ABSTRACT

On Birman–Hardy–Rellich-type Inequalities

Isaac B. Michael, Ph.D.

Advisor: Fritz Gesztesy, Ph.D.

In 1961, Birman proved a sequence of inequalities on the space of m-times continuously differentiable functions of compact support $C_0^m((0,\infty)) \subset L^2((0,\infty))$, containing the classical (integral) Hardy inequality and the well-known Rellich inequality. In this dissertation, we give a proof of this sequence of inequalities on a certain Hilbert space $H_L^m([0,\infty))$ as well as the standard Sobolev space $H_0^m((0,b))$ for $0 < b < \infty$. The Birman constants $[(2m-1)!!]^2/2^{2m}$ in each of these inequalities are sharp and the only function that gives equality in any of these inequalities is the trivial function in $L^2((0,\infty))$ (resp., $L^2((0,b))$). These Birman constants are closely related to the norm of generalized continuous Cesàro averaging operators whose spectral properties we determine in detail. We then revisit weighted Hardy-type inequalities employing an elementary ad hoc approach that yields explicit constants, then discuss the infinite sequence of power weighted Birman–Hardy–Rellich-type inequalities and derive an operator-valued version thereof. We further improve this sequence of inequalities by adding recursive logarithmic refinement terms with unrestricted weight and logarithmic parameters. In the multidimensional setting, we derive variants of Hardy's inequality involving power-type weights, radial derivatives and logarithmic refinements. Finally, we establish the multidimensional Birman–Hardy–Rellich-type inequalities with power-type weights and radial refinements.

On Birman–Hardy–Rellich-type Inequalities

by

Isaac B. Michael, B.S., M.S.

A Dissertation

Approved by the Department of Mathematics

Lance L. Littlejohn, Ph.D., Chairperson

Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

Approved by the Dissertation Committee

Fritz Gesztesy, Ph.D., Chairperson

Gregory Benesh, Ph.D.

Andrei Martinez-Finkelshtein, Ph.D.

Lance Littlejohn, Ph.D.

Accepted by the Graduate School
August 2019

J. Larry Lyon, Ph.D., Dean

Page bearing signatures is kept on file in the Graduate School.

TABLE OF CONTENTS

LIST OF FIGURES vi

ACKNOWLEDGMENTS vii

DEDICATION viii

1 Introduction 1

1.1 Background 1

1.2 Motivation 4

1.3 Attributions 6

2 On Birman's Sequence of Hardy–Rellich-type Inequalities 7

2.1 Introduction 7

2.2 An Integral Inequality 10

2.3 The Function Spaces $H_L^m([0,\infty))$ and $\widehat{H}_L^m((0,\infty))$ 11

2.4 A New Proof of Birman's Sequence of Hardy–Rellich-type Inequalities 19

2.5 Optimality of Constants 25

2.6 The Continuous Cesàro Operator T_1 and its Generalizations T_m . . . 27

2.7 The Birman Inequalities on the Finite Interval $[0,b]$ 41

2.8 The Vector-Valued Case 44

3 On Weighted Hardy-Type Inequalities 52

3.1 Introduction 52

3.2 Weighted Hardy-Type Inequalities Employing an Ad Hoc Approach . 56

3.3 More on Weighted Hardy-Type Inequalities 61

3.4 Some Applications to the Operator-Valued Case 70

4 On Power Weighted Birman–Hardy–Rellich-type Inequalities with Logarithmic Refinements via Hartman–Müeller-Pfeiffer Transformations 76
4.1 Introduction . . . 76
4.2 The Combined Hartman–Müeller-Pfeiffer Transformation . . . 78
4.3 Power-Weighted Birman–Hardy–Rellich-type Inequalities with Logarithmic Refinements . . . 81
4.4 The Vector-Valued Case . . . 92

5 On the Multidimensional Power Weighted Hardy Inequality with Radial and Logarithmic Refinements 96
5.1 Introduction . . . 96
5.2 Refinements of Hardy's inequality . . . 98

6 On Multidimensional Power Weighted Birman–Hardy–Rellich-type Inequalities with Radial Refinements 106
6.1 Introduction . . . 106
6.2 Integration with Polar Coordinates . . . 110
6.3 Radial Power Weighted Birman–Hardy–Rellich-type Inequalities . . . 111
6.3.1 Hardy and Rellich Inequalities . . . 111
6.3.2 Birman Inequality . . . 114
6.4 Weighted Sobolev Spaces and Distance to the Boundary . . . 115
6.4.1 Hardy and Rellich Inequalities . . . 118
6.4.2 Birman Inequality . . . 125
6.5 The Vector-Valued Case . . . 128

7 Conclusion 130

BIBLIOGRAPHY 137

LIST OF FIGURES

1.1 David Hilbert (1862-1943) 1

1.2 Godfrey Harold Hardy (1877-1947) 2

1.3 Franz Rellich (1906-1955) 3

1.4 Mikhail Shlemovich Birman (1928-2009) 4

2.1 $14\times$ Magnification of $\sigma(T_{100})$ 39

2.2 The Spectrum of T_m for certain $m \in \mathbb{N}$ 40

ACKNOWLEDGMENTS

First of all, I would like to extend a most sincere thanks to the Mathematics Department at Baylor University for accepting me into the doctoral program and changing the course of my life completely.

I am grateful to all of those whom I have had the pleasure to work with during this and other research projects. I wish to thank each member of the Dissertation Committee for their time and effort in these proceedings, several of whom have provided valuable insight through courses I've taken under them, projects we've worked on together, and lectures they've given. I would also like to extend my utmost gratitude to Paul Hagelstein for his very insightful discussions, as well as Katie Elliot for her valuable help throughout the project. I would especially like to thank my advisor Fritz Gesztesy and my co-advisor Lance Littlejohn. This research project would not have been possible without them, and their dedicated guidance and direction have greatly supported me during my academic pursuits throughout this program.

No one is more important to me than my family. I want to thank my loving parents, who greatly encouraged and supported me while pursuing graduate school. Their help was, is, and always will be, invaluable. Finally, I thank my amazing children, Serenity and Elijah, and my beautiful wife, Brenda. Their love has remained steadfast, even through the most difficult times.

"Half of mathematics is art. The other half is finding the truth of things."

