&\quad - \bigg[\sum_{N-1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k} + L_{N-2} \bigg] \bigg\} \\
&\qquad \times \bigg[\int_0^{\rho} dx\, x^{-1} \prod_{j=1}^{N-3} [\ln_j(\gamma/x)]^{-1} \prod_{j=N-2}^{N} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\psi(x)]^2 \bigg]^{-1} \bigg| \\
&\quad + \bigg|\bigg[\sum_{N-1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-2}(\underline{\varepsilon})\big)_{j,k} + L_{N-2} \bigg] \\
&\qquad \times \bigg[\int_0^{\rho} dx\, x^{-1} \prod_{j=1}^{N-3} [\ln_j(\gamma/x)]^{-1} \prod_{j=N-2}^{N} [\ln_j(\gamma/x)]^{-1-\varepsilon_j} [\psi(x)]^2 \bigg]^{-1} \\
&\quad - B(m,\alpha) \bigg| \\
&< \eta + 4\eta = 5\eta,
\end{aligned} \tag{2.108}$$

and

$$c_{20}\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1}<\eta, \tag{2.109}$$

as well as

$$\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2>1, \tag{2.110}$$

such that for all $R_{N-3}\in[-c_{20},c_{20}]$ one infers

$$\begin{aligned}
&\Bigg|\bigg\{\sum_{N-2\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-2},\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-3}(\underline{\varepsilon})\big)_{j,k}+R_{N-3}\bigg\}\\
&\qquad\times\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1}\\
&\quad-B(m,\alpha)\Bigg|\\
&\leqslant\Bigg|\bigg\{\sum_{N-2\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\dots,0,\varepsilon_{N-2},\varepsilon_{N-1},\varepsilon_N)\big(\Gamma_{N-3}(\underline{\varepsilon})\big)_{j,k}\bigg\}\\
&\qquad\times\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1}\\
&\quad-B(m,\alpha)\Bigg|\\
&\quad+c_{20}\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N-3}[\ln_j(\gamma/x)]^{-1}\prod_{j=N-2}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1}\\
&<5\eta+\eta=6\eta.
\end{aligned} \tag{2.111}$$

Repeating the argument above $N-1$ times (or if $N=1$) one arrives at the following fact: there exist $\varepsilon_1,\dots,\varepsilon_N\in(0,M)$ such that

$$\begin{aligned}
&\Bigg|\bigg\{\sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_N)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k}\bigg\}\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}\\
&\quad\times[\psi(x)]^2\bigg]^{-1}-B(m,\alpha)\Bigg|\leqslant(2N-1)\eta,
\end{aligned} \tag{2.112}$$

and

$$c_{20}\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1}<\eta, \tag{2.113}$$

as well as

$$\int_0^\rho dx\, x^{-1}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2>1, \tag{2.114}$$

so that for all $R_0 \in [-c_{20}, c_{20}]$ one obtains

$$\begin{aligned}
&\bigg|\bigg\{\sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_N)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k} + R_0\bigg\} \\
&\qquad \times \bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^N [\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1} - B(m,\alpha)\bigg| \\
&\quad \leqslant \bigg|\bigg\{\sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_N)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k}\bigg\}\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^N [\ln_j(\gamma/x)]^{-1-\varepsilon_j} \\
&\qquad \times [\psi(x)]^2\bigg]^{-1} - B(m,\alpha)\bigg| \\
&\qquad + c_{20}\bigg[\int_0^\rho dx\, x^{-1}\prod_{j=1}^N [\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1} \\
&\quad < (2N-1)\eta + \eta = 2N\eta.
\end{aligned} \tag{2.115}$$

Then, by Lemma 2.10, there exist $L_0 \in [-c_{20}, c_{20}]$ and a decreasing sequence $\{\varepsilon_{0,\ell}\}_{\ell=1}^\infty \subseteq (0,M)$ with $\lim_{\ell\uparrow\infty} \varepsilon_{0,\ell} = 0$ such that

$$\lim_{\ell\uparrow\infty} J_{N-1}[f_{\varepsilon_{0,\ell},\varepsilon_1,\dots,\varepsilon_N}] = \sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_N)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k} + L_0. \tag{2.116}$$

By (2.114) and monotone convergence, and replacing $\{\varepsilon_{0,\ell}\}_{\ell=1}^\infty$ by a subsequence if necessary, we can assume that

$$\int_0^\rho dx\, x^{-1+\varepsilon_{0,\ell}}\prod_{j=1}^N [\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2 > 1, \quad \ell \in \mathbb{N}. \tag{2.117}$$

Combining (2.112), (2.113), (2.115) with $R_0 = L_0$, (2.116), (2.117), and monotone convergence, there exists $\varepsilon_0 \in (0,M)$ satisfying

$$\begin{aligned}
&\bigg|J_{N-1}[f_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N}]\bigg[\int_0^\rho dx\, x^{-1+\varepsilon_0}\prod_{j=1}^N [\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1} \\
&\qquad - B(m,\alpha)\bigg| \\
&\quad \leqslant \bigg|\bigg[J_{N-1}[f_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N}] - \bigg\{\sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_n)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k} \\
&\qquad + L_0\bigg\}\bigg]\bigg[\int_0^\rho dx\, x^{-1+\varepsilon_0}\prod_{j=1}^N [\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1}\bigg| \\
&\qquad + \bigg|\bigg\{\sum_{1\leqslant j\leqslant k\leqslant N} b_{j,k}(0,\varepsilon_1,\dots,\varepsilon_n)\big(\Gamma_0(\underline{\varepsilon})\big)_{j,k} + L_0\bigg\} \\
&\qquad\quad \times \bigg[\int_0^\rho dx\, x^{-1+\varepsilon_0}\prod_{j=1}^N [\ln_j(\gamma/x)]^{-1-\varepsilon_j}[\psi(x)]^2\bigg]^{-1} - B(m,\alpha)\bigg| \\
&\quad < \eta + 2N\eta = (2N+1)\eta.
\end{aligned} \tag{2.118}$$

□

Lemma 2.14. *Suppose* $N = 0$ *and let* f_{ε_0} *be as defined on* (1.18). *Then*

$$\lim_{\varepsilon_0 \downarrow 0} \int_0^\rho dx\, x^\alpha \big|f_{\varepsilon_0}^{(m)}(x)\big|^2 \bigg[\int_0^\rho dx\, x^{\alpha-2m}|f_{\varepsilon_0}(x)|^2\bigg]^{-1} = A(m,\alpha). \tag{2.119}$$

Proof. By (1.10) we have

$$\lim_{\varepsilon_0 \downarrow 0} \int_0^\rho dx\, x^{\alpha-2m}|f_{\varepsilon_0}(x)|^2 \geqslant \lim_{\varepsilon_0 \downarrow 0} \int_0^{(0.8)\rho} dx\, x^{-1+\varepsilon_0} = \infty. \tag{2.120}$$

In addition, one has

$$f_{\varepsilon_0}^{(m)}(x) = \sum_{j=0}^m \binom{m}{j} P_j(\sigma_0(\varepsilon_0)) x^{\sigma_0(\varepsilon_0)-j} \psi^{(m-j)}(x), \quad 0 < x < \rho. \tag{2.121}$$

Thus, for all $0 < x < \rho$,

$$\begin{aligned} x^\alpha \big|f_{\varepsilon_0}^{(m)}(x)\big|^2 &= \sum_{j,k=0}^m \binom{m}{j}\binom{m}{k} P_j(\sigma_0(\varepsilon_0)) P_k(\sigma_0(\varepsilon_0)) x^{\alpha+2\sigma_0(\varepsilon_0)-j-k} \\ &\quad \times \psi^{(m-j)}(x)\psi^{(m-k)}(x) \\ &= \big[P_m(\sigma_0(\varepsilon_0))\big]^2 x^{-1+\varepsilon_0}[\psi(x)]^2 + G_{12}(\varepsilon_0,x) \\ &= A(m,\alpha-\varepsilon_0) x^{-1+\varepsilon_0}[\psi(x)]^2 + G_{12}(\varepsilon_0,x), \end{aligned} \tag{2.122}$$

where, again by (1.10),

$$\big|G_{12}(\varepsilon_0,x)\big| \leqslant c_{21}, \quad \varepsilon_0 \in (0,M),\ 0 < x < \rho \tag{2.123}$$

for some $c_{21} > 0$, independent of $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0,M)$. Hence,

$$\begin{aligned} \int_0^\rho dx\, x^\alpha \big|f_{\varepsilon_0}^{(m)}(x)\big|^2 &= A(m,\alpha-\varepsilon_0) \int_0^\rho dx\, x^{-1+\varepsilon_0}[\psi(x)]^2 \\ &\quad + \int_0^\rho dx\, G_{12}(\varepsilon_0,x), \end{aligned} \tag{2.124}$$

and the lemma follows by dividing both sides of (2.124) by

$$\int_0^\rho dx\, x^{\alpha-2m}|f_{\varepsilon_0}(x)|^2 = \int_0^\rho dx\, x^{-1+\varepsilon_0}[\psi(x)]^2 \tag{2.125}$$

and applying (2.120), (2.123). □

3. The Approximation Procedure

We start with some more notation. For the remainder of this paper we shall assume $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, \rho/20)$, that is, we shall assume $M = \rho/20$. Let $f_{\underline{\varepsilon}} = f_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N}$ be as defined in (1.18). Then for $\delta \in (0,\rho/20)$, we shall write, recalling $\underline{\varepsilon} = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N)$,

$$f_{(\delta),\underline{\varepsilon}}(x) = \begin{cases} 0, & x < \delta \text{ or } \rho \leqslant x, \\ f_{\underline{\varepsilon}}(x), & \delta \leqslant x < \rho. \end{cases} \tag{3.1}$$