- Fritz Gesztesy (February 16, 2018)

CHAPTER ONE

Introduction

1.1 Background

The majority of our research is based on extensions and improvements of the sequence of integral inequalities established by M. Š. Birman in 1961 (see [19]). However, the full history of the Birman–Hardy–Rellich inequalities actually dates as far back as 1915 (see [85] for a more in-depth historical discussion). Around that time, G. H. Hardy was working to find a new and elementary proof of Hilbert's inequality for double series (see [75]) which, in basic form, reads:

Theorem 1.1.1 (Hilbert's Inequality, 1906). *Let $a_k, b_k \in \mathbb{R}$ with $a_k, b_k \geqslant 0$ for all $k \in \mathbb{N}$. If $\sum_{j=1}^{\infty} a_j^2 < \infty$ and $\sum_{k=1}^{\infty} b_k^2 < \infty$ then*

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{a_j b_k}{j+k} \leqslant \pi \left(\sum_{j=1}^{\infty} a_j^2\right)^{1/2} \left(\sum_{k=1}^{\infty} b_k^2\right)^{1/2}. \tag{1.1.1}$$

The constant π is sharp.

Figure 1.1: David Hilbert (1862-1943)

Remark 1.1.2. The sharp constant π was established by Shur in [110] and Hilbert's version of (1.1.1) contained the non-optimal constant 2π.

To that end, Hardy sought a discrete inequality of the form

$$\sum_{k=1}^{\infty}\left|\frac{1}{k}\sum_{j=1}^{k}a_j\right|^2 \leqslant C_2\sum_{k=1}^{\infty}|a_k|^2. \tag{1.1.2}$$

Following a period of about 10 years, with contributions from several mathematicians such as E. Landau, G Pòlya, I. Shur, and M. Riesz, Hardy ultimately published his now famous 1925 paper [70], establishing that (1.1.1) follows from the inequality

$$\sum_{k=1}^{N}\left(\frac{1}{k}\sum_{j=1}^{k}a_j\right)^p \leqslant \left(\frac{p}{p-1}\right)^p\sum_{k=1}^{N}a_k^p, \quad a_k \geqslant 0,\ N\in\mathbb{N},\ p\in(1,\infty), \tag{1.1.3}$$

which, in turn, was shown to follow easily from its continuous analogue, the latter of which came to be known as the classical (integral) Hardy inequality, see also [71, Thms 3.26, 3.27, p. 240], shown below.

Theorem 1.1.3 (Hardy's Inequality, 1925). *Let $p \in (1,\infty)$, $f(x) \geqslant 0$ for all $x \in (0,\infty)$, and $F(x) = \int_0^x dt\, f(t)$, then*

$$\int_0^{\infty} dx \left(\frac{F(x)}{x}\right)^p < \left(\frac{p}{p-1}\right)^p \int_0^{\infty} dx\, f(x)^p, \tag{1.1.4}$$

unless $f \equiv 0$. The constant is sharp.

Figure 1.2: Godfrey Harold Hardy (1877-1947)

The differential form of (1.1.4) is given by

$$\int_0^{\infty} dx\, |f'(x)|^p > \left(\frac{p-1}{p}\right)^p \int_0^{\infty} dx\, x^{-p}|f(x)|^p, \quad p\in(1,\infty), \tag{1.1.5}$$

for all $f \not\equiv 0$ such that $f(0) = 0$ and the left side of (1.1.5) is finite.

In 1954 the first extension of (1.1.5) to higher-order derivatives was proven by F. Rellich (see [106, 107]) in the multidimensional setting for the case $p = 2$.

Theorem 1.1.4 (Rellich's Inequality, 1954). *Let $n \in \mathbb{N}$, $n \geqslant 5$, then*

$$\int_{\mathbb{R}^n} d^n x \, |\Delta f(x)|^2 > \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} d^n x \, |x|^{-4} |f(x)|^2, \tag{1.1.6}$$

for all $0 \neq f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$. The constant is sharp.

Figure 1.3: Franz Rellich (1906-1955)

Trivially, Theorem 1.1.4 contains the one-dimensional version

$$\int_0^\infty dx \, |f''(x)|^2 > \frac{9}{16} \int_0^\infty dx \, x^{-4} |f(x)|^2, \quad 0 \neq f \in C_0^\infty((0, \infty)), \tag{1.1.7}$$

as a special case.

Finally in 1961, M. Š. Birman established in [19], almost in passing, the sequence of integral inequalities containing the Hardy and one-dimensional Rellich inequality for the self-adjoint case $p = 2$. Although these integral inequalities in (1.1.8) have been well established, the fact that they originated from Birman seems to have been lost in the literature until recently.

Theorem 1.1.5 (Birman's Sequence of Inequalities, 1961). *Let* $m \in \mathbb{N}$. *Then*

$$\int_0^\infty dx \left|f^{(m)}(x)\right|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^\infty dx\, x^{-2m}|f(x)|^2, \quad f \in C_0^m((0,\infty)). \quad (1.1.8)$$

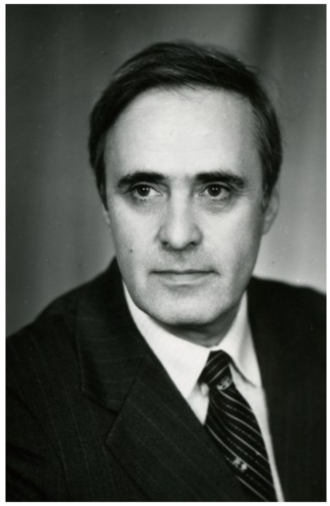

Figure 1.4: Mikhail Shlemovich Birman (1928-2009)

Since the establishment of (1.1.8), a great amount of research has been dedicated to further improving these inequalities, such as extending to $p \in [1,\infty)$, multi-dimensions, optimal function spaces, vector-valued functions, as well as adding weight functions and logarithmically weaker singular potentials.

1.2 Motivation

This research project focuses on Birman–Hardy–Rellich-type integral inequalities, invoking several tools from various analytic fields, including: real, complex, and functional analysis, operator theory, spectral theory, as well as ordinary and partial differential equations. The importance of inequalities in general and their wide range of applications in many areas of mathematics is well established. As shown in [86], during his Presidential Address at the meeting of the London Mathematical Society, on November 8, 1928, G. H. Hardy gave the following quote from Harald Bohr:

"All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."

Furthermore, inequalities that involve integrals of functions and their derivatives appear in areas of the theory of differential equations and approximation theory, see [101]. This topic has been further extended over the past few decades to weighted generalizations, giving applications in Sobolev embedding theorems in the context of weighted Lebesgue spaces, see for instance, [84] and [86]. Improved Hardy–Rellich-type inequalities have many applications in the study of elliptic and parabolic PDE's, particularly if singular potentials are present. In the case $m = 1$ (or $m = 2$) for example, one gains information on the existence of solutions and asymptotic behavior for $u_t = -\Delta + V$ (or $u_t = (-\Delta)^2 + V$). Such inequalities are also critical in determining the lower boundedness of Hamiltonians $\hat{H} = \hat{T} + \hat{V}$, and higher-order Hardy-type inequalities, beginning with Rellich's inequality, were investigated extensively, due to their vast implications in self-adjointness and spectral theory problems associated with second and higher-order differential operators containing strongly singular coefficients, see [50].