We shall let $h \in C^\infty(\mathbb{R})$ satisfy the following properties:

$$(i)\ h \text{ is even on } \mathbb{R}, \tag{3.2}$$

$$(ii)\ h(x) \geqslant 0, \quad x \in \mathbb{R}, \tag{3.3}$$

$$(iii)\ \operatorname{supp}(h) \subseteq (-1,1), \tag{3.4}$$

$$(iv)\ \int_{-1}^{1} dx\, h(x) = 1, \tag{3.5}$$

$$(v)\ h \text{ is non-increasing on } [0,\infty). \tag{3.6}$$

For $\varepsilon > 0$ we write

$$h_\varepsilon(x) = \varepsilon^{-1} h(x/\varepsilon), \quad x \in \mathbb{R}. \tag{3.7}$$

For $\delta \in (0,\rho/20)$ and $\varepsilon \in (0,\delta/4]$, we write

$$f_{(\delta,\varepsilon),\underline{\varepsilon}} = f_{(\delta),\underline{\varepsilon}} * h_\varepsilon. \tag{3.8}$$

Remark 3.1. (i) Since h is even, we have

$$\begin{aligned} f_{(\delta,\varepsilon),\underline{\varepsilon}}(x) &= \int_{-\infty}^{\infty} dt\, \varepsilon^{-1} h(t/\varepsilon) f_{(\delta),\underline{\varepsilon}}(x-t) \\ &= \int_{-\infty}^{\infty} dt\, \varepsilon^{-1} h(-t/\varepsilon) f_{(\delta),\underline{\varepsilon}}(x-t) = \int_{-\infty}^{\infty} du\, \varepsilon^{-1} h(u/\varepsilon) f_{(\delta),\underline{\varepsilon}}(x+u) \\ &= \int_{-\infty}^{\infty} dr\, \varepsilon^{-1} h((r-x)/\varepsilon) f_{(\delta),\underline{\varepsilon}}(r) = \int_{x-\varepsilon}^{x+\varepsilon} dr\, \varepsilon^{-1} h((r-x)/\varepsilon) f_{(\delta),\underline{\varepsilon}}(r), \quad x \in \mathbb{R}. \end{aligned} \tag{3.9}$$

(ii) Since $\varepsilon \in (0,\delta/4]$, $\operatorname{supp}(f_{(\delta,\varepsilon),\underline{\varepsilon}}) \subseteq [3\delta/4, 73\rho/80]$. Hence,

$$f_{(\delta,\varepsilon),\underline{\varepsilon}} \in C_0^\infty((0,\rho)). \tag{3.10}$$

(iii) Let $g \in L^\infty(\mathbb{R}), x \in \mathbb{R}, \tau \in \mathbb{R}\backslash\{0\}$. For $0 < \varepsilon \leqslant \delta/4 < \rho/80$, let $g_\varepsilon = h_\varepsilon * g$. By the sequence of change of variables in (3.9), we have

$$\begin{aligned} &\tau^{-1}[g_\varepsilon(x+\tau) - g_\varepsilon(x)] \\ &\quad = \int_{-\infty}^{\infty} dr\, (\tau\varepsilon)^{-1}\{h((r-x-\tau)/\varepsilon) - h((r-x)/\varepsilon)\} g(r) \\ &\quad = -\int_{-\infty}^{\infty} dr\, (\tau\varepsilon)^{-1} h'((r-x-\lambda(x,r,\tau)\tau)/\varepsilon)(\tau/\varepsilon) g(r) \\ &\quad = -\varepsilon^{-2} \int_{-\infty}^{\infty} dr\, h'((r-x-\lambda(x,r,\tau)\tau)/\varepsilon) g(r), \end{aligned} \tag{3.11}$$

where

$$0 \leqslant \lambda(x,r,\tau) \leqslant 1, \quad x, r \in \mathbb{R}. \tag{3.12}$$

Since $h', g \in L^\infty(\mathbb{R})$ and, for $-1 \leqslant \tau \leqslant 1$,

$$\operatorname{supp} h'([\,\cdot\, - x - \lambda(x,\,\cdot\,,\tau)\tau]/\varepsilon) \subseteq [x-\varepsilon-1, x+\varepsilon+1], \tag{3.13}$$

applying the dominated convergence theorem we get

$$\begin{aligned} g_\varepsilon'(x) &= \lim_{\tau\to 0} \tau^{-1}[g_\varepsilon(x+\tau) - g_\varepsilon(x)] \\ &= -\lim_{\tau\to 0} \varepsilon^{-2} \int_{-\infty}^{\infty} dr\, h'((r-x-\lambda(x,r,\tau)\tau)/\varepsilon) g(r) \\ &= -\varepsilon^{-2} \lim_{\tau\to 0} \int_{x-\varepsilon-1}^{x+\varepsilon+1} dr\, h'((r-x-\lambda(x,r,\tau)\tau)/\varepsilon) g(r) \\ &= -\varepsilon^{-2} \int_{x-\varepsilon-1}^{x+\varepsilon+1} dr\, h'((r-x)/\varepsilon) g(r) = -\varepsilon^{-2} \int_{x-\varepsilon}^{x+\varepsilon} dr\, h'((r-x)/\varepsilon) g(r). \end{aligned} \tag{3.14}$$

Let $\delta \in (0, \rho/20)$. For technical convenience, so that we can use the general theory of convolution, we shall write $\widetilde{f}_{(\delta),\underline{\varepsilon}}$ for a function in $C_0^\infty(\mathbb{R})$ satisfying:

$$\begin{aligned} &(i)\ \widetilde{f}_{(\delta),\underline{\varepsilon}}(x) = f_{(\delta),\underline{\varepsilon}}, \quad x \geqslant \delta, \\ &(ii)\ \widetilde{f}_{(\delta),\underline{\varepsilon}}(x) \geqslant 0, \quad -\infty < x < \infty. \end{aligned} \tag{3.15}$$

Constants denoted by $\nu_j, j \in \mathbb{N}$, will depend on $N \in \mathbb{N} \cup \{0\}$, $\gamma, \rho \in (0, \infty)$ with $\gamma \geqslant \rho e_{N+1}$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $h, \psi \in C^\infty(\mathbb{R})$, and $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N \in (0, \rho/20)$, but independent of $\delta \in (0, \rho/20)$ and $\varepsilon \in (0, \delta/4)$. ⋄

Lemma 3.2. *For all $k \in \mathbb{N} \cup \{0\}$ there exists $\nu_1 = \nu_1(k) > 0$ such that*

$$\big|f_{\underline{\varepsilon}}^{(k)}(x)\big| \leqslant \nu_1 x^{[2(m-k)-1-\alpha+(\varepsilon_0/2)]/2}, \quad 0 < x < \rho. \tag{3.16}$$

Proof. This lemma follows from Lemma 2.4, the product rule

$$f_{\underline{\varepsilon}}^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} v_{\underline{\varepsilon}}^{(k-j)}(x)\psi^{(j)}(x), \quad 0 < x < \rho, \tag{3.17}$$

and that, for all $\beta > 0$, the function $t \mapsto t^{-\beta}\ln(t)$ is bounded on $(1, \infty)$. □

Lemma 3.3. *For $j = 1, \dots, m$, and $x \in [3\delta/4, 5\delta/4]$, we have, writing $\theta = \delta/4$,*

$$\begin{aligned} f_{(\delta,\theta),\underline{\varepsilon}}^{(j)}(x) &= \sum_{k=1}^{j} (-1)^{k+1}\theta^{-k} h^{(k-1)}((\delta - x)/\theta) f_{(\delta),\underline{\varepsilon}}^{(j-k)}(\delta) \\ &\quad + \theta^{-1} \int_{\delta}^{x+\theta} dr\, h((r-x)/\theta) f_{(\delta),\underline{\varepsilon}}^{(j)}(r). \end{aligned} \tag{3.18}$$