Hardy-type inequalities in particular, apart from their intrinsic value, have significant applications in differential equations and mathematical physics. This fact is reflected in the extensive amount of literature on the subject, some of which are given in the bibliography. Extensions of Hardy-type inequalities to more general function spaces have been thoroughly studied and have made significant implications for these spaces, as well as important applications to differential equations. In fact, the special case ($p = 2$) of Hardy's multidimensional inequality has connections to the Heisenberg uncertainty principle in quantum mechanics, which asserts that it is impossible to simultaneously determine the position and momentum of a particle in space. In addition, the spectral analysis of quantum mechanical systems involving Coulomb forces between different particles naturally features this L^2 version of Hardy's inequality.

1.3 Attributions

Each publication used throughout this dissertation employed multiple roles crucial to rigorous mathematical research, including: planning, organization, supervision, literary and historical research, citation, notation, proofs, applications, LaTeX coding, styling and formatting, development, construction, proofreading, editing, submission, and revision.

Below we provide, in alphabetical order, the names of all authors listed within each publication used:

- Chian Chuah
- Fritz Gesztesy
- Lance Littlejohn
- Tao Mei
- Isaac Michael
- Michael Pang
- Richard Wellman

Furthermore, we confirm that each author contributed equally in all areas of research given above, and are listed alphabetically in each publication.

CHAPTER TWO

On Birman's Sequence of Hardy–Rellich-type Inequalities

The content of this chapter relies on (but is not identical to) the paper published as: F. Gesztesy, L. L. Littlejohn, I. Michael, and R. Wellman, *On Birman's Sequence of Hardy–Rellich-Type Inequalities*, J. Diff. Eq. **264(4)**, 2761–2801 (2018).

2.1 Introduction

In 1961, M. Š. Birman [19, p. 48], sketched a proof to establish the following sequence of inequalities

$$\int_0^\infty dx\, \left|f^{(m)}(x)\right|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^\infty dx\, \frac{|f(x)|^2}{x^{2m}}, \quad m \in \mathbb{N}, \tag{2.1.1}$$

valid for $f \in C_0^m((0,\infty))$, the space of m-times continuously differentiable complex-valued functions having compact support on $(0,\infty)$. Here we employed the well-known symbol, $(2m-1)!! := (2m-1)\cdot(2m-3)\cdots 3\cdot 1$. We denote the inequality in (2.1.1) by I_m. In particular, I_1 is the classical (integral) Hardy inequality (see [71, Sect. 7.3])

$$\int_0^\infty dx\, |f'(x)|^2 \geqslant \frac{1}{4}\int_0^\infty dx\, \frac{|f(x)|^2}{x^2}, \tag{2.1.2}$$

and I_2 is the Rellich inequality

$$\int_0^\infty dx\, |f''(x)|^2 \geqslant \frac{9}{16}\int_0^\infty dx\, \frac{|f(x)|^2}{x^4}. \tag{2.1.3}$$

We can find no reference in the literature to the general inequality (2.1.1) prior to the 1961 work of Birman cited above. In [60, pp. 83–84], Glazman gives a detailed proof of (2.1.1) using the ideas outlined in [19]. In [102, Lemma 2.1], Owen also establishes these inequalities. Each of these authors prove (2.1.1) for functions on $C_0^m((0,\infty))$. We note in passing that unless $f \equiv 0$, all inequalities (2.1.1)–(2.1.3) are strict.

In this paper we offer a new proof of (2.1.1) and confirm that the constant $[(2m-1)!!]^2/2^{2m}$ is best possible. We establish these inequalities for a general class of functions defined on $[0,\infty)$; the significance of this class is that we address the singularity at $x=0$, which is apparent on the right-hand side of (2.1.1), rather than deal with functions from $C_0^m((0,\infty))$. More specifically, we prove the inequalities in (2.1.1) are valid for all functions $f \in H_L^m([0,\infty))$, for $m \in \mathbb{N}$, where

$$\begin{aligned} H_L^m([0,\infty)) := \big\{ f : [0,\infty) \to \mathbb{C} \,\big|\, f^{(j)} \in AC_{loc}([0,\infty));\; f^{(m)} \in L^2((0,\infty)); \\ f^{(j)}(0) = 0,\; j = 0,1,\dots,m-1 \big\}. \end{aligned} \tag{2.1.4}$$

In [71, Sect. 7.3], Hardy, Littlewood, and Pólya established the classical Hardy inequality (2.1.2) on H_1. As we will see, the space $H_L^m([0,\infty))$ is a Hilbert space when endowed with the inner product

$$(f,g)_{H_L^m([0,\infty))} := \int_0^\infty dx\, \overline{f^{(m)}(x)}\, g^{(m)}(x), \quad f,g \in H_L^m([0,\infty)). \tag{2.1.5}$$

We also show that

$$H_L^m([0,\infty)) = \widehat{H}_L^m((0,\infty)), \tag{2.1.6}$$

where[1]

$$\begin{aligned} \widehat{H}_L^m((0,\infty)) := \big\{ f : (0,\infty) \to \mathbb{C} \,\big|\, f^{(j)} \in AC_{loc}((0,\infty)),\; j = 0,1,\dots,m-1; \\ f^{(m)}, f/x^m \in L^2((0,\infty)) \big\}. \end{aligned} \tag{2.1.7}$$

Upon first glance, it may seem unlikely that these spaces can be equal since one set deals with functions defined on $[0,\infty)$ while the other set has its functions defined on $(0,\infty)$. However, we will show that functions $f \in \widehat{H}_L^m((0,\infty))$, and their derivatives $f^{(j)}$ when $j = 0,1,\dots,m-1$, will have finite limits $f^{(j)}(0_+)$ at $x=0$.

There is an interesting connection with the spaces $\{H_L^m([0,\infty))\}_{m\in\mathbb{N}}$, namely that

$$f \in H_L^m([0,\infty)) \text{ implies } f' \in H_L^{m-1}([0,\infty)); \tag{2.1.8}$$

[1] We emphasize from the outset that despite the similarity of notation with Sobolev spaces, neither of the spaces $H_L^m([0,\infty))$ nor $\widehat{H}_L^m((0,\infty))$ coincides with the standard Sobolev space $H_0^m((0,\infty))$. (See, however, Theorem 2.7.1 in the finite interval context.)

this inclusion is important in establishing a new proof of (2.1.1) and in proving when equality in (2.1.1) occurs. Moreover, we will show that, in a sense, *each* of the inequalities I_m, $m \in \mathbb{N}$, follows from the classical Hardy inequality I_1.

In Section 2.2, we discuss a theorem attributed to a number of mathematicians, including Talenti, Tomaselli, Chisholm and Everitt, and Muckenhoupt; this result is useful in establishing various properties of functions in the spaces $H_L^m([0,\infty))$. These properties are dealt with in Section 2.3 where we establish the identity in (2.1.6). In Section 2.4, besides giving a slight extension of Glazman's proof of (2.1.1) including a power weight, we offer a new proof (Theorem 2.4.4) of (2.1.1) on the set $H_L^m([0,\infty)) = \widehat{H}_L^m((0,\infty))$. Each of these inequalities, when considered on $H_L^m([0,\infty))$, follows in a sense from the classical Hardy inequality (2.1.2). While the inequality in (2.1.1) does not imply that the Birman constant $[(2m-1)!!]^2/2^{2m}$ is sharp, the latter fact is well-known and in Section 2.5, we confirm that this constant is best possible in $H_L^m([0,\infty))$. In Section 2.6, we connect the Birman constants to the norms of generalized continuous Cesàro averaging operators T_m, $m \in \mathbb{N}$, and determine their spectra; in Section 2.7, we discuss the Birman inequalities on the finite interval $[0,b]$, $b \in (0,\infty)$. Finally, in Section 2.8 we derive Birman's sequence of Hardy–Rellich-type inequalities in the vector-valued case replacing complex-valued $f(x)$ by $f(x) \in \mathcal{H}$, with $\mathcal{H}$ a complex, separable Hilbert space.

Finally, a few comments on the notation used in this paper: $AC_{loc}((a,b))$ denotes the functions locally absolutely continuous on $(a,b) \subseteq \mathbb{R}$, while $AC_{loc}([a,b))$ represents absolutely continuous functions on $[a,c]$ for any $a < c < b$. Whenever possible we will omit Lebesgue measure dx in $L^p((a,b);dx)$ and simply write $L^p((a,b))$, $p \geqslant 1$, instead. We also abbreviate $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If T is a linear operator mapping (a subspace of) a Hilbert space into another, $\operatorname{dom}(T)$ denotes the domain and $\operatorname{ran}(T)$ is the range of T. The Banach space of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$.

The spectrum and point spectrum (i.e., the set of eigenvalues) of a closed operator T are denoted by $\sigma(T)$ and $\sigma_p(T)$. If N is normal in $\mathcal{H}$, the absolutely and singularly continuous spectrum of N are denoted by $\sigma_{ac}(N)$ and $\sigma_{sc}(N)$, respectively.

2.2 An Integral Inequality

The following theorem will be applied repeatedly in the next section to prove properties of functions in the space $H_L^m([0,\infty))$. This integral inequality in $L^2((a,b))$ was established by Talenti [113] and Tomaselli [116] in 1969. Unaware of their independent proofs, Chisholm and Everitt [27] established Theorem 2.2.1 in 1971; see also [28] for a more general result in the conjugate index case $1/p + 1/q = 1$. In addition, a 1972 paper by Muckenhoupt [95] has a result which contains Theorem 2.2.1. For further information, there is an excellent historical account of Theorem 2.2.1 in the book [85, Ch. 4, pp. 33–37].

Theorem 2.2.1. *Let* $(a,b) \subseteq \mathbb{R}$, $-\infty \leqslant a < b \leqslant \infty$, *and* $w : (a,b) \to \mathbb{R}$ *be Lebesgue measurable and nonnegative a.e. on* (a,b). *In addition, suppose* $\varphi, \psi : (a,b) \to \mathbb{C}$ *are Lebesgue measurable functions satisfying the following conditions:*

(i) $\varphi, \psi \in L^2_{loc}((a,b); w)$.

(ii) for some (and hence for all) $c \in (a,b)$,

$$\varphi \in L^2((a,c]; w), \;\; \psi \in L^2([c,b); w). \tag{2.2.1}$$

(iii) for all $[\alpha,\beta] \subset (a,b)$, *one has*

$$\int_a^\alpha dt\, w(t)\, |\varphi(t)|^2 > 0 \text{ and } \int_\beta^b dt\, w(t)\, |\psi(t)|^2 > 0. \tag{2.2.2}$$

Define the linear operators $A, B : L^2((a,b); w) \to L^2_{loc}((a,b); w)$ *by*

$$(Af)(x) := \varphi(x) \int_x^b dt\, \psi(t) w(t) f(t), \quad f \in L^2_{loc}((a,b); w), \tag{2.2.3}$$

and

$$(Bf)(x) := \psi(x) \int_a^x dt\, \varphi(t) w(t) f(t), \quad f \in L^2_{loc}((a,b); w), \tag{2.2.4}$$

and the function $K : (a,b) \to \mathbb{R}$ by

$$K(x) := \left(\int_a^x dt\, w(t)\, |\varphi(t)|^2\right)^{1/2} \left(\int_x^b dt\, w(t)\, |\psi(t)|^2\, dt\right)^{1/2}. \tag{2.2.5}$$

Then A and B are bounded linear operators in $L^2((a,b);w)$ if and only if

$$K := \sup_{x\in(a,b)} K(x) < \infty. \tag{2.2.6}$$

Moreover, if $K < \infty$, then A and B are adjoints of each other in $L^2((a,b);w)$, with

$$\|Af\|_{L^2((a,b);w)} = \|Bf\|_{L^2((a,b);w)} \leqslant 2K\|f\|_{L^2((a,b);w)}, \quad f \in L^2((a,b);w), \tag{2.2.7}$$

in particular,

$$\|A\|_{\mathcal{B}(L^2((a,b);w))} = \|B\|_{\mathcal{B}(L^2((a,b);w))} \leqslant 2K. \tag{2.2.8}$$