Proof. For $3\delta/4 \leqslant x \leqslant 5\delta/4$ we have, by (3.14)

$$\begin{aligned} f'_{(\delta,\theta),\underline{\varepsilon}}(x) &= -\theta^{-2} \int_{x-\theta}^{x+\theta} dr\, h'((r-x)/\theta) f_{(\delta),\underline{\varepsilon}}(r) \\ &= -\theta^{-2} \int_{\delta}^{x+\theta} dr\, h'((r-x)/\theta) f_{(\delta),\underline{\varepsilon}}(r) \\ &= -\theta^{-1} \int_{\delta}^{x+\theta} dr\, \tfrac{d}{dr}[h((r-x)/\theta)] f_{(\delta),\underline{\varepsilon}}(r) \\ &= -\theta^{-1} \bigg\{ h((r-x)/\theta) f_{(\delta),\underline{\varepsilon}}(r) \bigg|_{\delta}^{x+\theta} - \int_{\delta}^{x+\theta} dr\, h((r-x)/\theta) f'_{(\delta),\underline{\varepsilon}}(r) \bigg\} \\ &= -\theta^{-1} \bigg\{ - h((\delta-x)/\theta) f_{(\delta),\underline{\varepsilon}}(\delta) - \int_{\delta}^{x+\theta} dr\, h((r-x)/\theta) f'_{(\delta),\underline{\varepsilon}}(r) \bigg\} \\ &= \theta^{-1} h((\delta-x)/\theta) f_{(\delta),\underline{\varepsilon}}(\delta) + \theta^{-1} \int_{\delta}^{x+\theta} dr\, h((r-x)/\theta) f'_{(\delta),\underline{\varepsilon}}(r). \end{aligned} \tag{3.19}$$

Suppose $j \in \{1, \dots, m-1\}$ and that for all $x \in [3\delta/4, 5\delta/4]$ one has

$$\begin{aligned} f_{(\delta,\theta),\underline{\varepsilon}}^{(j)}(x) &= \sum_{k=1}^{j} (-1)^{k+1}\theta^{-k} h^{(k-1)}((\delta - x)/\theta) f_{(\delta),\underline{\varepsilon}}^{(j-k)}(\delta) \\ &\quad + \theta^{-1} \int_{\delta}^{x+\theta} dr\, h((r-x)/\theta) f_{(\delta),\underline{\varepsilon}}^{(j)}(r), \end{aligned} \tag{3.20}$$

then, by (3.14), one concludes

$$\begin{aligned}
f^{(j+1)}_{(\delta,\theta),\underline{\varepsilon}}(x) &= \sum_{k=1}^{j}(-1)^{k+1}\theta^{-k}(-1/\theta)h^{(k)}((\delta-x)/\theta)f^{(j-k)}_{(\delta),\underline{\varepsilon}}(\delta) \\
&\quad + \frac{d}{dx}\bigg(\theta^{-1}\int_{x-\theta}^{x+\theta} dr\, h((r-x)/\theta)f^{(j)}_{(\delta),\underline{\varepsilon}}(r)\bigg) \\
&= \sum_{k=1}^{j}(-1)^{k}\theta^{-(k+1)}h^{(k)}((\delta-x)/\theta)f^{(j-k)}_{(\delta),\underline{\varepsilon}}(\delta) \\
&\quad - \frac{1}{\theta^2}\int_{x-\theta}^{x+\theta} dr\, h'((r-x)/\theta)f^{(j)}_{(\delta),\underline{\varepsilon}}(r) \\
&= \sum_{k=2}^{j+1}(-1)^{k+1}\theta^{-k}h^{(k-1)}((\delta-x)/\theta)f^{(j+1-k)}_{(\delta),\underline{\varepsilon}}(\delta) \\
&\quad - \frac{1}{\theta}\int_{\delta}^{x+\theta} dr\,\bigg(\frac{d}{dr}[h((r-x)/\theta)]\bigg)f^{(j)}_{(\delta),\underline{\varepsilon}}(r) \\
&= \sum_{k=2}^{j+1}(-1)^{k+1}\theta^{-k}h^{(k-1)}((\delta-x)/\theta)f^{(j+1-k)}_{(\delta),\underline{\varepsilon}}(\delta) \\
&\quad - \frac{1}{\theta}\bigg\{h((r-x)/\theta)f^{(j)}_{(\delta),\underline{\varepsilon}}(r)\bigg|_{\delta}^{x+\theta} - \int_{\delta}^{x+\theta} dr\, h((r-x)/\theta)f^{(j+1)}_{(\delta),\underline{\varepsilon}}(r)\bigg\} \\
&= \sum_{k=2}^{j+1}(-1)^{k+1}\theta^{-k}h^{(k-1)}((\delta-x)/\theta)f^{(j+1-k)}_{(\delta),\underline{\varepsilon}}(\delta) \\
&\quad + \frac{1}{\theta}h((\delta-x)/\theta)f^{(j)}_{(\delta),\underline{\varepsilon}}(\delta) + \frac{1}{\theta}\int_{\delta}^{x+\theta} dr\, h((r-x)/\theta)f^{(j+1)}_{(\delta),\underline{\varepsilon}}(r) \\
&= \sum_{k=1}^{j+1}(-1)^{k+1}\theta^{-k}h^{(k-1)}((\delta-x)/\theta)f^{(j+1-k)}_{(\delta),\underline{\varepsilon}}(\delta) \\
&\quad + \frac{1}{\theta}\int_{\delta}^{x+\theta} dr\, h((r-x)/\theta)f^{(j+1)}_{(\delta),\underline{\varepsilon}}(r).
\end{aligned} \tag{3.21}$$

Hence, Lemma 3.3 follows by induction. □

Corollary 3.4. *There exists $\nu_2 > 0$ such that for all $\delta \in (0, \rho/20)$,*

$$\big|f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big| \leqslant \nu_2 x^{[-1-\alpha+(\varepsilon_0/2)]/2}, \quad 3\delta/4 \leqslant x \leqslant 5\delta/4. \tag{3.22}$$

Proof. Let

$$K_m = \sup\big\{\big|h^{(k)}(t)\big| \,\big|\, -1 \leqslant t \leqslant 1,\ k = 0, 1, \dots, m\big\}. \tag{3.23}$$

By Lemmas 3.2 and 3.3 we have for $x \in [3\delta/4, 5\delta/4]$,

$$\begin{aligned}
\big|f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big| &\leqslant \sum_{k=1}^{m} 4^k\delta^{-k}K_m\nu_1(m-k)\delta^{[2k-1-\alpha+(\varepsilon_0/2)]/2} \\
&\quad + 4\delta^{-1}(\tfrac{6\delta}{4} - \delta)K_m \sup\big\{\big|f^{(m)}_{(\delta),\underline{\varepsilon}}(r)\big| \,\big|\, \delta \leqslant r \leqslant 6\delta/4\big\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{m} 4^k K_m \nu_1(m-k) \delta^{[-1-\alpha+(\varepsilon_0/2)]/2} \\
&\quad + 2K_m \nu_1(m) \sup\big\{ r^{[-1-\alpha+(\varepsilon_0/2)]/2} \,\big|\, \delta \leqslant r \leqslant 6\delta/4 \big\} \\
&\leqslant K_m \bigg(\sum_{k=1}^{m} 4^k \nu_1(m-k) \bigg) \big\{ (4/3)^{[-1-\alpha+(\varepsilon_0/2)]/2} + (4/5)^{[-1-\alpha+(\varepsilon_0/2)]/2} \big\} \\
&\qquad \times x^{[-1-\alpha+(\varepsilon_0/2)]/2} \\
&\quad + 2K_m \nu_1(m) \big\{ (4/5)^{[-1-\alpha+(\varepsilon_0/2)]/2} + 2^{[-1-\alpha+(\varepsilon_0/2)]/2} \big\} x^{[-1-\alpha+(\varepsilon_0/2)]/2} \\
&= \nu_2 x^{[-1-\alpha+(\varepsilon_0/2)]/2},
\end{aligned} \tag{3.24}
$$

where

$$
\begin{aligned}
\nu_2 = {}& K_m \bigg(\sum_{k=1}^{m} 4^k \nu_1(m-k) \bigg) \big\{ (4/3)^{[-1-\alpha+(\varepsilon_0/2)]/2} + (4/5)^{[-1-\alpha+(\varepsilon_0/2)]/2} \big\} \\
& + 2K_m \nu_1(m) \big\{ (4/5)^{[-1-\alpha+(\varepsilon_0/2)]/2} + 2^{[-1-\alpha+(\varepsilon_0/2)]/2} \big\}.
\end{aligned} \tag{3.25}
$$