2.3 The Function Spaces $H_L^m([0,\infty))$ and $\widehat{H}_L^m((0,\infty))$

Let $H_L^m([0,\infty))$ and $\widehat{H}_L^m((0,\infty))$, $m \in \mathbb{N}$, be the spaces defined, respectively, in (2.1.4) and (2.1.7), that is,

$$\begin{aligned} H_L^m([0,\infty)) := \big\{f : [0,\infty) \to \mathbb{C} \,\big|\, f^{(j)} \in AC_{loc}([0,\infty));\ f^{(m)} \in L^2((0,\infty));\\ f^{(j)}(0) = 0,\ j = 0,1,\dots,m-1\big\}. \end{aligned} \tag{2.3.1}$$

and

$$\begin{aligned} \widehat{H}_L^m((0,\infty)) := \big\{f : (0,\infty) \to \mathbb{C} \,\big|\, f^{(j)} \in AC_{loc}((0,\infty)),\ j = 0,1,\dots,m-1;\\ f^{(m)}, f/x^m \in L^2((0,\infty))\big\}. \end{aligned} \tag{2.3.2}$$

With the inner product $(\,\cdot\,,\,\cdot\,)_{H_L^m([0,\infty))}$ as defined in (2.1.5), that is,

$$(f,g)_{H_L^m([0,\infty))} := \int_0^\infty dx\, \overline{f^{(m)}(x)}\, g^{(m)}(x), \quad f,g \in H_L^m([0,\infty)), \tag{2.3.3}$$

one observes that

$$\|f\|_{H_L^m([0,\infty))} = \big\|f^{(m)}\big\|_{L^2((0,\infty))}, \quad f \in H_L^m([0,\infty)). \tag{2.3.4}$$

Using (2.3.4), we now prove the following result.

Proposition 2.3.1. *The inner product space* $(H_L^m([0,\infty)), (\,\cdot\,,\,\cdot\,)_{H_L^m([0,\infty))})$ *is actually a Hilbert space. In addition,* $C_0^\infty((0,\infty))$ *is dense in* $(H_L^m([0,\infty)), (\,\cdot\,,\,\cdot\,)_{H_L^m([0,\infty))})$.

Proof. First we note that $f \in H_L^m([0,\infty))$ and $\|f\|_{H_L^m([0,\infty))} = 0$ implies $f^{(m)} = 0$ a.e. on $(0,\infty)$ and hence $f = 0$ as $f^{(j)}(0) = 0$ for $j = 0, 1, \dots, m-1$.

Next, let $\{f_m\}_{k=1}^\infty \subset H_L^m([0,\infty))$ be a Cauchy sequence. Then, from (2.3.4), one infers that $\{f_k^{(m)}\}_{k=1}^\infty$ is Cauchy in $L^2((0,\infty))$. Consequently, there exists $g \in L^2((0,\infty))$ such that

$$f_k^{(m)} \underset{k\uparrow\infty}{\longrightarrow} g \text{ in } L^2((0,\infty)). \tag{2.3.5}$$

Define

$$f(x) := \int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m\, g(t_m), \quad x \in [0,\infty). \tag{2.3.6}$$

Noting that $f^{(j)} \in AC_{loc}([0,\infty))$ and $f^{(j)}(0) = 0$ for $j = 0, 1, \dots, m-1$, and $f^{(m)} = g$ a.e. on $(0,\infty)$, one obtains $f \in H_L^m([0,\infty))$. Furthermore, by (2.3.5),

$$\|f_m - f\|_{H_L^m([0,\infty))} = \big\|f_k^{(m)} - f^{(m)}\big\|_{L^2((0,\infty))} = \big\|f_k^{(m)} - g\big\|_{L^2((0,\infty))} \underset{k\uparrow\infty}{\longrightarrow} 0. \tag{2.3.7}$$

This completes the proof that $(H_L^m([0,\infty)), (\,\cdot\,,\,\cdot\,)_{H_L^m([0,\infty))})$ is a Hilbert space.

To prove density of $C_0^\infty((0,\infty))$ in $H_L^m([0,\infty))$ we assume that $g_0 \in H_L^m([0,\infty))$ is perpendicular to $C_0^\infty((0,\infty))$ with respect to the inner product introduced in (2.3.3). Viewing $\overline{g_0}$ as a regular distribution $T_{\overline{g_0}}$ yields

$$T_{\overline{g_0}}(\varphi) := \int_0^\infty dx\, \overline{g_0(x)}\varphi(x), \quad \varphi \in C_0^\infty((0,\infty)). \tag{2.3.8}$$

Since $g_0 \perp C_0^\infty((0,\infty))$ one concludes that

$$(g_0, \varphi)_{H_L^m([0,\infty))} = (g_0^{(m)}, \varphi^{(m)})_{L^2((0,\infty))} = 0, \quad \varphi \in C_0^\infty((0,\infty)). \tag{2.3.9}$$

Since $g_0^{(j)} \in AC_{loc}([0,\infty))$ for $j = 0, \dots, m-1$, one can integrate by parts m times to yield

$$\int_0^\infty dx\, \overline{g_0(x)}\varphi^{(2m)}(x) = (-1)^m \int_0^\infty dx\, \overline{g_0^{(m)}(x)}\varphi^{(m)}(x) \tag{2.3.10}$$

$$= (-1)^m (g_0^{(m)}, \varphi^{(m)})_{L^2((0,\infty))} = (-1)^m (g_0, \varphi)_{H_L^m([0,\infty))} = 0, \quad \varphi \in C_0^\infty((0,\infty)).$$

The left-hand side of (2.3.10) is the $2m^{\text{th}}$-distributional derivative of $\overline{g_0}$. Hence,

$$T_{\overline{g_0}}^{(2m)}(\varphi) = T_{\overline{g_0}}(\varphi^{(2m)}) = (-1)^m (g_0, \varphi)_{H_L^m([0,\infty))} = 0, \quad \varphi \in C_0^\infty((0,\infty)). \quad (2.3.11)$$

Thus, by [91, Thm. 6.11 and Exercise 6.12], it follows that $T_{\overline{g_0}}$, or rather g_0, is a polynomial of degree at most $2m-1$,

$$g_0(x) = \sum_{k=0}^{2m-1} c_k x^k. \quad (2.3.12)$$

However, as $g_0 \in H_L^m([0,\infty))$, it follows that $g_0 \equiv 0$. Indeed, as $g_0^{(j)}(0) = 0$ for $j = 0, \ldots, m-1$ we have

$$c_0 = c_1 = \cdots = c_{m-1} = 0 \quad (2.3.13)$$

Furthermore, the condition $g_0^{(m)} \in L^2((0,\infty))$ yields

$$c_m = c_{m+1} = \cdots = c_{2m-1} = 0, \quad (2.3.14)$$

completing the proof. □

For general statements concerning completeness, see also [24, p. 31].

Using Theorem 2.2.1, we now prove the following theorem. The results of this theorem will be used in our new proof of the Birman inequalities, defined in (2.1.1), on $H_L^m([0,\infty))$ in the next section.

Theorem 2.3.2. *Let $f \in H_L^m([0,\infty))$. Then the following items (i)–(iii) hold:*

(i) $f^{(m-j)}/x^j \in L^2((0,\infty))$, $j = 0, 1, \ldots, m$.

(ii) $\lim_{x\uparrow\infty} \dfrac{|f^{(j)}(x)|^2}{x^{2m-2j-1}} = 0$, $j = 0, 1, \ldots, m-1$.

(iii) $\lim_{x\downarrow 0} \dfrac{|f^{(j)}(x)|^2}{x^{2m-2j-1}} = 0$, $j = 0, 1, \ldots, m-1$.

In addition,

$$f \in \widehat{H}_L^m((0,\infty)) \ \textit{implies} \ f' \in \widehat{H}_L^{m-1}((0,\infty)). \quad (2.3.15)$$

Proof. We may assume without loss of generality that $f \in H_L^m([0,\infty))$ is real-valued. To prove item (i), one notes that the case $j = 0$ is valid by definition of $H_L^m([0,\infty))$. For $j = 1$, one uses Theorem 2.2.1. Since $f^{(m-1)}(0) = 0$, one has the identity

$$\frac{f^{(m-1)}(x)}{x} = \frac{1}{x}\int_0^x dt\, f^{(m)}(t). \tag{2.3.16}$$

Next, one applies Theorem 2.2.1 with $\varphi(x) = 1$, $\psi(x) = 1/x$ and $w(x) = 1$. Since

$$\int_0^x dt\, 1^2 \int_x^\infty dt\, \frac{1}{t^2} = 1, \tag{2.3.17}$$

one infers from (2.3.16) that $f^{(m-1)}/x \in L^2((0,\infty))$ and this establishes (i) for $j = 1$. For $j = 2$, one obtains

$$\frac{f^{(m-2)}(x)}{x^2} = \frac{1}{x^2}\int_0^x dt\, t \frac{f^{(m-1)}(t)}{t}. \tag{2.3.18}$$

Again, with $\varphi(x) = x$, $\psi(x) = 1/x^2$ and $w(x) = 1$ and noting that

$$\int_0^x dt\, t^2 \int_x^\infty dt\, \frac{1}{t^4} = \frac{1}{9}, \tag{2.3.19}$$

one concludes from Theorem 2.2.1 that $f^{(m-2)}/x^2 \in L^2((0,\infty))$. By induction, for $j = 0, 1, \dots, m-1$, one obtains

$$\frac{f^{(m-j-1)}(x)}{x^{j+1}} = \frac{1}{x^{j+1}}\int_0^x dt\, t^j \frac{f^{(m-j)}(t)}{t^j}, \tag{2.3.20}$$

assuming $f^{(m-j)}/x^j \in L^2((0,\infty))$ (and $f^{(m-j-1)}(0) = 0$). Since

$$\int_0^x dt\, t^{2j} \int_x^\infty dt\, \frac{1}{t^{2j+2}} = \frac{1}{(2j+1)^2}, \tag{2.3.21}$$

one obtains from Theorem 2.2.1 that $f^{(m-j-1)}/x^{j+1} \in L^2((0,\infty))$, completing the proof of item (i). In particular, $f^{(j)}/x^{m-j}$, $f^{(j+1)}/x^{m-j-1} \in L^2((0,\infty))$ and Hölder's inequality implies

$$\frac{f^{(j)} f^{(j+1)}}{x^{2m-2j-1}} \in L^1((0,\infty)). \tag{2.3.22}$$

Using integration by parts one obtains for any $[a,b] \subset (0,\infty)$ and $j = 0, 1, \dots, m-1$,

$$\int_a^b dx\, \frac{[f^{(j)}(x)]^2}{x^{2(m-j)}} = -\frac{1}{2m-2j-1}\int_a^b dx\, [f^{(j)}(x)]^2 \left(\frac{1}{x^{2m-2j-1}}\right)'$$

$$= -\frac{1}{2m-2j-1}\left(\frac{[f^{(j)}(x)]^2}{x^{2m-2j-1}}\bigg|_a^b - 2\int_a^b dx\, \frac{f^{(j)}(x)f^{(j+1)}(x)}{x^{2m-2j-1}}\right). \tag{2.3.23}$$

From part (i) and (2.3.22), both integral terms in the identity in (2.3.23) have finite limits as $a \downarrow 0$ or $b \uparrow \infty$; hence both limits

$$\lim_{x\uparrow\infty} \frac{[f^{(j)}(x)]^2}{x^{2m-2j-1}} \text{ and } \lim_{x\downarrow 0} \frac{[f^{(j)}(x)]^2}{x^{2m-2j-1}}, \quad j = 0, 1, \dots, m-1, \tag{2.3.24}$$

exist and are finite. We now establish part (ii). Suppose, to the contrary that for some $j \in \{0, 1, \dots, m-1\}$,

$$\lim_{x\uparrow\infty} \frac{[f^{(j)}(x)]^2}{x^{2m-2j-1}} = c > 0. \tag{2.3.25}$$

Then there exists $X > 0$ such that

$$\frac{[f^{(j)}(x)]^2}{x^{2m-2j-1}} \geqslant \frac{c}{2}, \quad x \geqslant X. \tag{2.3.26}$$

Multiplying the inequality by $1/x$, integrating and applying item (i) yields

$$\infty > \int_X^\infty dx\, \frac{[f^{(j)}(x)]^2}{x^{2(m-j)}} \geqslant \frac{c}{2}\int_X^\infty dx\, \frac{1}{x} = \infty, \tag{2.3.27}$$

a contradiction. This forces $c = 0$ and proves item (ii). A similar argument proves part (iii).

The claim (2.3.15) is proved as in part (i), choosing $j = m-1$. $\square$

Remark 2.3.3. We emphasize that

$$H_L^m([0,\infty)) \neq H_0^m((0,\infty)), \quad m \in \mathbb{N}, \tag{2.3.28}$$

with $H_0^m((0,\infty))$ denoting the standard Sobolev space obtained upon completing $C_0^\infty((0,\infty))$ in the norm of $H^m((0,\infty))$. (See, however, Theorem 2.7.1 in the finite interval context.)