□

Lemma 3.5. *There exists $\nu_3 > 0$ such that for all $\delta \in (0, \rho/20)$ we have*

$$
\big| f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x) \big| \leqslant \nu_3 x^{[-1-\alpha+(\varepsilon_0/2)]/2}, \quad 5\delta/4 \leqslant x \leqslant \rho. \tag{3.26}
$$

Proof. We first note that, for $5\delta/4 \leqslant x \leqslant 73\rho/80$,

$$
\begin{aligned}
f_{(\delta,\delta/4),\underline{\varepsilon}}(x) &= \int_{x-\delta/4}^{x+\delta/4} dr\, (4/\delta) h(4(r-x)/\delta) f_{(\delta),\underline{\varepsilon}}(r) \\
&= \int_{x-\delta/4}^{x+\delta/4} dr\, (4/\delta) h(4(r-x)/\delta) \widetilde{f}_{(\delta),\underline{\varepsilon}}(r) \\
&= \big(h_{\delta/4} * \widetilde{f}_{(\delta),\underline{\varepsilon}} \big)(x),
\end{aligned} \tag{3.27}
$$

hence

$$
\begin{aligned}
f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x) &= \Big(h_{\delta/4} * \widetilde{f}^{(m)}_{(\delta),\underline{\varepsilon}} \Big)(x) \\
&= 4\delta^{-1} \int_{x-\delta/4}^{x+\delta/4} dr\, h(4(r-x)/\delta) \widetilde{f}^{(m)}_{(\delta),\underline{\varepsilon}}(r) \\
&= 4\delta^{-1} \int_{x-\delta/4}^{x+\delta/4} dr\, h(4(r-x)/\delta) f^{(m)}_{(\delta),\underline{\varepsilon}}(r),
\end{aligned} \tag{3.28}
$$

therefore, by Lemma 3.2,

$$
\begin{aligned}
\big| f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x) \big| &\leqslant \sup\big\{ \big| f^{(m)}_{(\delta),\underline{\varepsilon}}(r) \big| \,\big|\, x-(\delta/4) \leqslant r \leqslant x+(\delta/4) \big\} \\
&\leqslant \nu_1(m) \sup\big\{ r^{[-1-\alpha+(\varepsilon_0/2)]/2} \,\big|\, x-(\delta/4) \leqslant r \leqslant x+(\delta/4) \big\} \\
&\leqslant \nu_1(m) \sup\big\{ r^{[-1-\alpha+(\varepsilon_0/2)]/2} \,\big|\, 3x/4 \leqslant r \leqslant 5x/4 \big\} \\
&\leqslant \nu_1(m) \big\{ (3/4)^{[-1-\alpha+(\varepsilon_0/2)]/2} + (5/4)^{[-1-\alpha+(\varepsilon_0/2)]/2} \big\} x^{[-1-\alpha+(\varepsilon_0/2)]/2}.
\end{aligned} \tag{3.29}
$$

By Remark 3.1 (*ii*), $\operatorname{supp}(f_{(\delta,\delta/4),\underline{\varepsilon}}) \subseteq [3\delta/4, 73\rho/80]$. So (3.26) holds for $x \in [73\rho/80, \rho]$, completing the proof. □

Lemma 3.6. *On any compact interval* $[a,b] \subseteq (0,\rho]$*,* $f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}$ *converges to* $f^{(m)}_{\underline{\varepsilon}}$ *uniformly as* $\delta \downarrow 0$*.*

Proof. Choose $\delta_0 \in (0,\rho/20)$ such that $0 < 5\delta_0/4 < a$. Then for all $0 < \delta < \delta_0$ and $x \in [a,b]$,

$$\begin{aligned} f_{(\delta,\delta/4),\underline{\varepsilon}}(x) &= 4\delta^{-1}\int_{x-\delta/4}^{x+\delta/4} dr\, h(4(r-x)/\delta) f_{(\delta),\underline{\varepsilon}}(r) \\ &= 4\delta^{-1}\int_{x-\delta/4}^{x+\delta/4} dr\, h(4(r-x)/\delta) f_{(\delta_0),\underline{\varepsilon}}(r) \\ &= 4\delta^{-1}\int_{x-\delta/4}^{x+\delta/4} dr\, h(4(r-x)/\delta) \widetilde{f}_{(\delta_0),\underline{\varepsilon}}(r) \\ &= \big(h_{\delta/4} * \widetilde{f}_{(\delta_0),\underline{\varepsilon}}\big)(x). \end{aligned} \tag{3.30}$$

Since $\widetilde{f}_{(\delta_0),\underline{\varepsilon}} \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned} f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x) &= \Big(h_{\delta/4} * \widetilde{f}^{(m)}_{(\delta_0),\underline{\varepsilon}}\Big)(x), \quad x \in [a,b], \\ &\underset{\delta\downarrow 0}{\longrightarrow} \widetilde{f}^{(m)}_{(\delta_0),\underline{\varepsilon}}(x) \quad \text{uniformly for } x \in [a,b], \\ &= f^{(m)}_{\underline{\varepsilon}}(x). \end{aligned} \tag{3.31}$$

□

Corollary 3.7. *We have*

$$\lim_{\delta\downarrow 0}\int_0^\rho dx\, x^\alpha \big|f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big|^2 = \int_0^\rho dx\, x^\alpha \big|f^{(m)}_{\underline{\varepsilon}}(x)\big|^2. \tag{3.32}$$

Proof. Let $\nu_4 = \max\{\nu_2,\nu_3\} > 0$. Then by Corollary 3.4 and Lemma 3.5, we have, for all $\delta \in (0,\rho/20)$,

$$x^\alpha \big|f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big|^2 \leqslant \nu_4^2 x^{-1+(\varepsilon_0/2)}, \quad 0 < x < \rho. \tag{3.33}$$

By Lemma 3.6 we have

$$\lim_{\delta\downarrow 0} x^\alpha \big|f^{(m)}_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big|^2 = x^\alpha \big|f^{(m)}_{\underline{\varepsilon}}(x)\big|^2, \quad 0 < x < \rho. \tag{3.34}$$

Since $x \mapsto \nu_4 x^{-1+(\varepsilon_0/2)}$ is integrable on $(0,\rho)$, the corollary now follows by dominated convergence. □

Lemma 3.8. *There exists* $\nu_5 > 0$ *such that for all* $\delta \in (0,\rho/20)$ *we have*

$$|f_{(\delta,\delta/4),\underline{\varepsilon}}(x)| \leqslant \nu_5 x^{[2m-1-\alpha+(\varepsilon_0/2)]/2}, \quad 3\delta/4 \leqslant x \leqslant 5\delta/4. \tag{3.35}$$

Proof. For $3\delta/4 \leqslant x \leqslant 5\delta/4$ we have

$$\begin{aligned} &|f_{(\delta,\delta/4),\underline{\varepsilon}}(x)| = \bigg|4\delta^{-1}\int_\delta^{x+\delta/4} dr\, h(4(r-x)/\delta) f_{(\delta),\underline{\varepsilon}}(r)\bigg| \\ &\quad \leqslant \sup\{|f_{(\delta),\underline{\varepsilon}}(r)| \,|\, \delta \leqslant r \leqslant 6\delta/4\} \\ &\quad = \sup\{|f_{\underline{\varepsilon}}(r)| \,|\, \delta \leqslant r \leqslant 3\delta/2\} \\ &\quad \leqslant \nu_1(0)\sup\big\{r^{[2m-1-\alpha+(\varepsilon_0/2)]/2} \,\big|\, \delta \leqslant r \leqslant 3\delta/2\big\} \\ &\quad \leqslant \nu_1(0)\big\{(4/5)^{[2m-1-\alpha+(\varepsilon_0/2)]/2} + 2^{[2m-1-\alpha+(\varepsilon_0/2)]/2}\big\} x^{[2m-1-\alpha+(\varepsilon_0/2)]/2}. \end{aligned} \tag{3.36}$$

□

Lemma 3.9. *There exists $\nu_6 > 0$ such that for all $\delta \in (0, \rho/20)$ we have*

$$|f_{(\delta,\delta/4),\underline{\varepsilon}}(x)| \leqslant \nu_6 x^{[2m-1-\alpha+(\varepsilon_0/2)]/2}, \quad 5\delta/4 \leqslant x < \rho. \tag{3.37}$$