Indeed, $f \in H_L^m([0,\infty))$ does not necessarily imply that some, or all, of the functions f, $f', \dots, f^{(m-1)}$ belong to $L^2((0,\infty))$. In fact, define

$$\widetilde{f}(x) = \begin{cases} 0, & x \text{ near } 0, \\ x^{(2m-1)/2}/\ln(x), & x \text{ near } \infty, \end{cases} \tag{2.3.29}$$

such that

$$\widetilde{f}^{(j)} \in AC_{loc}([0,\infty)), \quad j = 0,1,\dots,m. \tag{2.3.30}$$

Calculations show that $\widetilde{f} \in H_L^m([0,\infty))$, but $\widetilde{f}^{(j)} \notin L^2((0,\infty))$, $j = 0,1,\dots,m-1$.

$\diamond$

For $m \in \mathbb{N}$, let $\widehat{H}_L^m((0,\infty))$ be as in (2.1.7) and pick $f \in \widehat{H}_L^m((0,\infty))$. Then

$$f^{(m-1)}(1) - f^{(m-1)}(x) = \int_x^1 dt\, f^{(m)}(t) \underset{x\downarrow 0}{\longrightarrow} \int_0^1 dt\, f^{(m)}(t); \tag{2.3.31}$$

hence $f^{(m-1)}(0_+) = \lim_{x\downarrow 0} f^{(m-1)}(x)$ exists and is finite. By defining $f^{(m-1)}(0) := f^{(m-1)}(0_+)$, we see that $f^{(m-1)} \in AC_{loc}([0,\infty))$. A similar argument shows that

$$f \in \widehat{H}_L^m((0,\infty)) \text{ implies } f^{(j)} \in AC_{loc}([0,\infty)),\ j = 0,1,\dots,m-1. \tag{2.3.32}$$

We now prove the following result.

Theorem 2.3.4. *For each $m \in \mathbb{N}$,*

$$H_L^m([0,\infty)) = \widehat{H}_L^m((0,\infty)) \tag{2.3.33}$$

as sets.

Proof. Let $m \in \mathbb{N}$. If $f \in H_L^m([0,\infty))$, one concludes by Theorem 2.3.2 that $f/x^m \in L^2((0,\infty))$ and hence $H_L^m([0,\infty)) \subseteq \widehat{H}_L^m((0,\infty))$. Next, we show that

$$\widehat{H}_L^m((0,\infty)) \subseteq H_L^m([0,\infty)). \tag{2.3.34}$$

One notes that

$$f \in \widehat{H}_L^m((0,\infty)) \text{ implies } f' \in \widehat{H}_L^{m-1}((0,\infty)); \tag{2.3.35}$$

indeed, this follows from Theorem 2.3.2 (i) with $j = m-1$. Repeated application of (2.3.35) yields

$$f \in \widehat{H}_L^m((0,\infty)) \text{ implies } f^{(j)} \in \widehat{H}_L^{m-j}((0,\infty)), \quad j = 0, 1, \dots, m-1. \tag{2.3.36}$$

Next, we claim that

$$f \in \widehat{H}_L^m((0,\infty)) \text{ implies } f(0) = 0 \text{ (cf. (2.3.32));} \tag{2.3.37}$$

to prove (2.3.37), suppose $|f(0)| = c > 0$. By continuity, there exists $\delta > 0$ such that

$$|f(x)| \geqslant c/2 \text{ for all } x \in [0, \delta]. \tag{2.3.38}$$

Then

$$\infty > \int_0^\infty dx\, \frac{|f(x)|^2}{x^{2m}} \geqslant \int_0^\delta dx\, \frac{|f(x)|^2}{x^{2m}} \geqslant \frac{c^2}{4} \int_0^\delta dx\, \frac{1}{x^{2m}} = \infty, \tag{2.3.39}$$

a contradiction. Hence, $f(0) = 0$ proving (2.3.37). Applying this argument to the implication in (2.3.36) yields

$$f \in \widehat{H}_L^m((0,\infty)) \text{ implies } f^{(j)}(0) = 0, \quad j = 0, 1, \dots, m-1, \tag{2.3.40}$$

proving (2.3.34). $\square$

Next, we offer one more characterization of $H_L^m([0,\infty))$. Define for each $m \in \mathbb{N}$,

$$D_m([0,\infty)) := \left\{ \int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m\, f(t_m) \,\middle|\, f \in L^2((0,\infty)) \right\}. \tag{2.3.41}$$

In particular, $D_1([0,\infty)) = \left\{ \int_0^x dt\, f(t) \,\middle|\, f \in L^2((0,\infty)) \right\}$.

Theorem 2.3.5. *For each* $m \in \mathbb{N}$, $H_L^m([0,\infty)) = D_m([0,\infty))$.

Proof. Following the discussion in the proof of Proposition 2.3.1, one concludes that $D_m([0,\infty)) \subseteq H_L^m([0,\infty))$, and hence it suffices to show $H_L^m([0,\infty)) \subseteq D_m([0,\infty))$. To this end it is instructive to first consider the case $m = 1$. Let $f \in H_1([0,\infty))$ so

$f' \in L^2((0,\infty))$. By Hölder's inequality, $f' \in L^1_{loc}((0,\infty))$; indeed, for $0 \leqslant x < y < \infty$,

$$\begin{aligned} \int_x^y dt\ |f'(t)| &\leqslant \left(\int_x^y dt\ |f'(t)|^2\right)^{1/2} \left(\int_x^y dt\, 1^2\right)^{1/2} \\ &\leqslant \left(\int_0^\infty dt\ |f'(t)|^2\right)^{1/2} |y-x|^{1/2} < \infty. \end{aligned} \tag{2.3.42}$$

For $x \geqslant 0$, let $h(x) := \int_0^x dt\, f'(t)$. Then $h \in D_1([0,\infty)) \cap H_1([0,\infty))$. By standard integration arguments, $h = f + C$ on $[0,\infty)$ for some constant C. Since $f(0) = h(0) = 0$, $C = 0$ and thus $f = h \in D_1([0,\infty))$.

In general, let $f \in H^m_L([0,\infty))$. Then $f^{(m)} \in L^2((0,\infty))$ and, as above, $f^{(m)} \in L^1_{loc}((0,\infty))$. Define, for $x \geqslant 0$,

$$h(x) := \int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m\, f^{(m)}(t_m), \tag{2.3.43}$$

so $h \in D_m([0,\infty))$ and $h^{(m)}(x) = f^{(m)}(x)$ for *a.e.* $x \geqslant 0$. Mimicking the argument for $m = 1$, it follows that $h = f + p$ on $[0,\infty)$ for some polynomial p of degree less than or equal to $m-1$. However, $h^{(j)}(0) = f^{(j)}(0)$ for $j = 0,1,\ldots,m-1$; that is to say, $p^{(j)}(0) = 0$ for $j = 0,1,\ldots,m-1$. Hence $p \equiv 0$ and thus $f = h \in D_m([0,\infty))$. □

We conclude this section with the following result which is interesting in its own right; the proof of part (i) is contained in the proof of Theorem 2.3.4 and the proof of part (ii) follows from Theorem 2.3.2.