Proof. For $x \in [5\delta/4, \rho)$ we have

$$\begin{aligned}
|f_{(\delta,\delta/4),\underline{\varepsilon}}(x)| &= \left| 4\delta^{-1} \int_{x-\delta/4}^{x+\delta/4} dr\, h(4(r-x)/\delta) f_{(\delta),\underline{\varepsilon}}(r) \right| \\
&\leqslant \sup\{|f_{(\delta),\underline{\varepsilon}}(r)| \,|\, x-\delta/4 \leqslant r \leqslant x+\delta/4\} \\
&\leqslant \nu_1(0) \sup\left\{ r^{[2m-1-\alpha+(\varepsilon_0/2)]/2} \,\middle|\, 3x/4 \leqslant r \leqslant 5x/4 \right\} \\
&\leqslant \nu_1(0) \big\{ (3/4)^{[2m-1-\alpha+(\varepsilon_0/2)]/2} + (5/4)^{[2m-1-\alpha+(\varepsilon_0/2)]/2} \big\} x^{[2m-1-\alpha+(\varepsilon_0/2)]/2}.
\end{aligned} \tag{3.38}$$

□

Lemma 3.10. *On any compact interval $[a, b] \subseteq (0, \rho]$, $f_{(\delta,\delta/4),\underline{\varepsilon}}$ converges to $f_{\underline{\varepsilon}}$ uniformly as $\delta \downarrow 0$.*

Proof. Choose $\delta_0 \in (0, \rho/20)$ with $0 < 5\delta_0/4 < a$. By (3.30), for all $0 < \delta < \delta_0$, we have

$$f_{(\delta,\delta/4),\underline{\varepsilon}}(x) = \big(h_{\delta/4} * \widetilde{f}_{(\delta_0),\underline{\varepsilon}}\big)(x), \quad a \leqslant x \leqslant b. \tag{3.39}$$

Since $\widetilde{f}_{(\delta_0),\underline{\varepsilon}} \in C_0^\infty(\mathbb{R})$, we have

$$\begin{aligned}
f_{(\delta,\delta/4),\underline{\varepsilon}}(x) &= \big(h_{\delta/4} * \widetilde{f}_{(\delta_0),\underline{\varepsilon}}\big)(x) \\
&\underset{\delta\downarrow 0}{\longrightarrow} \widetilde{f}_{(\delta_0),\underline{\varepsilon}}(x) \quad \text{uniformly for } x \in [a,b] \\
&= f_{\underline{\varepsilon}}(x).
\end{aligned} \tag{3.40}$$

□

Corollary 3.11. *For $k \in \{0, 1, \dots, N\}$ we have*

$$\begin{aligned}
&\lim_{\delta\downarrow 0} \int_0^\rho dx\, x^{\alpha-2m} \prod_{j=1}^k [\ln_j(\gamma/x)]^{-2} \big|f_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big|^2 \\
&\quad = \int_0^\rho dx\, x^{\alpha-2m} \prod_{j=1}^k [\ln_j(\gamma/x)]^{-2} \big|f_{\underline{\varepsilon}}(x)\big|^2.
\end{aligned} \tag{3.41}$$

Proof. Let $\nu_7 = \max\{\nu_5, \nu_6\} > 0$. By Lemmas 3.8 and 3.9 we have, for all $\delta \in (0, \rho/20)$ and $x \in (0, \rho)$,

$$x^{\alpha-2m} \prod_{j=1}^k [\ln_j(\gamma/x)]^{-2} \big|f_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big|^2 \leqslant \nu_7^2 x^{-1+(\varepsilon_0/2)} \prod_{j=1}^k [\ln_j(\gamma/x)]^{-2}. \tag{3.42}$$

By Lemma 3.10 we have for $x \in (0, \rho)$,

$$\begin{aligned}
&\lim_{\delta\downarrow 0} x^{\alpha-2m} \prod_{j=1}^k [\ln_j(\gamma/x)]^{-2} \big|f_{(\delta,\delta/4),\underline{\varepsilon}}(x)\big|^2 \\
&\quad = x^{\alpha-2m} \prod_{j=1}^k [\ln_j(\gamma/x)]^{-2} \big|f_{\underline{\varepsilon}}(x)\big|^2.
\end{aligned} \tag{3.43}$$

Since $x \mapsto x^{-1+(\varepsilon_0/2)}\bigg(\prod_{j=1}^{k}[\ln_j(\gamma/x)]^{-2}\bigg)$ is integrable on $(0,\rho)$, the corollary now follows by dominated convergence. □

Corollary 3.12. *Suppose $N \in \mathbb{N}$. Then there exists a family $\{g_{\delta,\underline{\varepsilon}}\}_{\delta\in(0,(0.05)\rho)} \subseteq C_0^\infty((0,\rho))$ such that*

$$\begin{aligned} &\lim_{\delta\downarrow 0} J_{N-1}[g_{\delta,\underline{\varepsilon}}]\bigg(\int_0^\rho dx\, x^{\alpha-2m}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-2}\big|g_{\delta,\underline{\varepsilon}}(x)\big|^2\bigg)^{-1} \\ &\quad = J_{N-1}[f_{\underline{\varepsilon}}]\bigg(\int_0^\rho dx\, x^{\alpha-2m}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-2}\big|f_{\underline{\varepsilon}}(x)\big|^2\bigg)^{-1}. \end{aligned} \tag{3.44}$$

Proof. For $\delta \in (0,\rho/20)$ put $g_{\delta,\underline{\varepsilon}} = f_{(\delta,\delta/4),\underline{\varepsilon}}$. Then $g_{\delta,\underline{\varepsilon}} \in C_0^\infty((0,\rho))$ by Remark 3.1 (ii). The result now follows from Corollaries 3.7 and 3.11. □

Corollary 3.13. *Suppose $N = 0$. Then there exists a family $\{g_{\delta,\underline{\varepsilon}}\}_{\delta\in(0,(0.05)\rho)} \subseteq C_0^\infty((0,\rho))$ such that*

$$\begin{aligned} &\lim_{\delta\downarrow 0}\int_0^\rho dx\, x^{\alpha}\big|g_{\delta,\underline{\varepsilon}}^{(m)}(x)\big|^2\bigg(\int_0^\rho dx\, x^{\alpha-2m}\big|g_{\delta,\underline{\varepsilon}}(x)\big|^2\bigg)^{-1} \\ &\quad = \int_0^\rho dx\, x^{\alpha}\big|f_{\underline{\varepsilon}}^{(m)}(x)\big|^2\bigg(\int_0^\rho dx\, x^{\alpha-2m}\big|f_{\underline{\varepsilon}}(x)\big|^2\bigg)^{-1}. \end{aligned} \tag{3.45}$$

Proof. The proof of this corollary is the same as that of Corollary 3.12. □

4. Principal Results on Optimal Constants

In our final section we now prove optimality of the constants $A(m,\alpha)$ and $B(m,\alpha)$.

Starting with the interval $(0,\rho)$, we first establish optimality of $A(m,\alpha)$ in (1.1).

Theorem 4.1. *Suppose that $N = 0$. Then, given any $\eta > 0$, there exists $g \in C_0^\infty((0,\rho))$ such that*

$$\bigg|\int_0^\rho dx\, x^{\alpha}\big|g^{(m)}(x)\big|^2\bigg[\int_0^\rho dx\, x^{\alpha-2m}\big|g(x)\big|^2\bigg]^{-1} - A(m,\alpha)\bigg| \leqslant \eta. \tag{4.1}$$

In particular, the constant $A(m,\alpha)$ in (1.1) is sharp.

Proof. Given any $\eta > 0$ there exists $\varepsilon_0 \in (0,\rho/20)$ such that

$$\bigg|\int_0^\rho dx\, x^{\alpha}\big|f_{\varepsilon_0}^{(m)}(x)\big|^2\bigg[\int_0^\rho dx\, x^{\alpha-2m}\big|f_{\varepsilon_0}(x)\big|^2\bigg]^{-1} - A(m,\alpha)\bigg| \leqslant \eta/2, \tag{4.2}$$

by Lemma 2.14. With this value of $\varepsilon_0 \in (0,\rho/20)$, Corollary 3.13 implies that there exists $g \in C_0^\infty((0,\rho))$ such that

$$\begin{aligned} &\bigg|\int_0^\rho dx\, x^{\alpha}\big|g^{(m)}(x)\big|^2\bigg[\int_0^\rho dx\, x^{\alpha-2m}\big|g(x)\big|^2\bigg]^{-1} \\ &\quad - \int_0^\rho dx\, x^{\alpha}\big|f_{\varepsilon_0}^{(m)}(x)\big|^2\bigg[\int_0^\rho dx\, x^{\alpha-2m}\big|f_{\varepsilon_0}(x)\big|^2\bigg]^{-1}\bigg| \leqslant \eta/2. \end{aligned} \tag{4.3}$$

Theorem 4.1 now follows from (4.2), (4.3). □

Next, we prove optimality of the N constants $B(m,\alpha)$ in (1.1):

Theorem 4.2. *Suppose that $N \in \mathbb{N}$. Then for any $\eta > 0$, there exists $g \in C_0^\infty((0,\rho))$ such that*

$$\bigg|\bigg[\int_0^\rho dx\, x^\alpha \big|g^{(m)}(x)\big|^2 - A(m,\alpha)\int_0^\rho dx\, x^{\alpha-2m}\big|g(x)\big|^2 \\ - B(m,\alpha)\sum_{k=1}^{N-1}\int_0^\rho dx\, x^{\alpha-2m}\big|g(x)\big|^2\prod_{p=1}^{k}[\ln_p(\gamma/x)]^{-2}\bigg] \\ \times\bigg[\int_0^\rho dx\, x^{\alpha-2m}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-2}\big|g(x)\big|^2\bigg]^{-1} - B(m,\alpha)\bigg| \leqslant \eta. \tag{4.4}$$

In particular, successively increasing N through $1,2,3,\dots,$ demonstrates that the N constants $B(m,\alpha)$ in (1.1) are sharp. Together with Theorem 4.1, this theorem asserts that the $N+1$ constants, $A(m,\alpha)$ and the N constants $B(m,\alpha)$, in (1.1) are sharp.

Proof. Given any $\eta > 0$ there exist $\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N \in (0,\rho/20)$ such that, writing $f_{\underline{\varepsilon}} = f_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N}$,

$$\bigg|J_{N-1}[f_{\underline{\varepsilon}}]\bigg[\int_0^\rho dx\, x^{\alpha-2m}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-2}\big|f_{\underline{\varepsilon}}(x)\big|^2\bigg]^{-1} - B(m,\alpha)\bigg| \leqslant \eta/2, \tag{4.5}$$

by Lemma 2.13. With these values of $\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N \in (0,\rho/20)$, Corollary 3.12 implies that there exists $g \in C_0^\infty((0,\rho))$ such that

$$\bigg|J_{N-1}[g]\bigg[\int_0^\rho dx\, x^{\alpha-2m}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-2}\big|g(x)\big|^2\bigg]^{-1} \\ - J_{N-1}[f_{\underline{\varepsilon}}]\bigg[\int_0^\rho dx\, x^{\alpha-2m}\prod_{j=1}^{N}[\ln_j(\gamma/x)]^{-2}\big|f_{\underline{\varepsilon}}(x)\big|^2\bigg]^{-1}\bigg| \leqslant \eta/2. \tag{4.6}$$

Theorem 4.2 now follows from (4.5), (4.6). □

Next we turn to analogous results for the half line (r,∞). We start with some preparations.

Writing

$$Q_{m,\alpha}(\lambda) = \bigg(\lambda^2 - \frac{(1-\alpha)^2}{4}\bigg)\bigg(\lambda^2 - \frac{(3-\alpha)^2}{4}\bigg)\cdots\bigg(\lambda^2 - \frac{(2m-1-\alpha)^2}{4}\bigg) \\ = \prod_{j=1}^{m}\bigg(\lambda^2 - \frac{(2j-1-\alpha)^2}{4}\bigg) = \sum_{\ell=0}^{2m} k_\ell(m,\alpha)\lambda^\ell, \tag{4.7}$$

one infers that

$$(i)\ k_{2j-1}(m,\alpha) = 0, \quad j = 1,\dots,m, \tag{4.8}$$

$$(ii)\ k_{2j}(m,\alpha) = (-1)^{m-j}|k_{2j}(m,\alpha)|, \quad j = 0,1,\dots,m, \tag{4.9}$$

and thus,

$$Q_{m,\alpha}(\lambda) = \sum_{j=0}^{m}(-1)^{m-j}|k_{2j}(m,\alpha)|\lambda^{2j}. \tag{4.10}$$

Lemma 4.3. ([41, Sect. 2 and proof of Theorem 3.1 (*i*)])
Suppose $\hat\rho > e_{N+1}$ and $\alpha \in \mathbb{R}\backslash\{1,\dots,2m-1\}$. For $g \in C_0^\infty((\hat\rho,\infty))$ let $w = w_g \in C_0^\infty((\ln(\hat\rho),\infty))$ be defined by

$$g(e^t) = e^{[(2m-1-\alpha)/2]t} w(t), \quad t \in (\ln(\hat\rho),\infty). \tag{4.11}$$

Then for all $g \in C_0^\infty((\hat\rho,\infty))$,

$$\begin{aligned} \int_{\hat\rho}^\infty dy\, y^\alpha \big|g^{(m)}(y)\big|^2 &= \int_{\ln(\hat\rho)}^\infty dt \sum_{j=0}^m |k_{2j}(m,\alpha)| \big|w^{(j)}(t)\big|^2, \\ \int_{\hat\rho}^\infty dy\, y^{\alpha-2m} |g(y)|^2 &= \int_{\ln(\hat\rho)}^\infty dt\, |w(t)|^2, \end{aligned} \tag{4.12}$$

and, if $N \in \mathbb{N}$, one also has, for $k = 1,\dots,N$,

$$\begin{aligned} &\big(e^t\big)^{\alpha-2m} \big|g(e^t)\big|^2 \prod_{p=1}^k [\ln_p(e^t)]^{-2} \\ &\quad = e^{-t} |w(t)|^2 t^{-2} \prod_{p=1}^{k-1} [\ln_p(t)]^{-2}, \quad t \in (\ln(\hat\rho),\infty). \end{aligned} \tag{4.13}$$

Hence, if $N \in \mathbb{N}$,

$$\begin{aligned} &\bigg[\int_{\hat\rho}^\infty dy\, y^\alpha \big|g^{(m)}(y)\big|^2 - A(m,\alpha) \int_{\hat\rho}^\infty dy\, y^{\alpha-2m} |g(y)|^2 \\ &\qquad - B(m,\alpha) \int_{\hat\rho}^\infty dy\, y^{\alpha-2m} |g(y)|^2 \sum_{k=1}^{N-1} \prod_{p=1}^k [\ln_p(y)]^{-2}\bigg] \\ &\quad \times \bigg[\int_{\hat\rho}^\infty dy\, y^{\alpha-2m} |g(y)|^2 \prod_{p=1}^N [\ln_p(y)]^{-2}\bigg]^{-1} \\ &= \bigg[\int_{\ln(\hat\rho)}^\infty dt \sum_{j=0}^m |k_{2j}(m,\alpha)| \big|w^{(j)}(t)\big|^2 - A(m,\alpha) \int_{\ln(\hat\rho)}^\infty dt\, |w(t)|^2 \\ &\qquad - B(m,\alpha) \int_{\ln(\hat\rho)}^\infty dt\, |w(t)|^2 t^{-2} \sum_{k=1}^{N-1} \prod_{p=1}^{k-1} [\ln_p(t)]^{-2}\bigg] \\ &\quad \times \bigg[\int_{\ln(\hat\rho)}^\infty dt\, |w(t)|^2 t^{-2} \prod_{p=1}^{N-1} [\ln_p(t)]^{-2}\bigg]^{-1}, \quad g \in C_0^\infty((\hat\rho,\infty)). \end{aligned} \tag{4.14}$$

Corollary 4.4. *Lemma* 4.3 *holds for all $\alpha \in \mathbb{R}$, that is, it holds without the restriction $\alpha \in \mathbb{R}\backslash\{1,\dots,2m-1\}$.*

Proof. We first note that by (4.7), for $\ell = 0,1,\dots,2m$, $k_\ell(m,\alpha)$ is a polynomial in α and so it is continuous in α. For $g \in C_0^\infty((\hat\rho,\infty))$, to emphasize that the definition of $w = w_g \in C_0^\infty((\ln(\hat\rho),\infty))$ in (4.11) depends also on α, we shall write, for all $\alpha \in \mathbb{R}$,

$$w_\alpha(t) = e^{-[(2m-1-\alpha)/2]t} g(e^t), \quad t \in (\ln(\hat\rho),\infty). \tag{4.15}$$

Then, for $j = 0, 1, \dots, m$, one gets

$$w_\alpha^{(j)}(t) = \sum_{k=0}^{j} S(j,k,\alpha,t) g^{(k)}(e^t), \quad t \in (\ln(\hat{\rho}), \infty), \tag{4.16}$$

where, for $j \in \{0,1,\dots,m\}$, $k \in \{0,1,\dots,j\}$, and $t \in (\ln(\hat{\rho}), \infty)$, $\alpha \mapsto S(j,k,\alpha,t)$ is continuous in α. We also note that, for $g \in C_0^\infty((\hat{\rho},\infty))$,

$$\operatorname{supp}(w_\alpha) = \big\{t \in (\ln(\hat{\rho}), \infty) \,|\, e^t \in \operatorname{supp}(g)\big\} \tag{4.17}$$

is independent of $\alpha \in \mathbb{R}$. Now let $\alpha \in \{1, \dots, 2m-1\}$. Then, by dominated convergence, for $g \in C_0^\infty((\hat{\rho},\infty))$,

$$\begin{aligned} \lim_{\beta\to\alpha} \int_{\hat{\rho}}^\infty dy\, y^\beta \big|g^{(m)}(y)\big|^2 &= \int_{\hat{\rho}}^\infty dy\, y^\alpha \big|g^{(m)}(y)\big|^2, \\ \lim_{\beta\to\alpha} \int_{\hat{\rho}}^\infty dy\, y^{\beta-2m} |g(y)|^2 &= \int_{\hat{\rho}}^\infty dy\, y^{\alpha-2m} |g(y)|^2, \end{aligned} \tag{4.18}$$

and, if $N \in \mathbb{N}$, one obtains

$$\begin{aligned} &\lim_{\beta\to\alpha} \int_{\hat{\rho}}^\infty dy\, y^{\beta-2m} |g(y)|^2 \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_p(y)]^{-2} \\ &\quad = \int_{\hat{\rho}}^\infty dy\, y^{\alpha-2m} |g(y)|^2 \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_p(y)]^{-2}, \\ &\lim_{\beta\to\alpha} \int_{\hat{\rho}}^\infty dy\, y^{\beta-2m} |g(y)|^2 \prod_{p=1}^{N} [\ln_p(y)]^{-2} = \int_{\hat{\rho}}^\infty dy\, y^{\alpha-2m} |g(y)|^2 \prod_{p=1}^{N} [\ln_p(y)]^{-2}. \end{aligned} \tag{4.19}$$

Similarly, for $g \in C_0^\infty((\hat{\rho},\infty))$,

$$\begin{aligned} &\lim_{\beta\to\alpha} \int_{\ln(\hat{\rho})}^\infty dt\, \sum_{j=0}^{m} |k_{2j}(m,\beta)| \big|w_\beta^{(j)}(t)\big|^2 = \int_{\ln(\hat{\rho})}^\infty dt\, \sum_{j=0}^{m} |k_{2j}(m,\alpha)| \big|w_\alpha^{(j)}(t)\big|^2, \\ &\lim_{\beta\to\alpha} A(m,\beta) \int_{\ln(\hat{\rho})}^\infty dy\, |w_\beta(t)|^2 = A(m,\alpha) \int_{\ln(\hat{\rho})}^\infty dy\, |w_\alpha(t)|^2, \end{aligned} \tag{4.20}$$

and, if $N \in \mathbb{N}$, one has

$$\begin{aligned} &\lim_{\beta\to\alpha} B(m,\beta) \int_{\ln(\hat{\rho})}^\infty dt\, |w_\beta(t)|^2 t^{-2} \sum_{k=1}^{N-1} \prod_{p=1}^{k-1} [\ln_p(t)]^{-2} \\ &\quad = B(m,\alpha) \int_{\ln(\hat{\rho})}^\infty dt\, |w_\alpha(t)|^2 t^{-2} \sum_{k=1}^{N-1} \prod_{p=1}^{k-1} [\ln_p(t)]^{-2}, \end{aligned} \tag{4.21}$$

$$\lim_{\beta\to\alpha} \int_{\ln(\hat{\rho})}^\infty dt\, |w_\beta(t)|^2 t^{-2} \prod_{p=1}^{N-1} [\ln_p(t)]^{-2} = \int_{\ln(\hat{\rho})}^\infty dt\, |w_\alpha(t)|^2 t^{-2} \prod_{p=1}^{N-1} [\ln_p(t)]^{-2}. \tag{4.22}$$

The corollary now follows from (4.18)–(4.22) and Lemma 4.3. $\square$

Lemma 4.5. ([41, Sect. 2 and proof of Theorem 3.1 (*iii*)])
Suppose $1/\widetilde{\rho} > e_{N+1}$ and $\alpha \in \mathbb{R}\backslash\{1, \dots, 2m-1\}$. For $g \in C_0^\infty((0,\widetilde{\rho}))$ let $u = u_g \in C_0^\infty((\ln(1/\widetilde{\rho}),\infty))$ be defined by

$$g(e^{-t}) = e^{-[(2m-1-\alpha)/2]t} u(t), \quad t \in (\ln(1/\widetilde{\rho}), \infty). \tag{4.23}$$

Then, for all $g \in C_0^\infty((0,\widetilde{\rho}))$,

$$\begin{aligned}
&\int_0^{\widetilde{\rho}} dy\, y^\alpha \big|g^{(m)}(y)\big|^2 = \int_{\ln(1/\widetilde{\rho})}^\infty dt \sum_{j=0}^m |k_{2j}(m,\alpha)|\big|u^{(j)}(t)\big|^2, \\
&\int_0^{\widetilde{\rho}} dy\, y^{\alpha-2m} |g(y)|^2 = \int_{\ln(1/\widetilde{\rho})}^\infty dt\, |u(t)|^2,
\end{aligned} \tag{4.24}$$

and, if $N \in \mathbb{N}$, we also have, for $k = 1,\dots,N$,

$$\begin{aligned}
&\big(e^{-t}\big)^{\alpha-2m} \big|g(e^{-t})\big|^2 \prod_{p=1}^k [\ln_p(e^t)]^{-2} \\
&\quad = e^t |u(t)|^2 t^{-2} \prod_{p=1}^{k-1} [\ln_p(t)]^{-2}, \quad t \in (\ln(1/\widetilde{\rho}),\infty).
\end{aligned} \tag{4.25}$$

Hence, if $N \in \mathbb{N}$,

$$\begin{aligned}
&\bigg[\int_0^{\widetilde{\rho}} dy\, y^\alpha \big|g^{(m)}(y)\big|^2 - A(m,\alpha) \int_0^{\widetilde{\rho}} dy\, y^{\alpha-2m} |g(y)|^2 \\
&\qquad - B(m,\alpha) \int_0^{\widetilde{\rho}} dy\, y^{\alpha-2m} |g(y)|^2 \sum_{k=1}^{N-1} \prod_{p=1}^k [\ln_p(1/y)]^{-2}\bigg] \\
&\qquad \times \bigg[\int_0^{\widetilde{\rho}} dy\, y^{\alpha-2m} |g(y)|^2 \prod_{p=1}^N [\ln_p(1/y)]^{-2}\bigg]^{-1} \\
&\quad = \bigg[\int_{\ln(1/\widetilde{\rho})}^\infty dt \sum_{j=0}^m |k_{2j}(m,\alpha)|\big|u^{(j)}(t)\big|^2 - A(m,\alpha) \int_{\ln(1/\widetilde{\rho})}^\infty dt\, |u(t)|^2 \\
&\qquad - B(m,\alpha) \int_{\ln(1/\widetilde{\rho})}^\infty dt\, |u(t)|^2 t^{-2} \sum_{k=1}^{N-1} \prod_{p=1}^{k-1} [\ln_p(t)]^{-2}\bigg] \\
&\qquad \times \bigg[\int_{\ln(1/\widetilde{\rho})}^\infty dt\, |u(t)|^2 t^{-2} \prod_{p=1}^{N-1} [\ln_p(t)]^{-2}\bigg]^{-1}, \quad g \in C_0^\infty((0,\widetilde{\rho})).
\end{aligned} \tag{4.26}$$

Corollary 4.6. *Lemma 4.5 holds for all $\alpha \in \mathbb{R}$, that is, it holds without the restriction $\alpha \in \mathbb{R}\backslash\{1,\dots,2m-1\}$.*

As the proof of this corollary is very similar to that of Corollary 4.4 we shall omit it.

At this point we are ready to establish optimality of $A(m,\alpha)$ on the interval (r,∞) in (1.2).

Theorem 4.7. *Suppose that $N = 0$. Let $r \in (1,\infty)$. Then, for any $\eta > 0$, there exists $\varphi \in C_0^\infty((r,\infty))$ such that*

$$\bigg|\int_r^\infty dx\, x^\alpha \big|\varphi^{(m)}(x)\big|^2 \bigg[\int_r^\infty dx\, x^{\alpha-2m} |\varphi(x)|^2\bigg]^{-1} - A(m,\alpha)\bigg| \leqslant \eta. \tag{4.27}$$

In particular, the constant $A(m,\alpha)$ in (1.2) is sharp.

Proof. Put $\rho = 1/r$ so that $1 > \rho$. Applying Theorem 4.1, there exists $g \in C_0^\infty((0,\rho))$ such that

$$\left| \int_0^\rho dy\, y^\alpha \big|g^{(m)}(y)\big|^2 \left[\int_0^\rho dy\, y^{\alpha-2m} |g(y)|^2 \right]^{-1} - A(m,\alpha) \right| \leqslant \eta. \tag{4.28}$$

By Corollary 4.6, writing

$$u(t) = e^{[(2m-1-\alpha)/2]t} g(e^{-t}), \quad t \in (\ln(1/\rho), \infty), \tag{4.29}$$

one obtains

$$\left| \int_{\ln(1/\rho)}^\infty dt \sum_{j=0}^m |k_{2j}(m,\alpha)| \big|u^{(j)}(t)\big|^2 \left[\int_{\ln(1/\rho)}^\infty dt\, |u(t)|^2 \right]^{-1} - A(m,\alpha) \right| \leqslant \eta. \tag{4.30}$$

Introducing

$$\varphi(x) = x^{(2m-1-\alpha)/2} u(\ln(x)), \quad x \in (1/\rho, \infty) = (r, \infty), \tag{4.31}$$

Corollary 4.4 implies

$$\left| \int_r^\infty dx\, x^\alpha \big|\varphi^{(m)}(x)\big|^2 \left[\int_r^\infty dx\, x^{\alpha-2m} |\varphi(x)|^2 \right]^{-1} - A(m,\alpha) \right| \leqslant \eta, \tag{4.32}$$

concluding the proof since $\varphi \in C_0^\infty((r,\infty))$. □

Next, we prove optimality of the N constants $B(m,\alpha)$ in (1.2):

Theorem 4.8. *Suppose that $N \in \mathbb{N}$. Let $r, \Gamma \in (0,\infty)$ satisfy $r > \Gamma e_{N+1}$. Then, for any $\eta > 0$, there exists $\varphi \in C_0^\infty((r,\infty))$ such that*

$$\begin{aligned} &\Bigg| \Bigg[\int_r^\infty dx\, x^\alpha \big|\varphi^{(m)}(x)\big|^2 - A(m,\alpha) \int_r^\infty dx\, x^{\alpha-2m} |\varphi(x)|^2 \\ &\quad - B(m,\alpha) \sum_{k=1}^{N-1} \int_r^\infty dx\, x^{\alpha-2m} |\varphi(x)|^2 \prod_{p=1}^k [\ln_p(x/\Gamma)]^{-2} \Bigg] \\ &\quad \times \left[\int_r^\infty dx\, x^{\alpha-2m} |\varphi(x)|^2 \prod_{p=1}^N [\ln_p(x/\Gamma)]^{-2} \right]^{-1} - B(m,\alpha) \Bigg| \leqslant \eta. \end{aligned} \tag{4.33}$$

In particular, successively increasing N through $1, 2, 3\ldots$, demonstrates that the N constants $B(m,\alpha)$ in (1.2) are sharp. Together with Theorem 4.7, this theorem asserts that the $N+1$ constants, $A(m,\alpha)$ and the N constants $B(m,\alpha)$, in (1.2) are sharp.

Proof. Put $\rho = \Gamma/r$ so that $1 > \rho e_{N+1}$. Applying Theorem 4.2 with $\gamma = 1$, there exists $g \in C_0^\infty((0,\rho))$ such that

$$\begin{aligned} &\Bigg| \Bigg[\int_0^\rho dy\, y^\alpha \big|g^{(m)}(y)\big|^2 - A(m,\alpha) \int_0^\rho dy\, y^{\alpha-2m} |g(y)|^2 \\ &\quad - B(m,\alpha) \int_0^\rho dy\, y^{\alpha-2m} |g(y)|^2 \sum_{k=1}^{N-1} \prod_{p=1}^k [\ln_p(1/y)]^{-2} \Bigg] \\ &\quad \times \left[\int_0^\rho dy\, y^{\alpha-2m} |g(y)|^2 \prod_{p=1}^N [\ln_p(1/y)]^{-2} \right]^{-1} - B(m,\alpha) \Bigg| \leqslant \eta. \end{aligned} \tag{4.34}$$

By Corollary 4.6, writing

$$u(t) = e^{[(2m-1-\alpha)/2]t} g(e^{-t}), \quad t \in (\ln(1/\rho), \infty), \tag{4.35}$$

one has

$$\begin{aligned} &\bigg| \bigg[\int_{\ln(1/\rho)}^{\infty} dt \sum_{j=0}^{m} |k_{2j}(m,\alpha)| \big| u^{(j)}(t) \big|^2 - A(m,\alpha) \int_{\ln(1/\rho)}^{\infty} dt\, |u(t)|^2 \\ &\quad - B(m,\alpha) \int_{\ln(1/\rho)}^{\infty} dt\, |u(t)|^2 t^{-2} \sum_{k=1}^{N-1} \prod_{p=1}^{k-1} [\ln_p(t)]^{-2} \bigg] \\ &\quad \times \bigg[\int_{\ln(1/\rho)}^{\infty} dt\, |u(t)|^2 t^{-2} \prod_{p=1}^{N-1} [\ln_p(t)]^{-2} \bigg]^{-1} - B(m,\alpha) \bigg| \leqslant \eta. \end{aligned} \tag{4.36}$$

Introducing

$$\widetilde{\varphi}(\xi) = \xi^{(2m-1-\alpha)/2} u(\ln(\xi)), \quad \xi \in (1/\rho, \infty), \tag{4.37}$$

Corollary 4.4 implies

$$\begin{aligned} &\bigg| \bigg[\int_{1/\rho}^{\infty} d\xi\, \xi^{\alpha} \big| \widetilde{\varphi}^{(m)}(\xi) \big|^2 - A(m,\alpha) \int_{1/\rho}^{\infty} d\xi\, \xi^{\alpha-2m} |\widetilde{\varphi}(\xi)|^2 \\ &\quad - B(m,\alpha) \int_{1/\rho}^{\infty} d\xi\, \xi^{\alpha-2m} |\widetilde{\varphi}(\xi)|^2 \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_p(\xi)]^{-2} \bigg] \\ &\quad \times \bigg[\int_{1/\rho}^{\infty} d\xi\, \xi^{\alpha-2m} |\widetilde{\varphi}(\xi)|^2 \prod_{p=1}^{N} [\ln_p(\xi)]^{-2} \bigg]^{-1} - B(m,\alpha) \bigg| \leqslant \eta. \end{aligned} \tag{4.38}$$

Putting

$$\varphi(x) = \widetilde{\varphi}(x/\Gamma), \quad x \in (\Gamma/\rho, \infty) = (r, \infty), \tag{4.39}$$

one infers

$$\begin{aligned} &\bigg| \bigg[\Gamma^{2m-\alpha-1} \bigg\{ \int_{r}^{\infty} dx\, x^{\alpha} \big| \varphi^{(m)}(x) \big|^2 - A(m,\alpha) \int_{r}^{\infty} dx\, x^{\alpha-2m} |\varphi(x)|^2 \\ &\quad - B(m,\alpha) \int_{r}^{\infty} dx\, x^{\alpha-2m} |\varphi(x)|^2 \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_p(x/\Gamma)]^{-2} \bigg\} \bigg] \\ &\quad \times \bigg[\Gamma^{2m-\alpha-1} \int_{r}^{\infty} dx\, x^{\alpha-2m} |\varphi(x)|^2 \prod_{p=1}^{N} [\ln_p(x/\Gamma)]^{-2} \bigg]^{-1} - B(m,\alpha) \bigg| \leqslant \eta, \end{aligned} \tag{4.40}$$

finishing the proof since $\varphi \in C_0^{\infty}((r,\infty))$. □

Remark 4.9. (*i*) Theorem 4.1 (resp., Theorem 4.7) extends to $\rho = \infty$ (resp., $r = 0$) upon disregarding all logarithmic terms (i.e., upon putting $B(m,\alpha) = 0$), we omit the details.

(*ii*) The sequence of logarithmically refined power-weighted Birman–Hardy–Rellich inequalities underlying Theorems 4.1, 4.2, 4.7, and 4.8, extend from C_0^{∞}-functions to functions in appropriately weighted (homogeneous) Sobolev spaces as shown in detail in [41, Sect. 3]. In the course of this extension, the constants $A(m,\alpha)$ and the N constants $B(m,\alpha)$ remain the same and hence optimal.

(*ii*) We note once more that Theorems 4.1 and 4.7 were proved in [41, Theorem A.1] using a different method.

(iv) Both Theorems 4.2 and 4.8 still hold if the repeated log-terms $ln_p(\,\cdot\,)$ are replaced by the type of repeated log-terms used in [15, 16, 17, 90]. Detailed proofs of Theorems 4.2 and 4.8 for the type of repeated log-terms used in [15, 16, 17, 90] are available upon request from the authors. $\diamond$

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