Theorem 2.3.6. *Let $m \in \mathbb{N}$. Suppose $f : [0,\infty) \to \mathbb{C}$ satisfies $f^{(j)} \in AC_{loc}([0,\infty))$, $j = 0,1,\ldots,m-1$. Then the following assertions (i) and (ii) hold:*

(i) If $f/x^m, f^{(m)} \in L^2((0,\infty))$, then $f^{(j)}(0) = 0$, $j = 0,1,\ldots,m-1$.

(ii) If $f^{(m)} \in L^2((0,\infty))$ and $f^{(j)}(0) = 0$, $j = 0,1,\ldots,m-1$, then $f/x^m \in L^2((0,\infty))$. In fact, $f^{(m-j)}/x^j \in L^2((0,\infty))$, $j = 0,1,\ldots,m$.

For the sake of completeness, we first give Glazman's proof (see [60, pp. 83–84]) of the Birman inequalities in (2.1.1); actually, we provide a slight generalization including a power weight. Birman does not give these explicit details in [19], but it is clear that he knew this proof. We note that another proof of the inequalities in (2.1.1), for $f \in C_0^m((0,\infty))$, follows from repeated applications of Lemmas 5.3.1 and 5.3.3 in Davies' text [32, pp. 104–105]. Subsequently, we present a new proof of the Birman inequalities whose interest lies in the fact that it essentially consists of repeated use of Hardy's inequality.

We start with a slight extension of Glazman's result, [60, pp. 83–84]:

Theorem 2.4.1. *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $f \in C_0^m((0,\infty))$ be real-valued. Then*

$$\int_0^\infty dx\, x^\alpha \big[f^{(m)}(x)\big]^2 \geqslant \frac{\big[\prod_{j=1}^m (2m+1-2j-\alpha)\big]^2}{2^{2m}} \int_0^\infty dx\, \frac{[f(x)]^2}{x^{2m-\alpha}}. \tag{2.4.1}$$

Moreover, if $f \not\equiv 0$, the inequalities (2.4.1) *are strict.*

Proof. Since

$$[f(x)]^2 = 2\int_0^x dt\, f(t)f'(t), \tag{2.4.2}$$

one infers that

$$\begin{aligned} \int_0^\infty dx\, \frac{[f(x)]^2}{x^{2m-\alpha}} &= 2\int_0^\infty dx\, x^{\alpha-2m}\left(\int_0^x dt\, f(t)f'(t)\right) \\ &= 2\int_0^\infty dt\, f(t)f'(t)\left(\int_t^\infty dx\, x^{\alpha-2m}\right) \\ &= \frac{2}{2m-1-\alpha}\int_0^\infty dt\, t^{\alpha+1-2m} f(t)f'(t) \\ &\leqslant \frac{2}{|2m-1-\alpha|}\left(\int_0^\infty dx\, \frac{[f(x)]^2}{x^{2m-\alpha}}\right)^{1/2}\left(\int_0^\infty dx\, \frac{[f'(x)]^2}{x^{2(m-1)-\alpha}}\right)^{1/2}. \end{aligned} \tag{2.4.3}$$

Here we used the elementary fact

$$\int_0^y dx\, f_1(x)\bigg(\int_0^x dt\, f_2(t)\bigg) = \int_0^y dt\, f_2(t)\bigg(\int_t^y dx\, f_1(x)\bigg)dt, \quad y > 0 \tag{2.4.4}$$

(verified, e.g., by differentiating with respect to y, assuming appropriate integrability conditions on f_1, f_2), in the second line of (2.4.3), and employed Cauchy–Schwarz in the final step of (2.4.3). This implies

$$\int_0^\infty dx\, \frac{[f(x)]^2}{x^{2m-\alpha}} \leqslant \left(\frac{2}{2m-1-\alpha}\right)^2 \int_0^\infty dx\, \frac{[f'(x)]^2}{x^{2(m-1)-\alpha}}. \tag{2.4.5}$$

By iteration, for $\alpha \neq 2m+1-2j$, $j = 0, 1, \ldots, m$, one obtains

$$\begin{aligned} &\frac{((2m-1-\alpha)(2m-3-\alpha)\cdots(2m+1-2j-\alpha))^2}{2^{2j}} \int_0^\infty dx\, \frac{[f(x)]^2}{x^{2m-\alpha}} \\ &\quad \leqslant \int_0^\infty dx\, \frac{[f^{(j)}(x)]^2}{x^{2(m-j)-\alpha}}. \end{aligned} \tag{2.4.6}$$

Letting $j = m$ in (2.4.6) implies (2.4.1) on $C_0^m((0,\infty))$. Inequality (2.4.5) becomes trivial if $\alpha = 2m+1-2j$, $1 \leqslant j \leqslant m$.

To prove that all inequalities are strict unless $f \equiv 0$, one just has to check the case of equality in all the Cauchy inequalities involved. The latter are of the type

$$\begin{aligned} &\int_0^\infty dx\, \frac{f^{(j-1)}(x)}{x^{m-(j-1)-(\alpha/2)}} \frac{f^{(j)}(x)}{x^{m-j-(\alpha/2)}} \\ &\leqslant \left(\int_0^\infty dx\, \frac{\big[f^{(j-1)}(x)\big]^2}{x^{2m-2(j-1)-\alpha}}\right)^{1/2} \left(\int_0^\infty dx\, \frac{\big[f^{(j)}(x)\big]^2}{x^{2m-2j-\alpha}}\right)^{1/2}, \quad 1 \leqslant j \leqslant m. \end{aligned} \tag{2.4.7}$$

Thus, equality in (2.4.7) holds if and only if there exists some $\beta^2 \in [0,\infty)$ such that a.e. on $(0,\infty)$,

$$\pm\beta f^{(j-1)}(x) = x f^{(j)}(x), \quad 1 \leqslant j \leqslant m, \tag{2.4.8}$$

with general solution of the form

$$f_j(x) = c_{j-1} x^{\pm\beta+j-1} + c_{j-2} x^{j-2} + c_{j-3} x^{j-3} + \cdots + c_1 x + c_0, \quad 1 \leqslant j \leqslant m. \tag{2.4.9}$$

The right-hand side in (2.4.9) is not compactly supported, completing the proof. $\square$

Remark 2.4.2. If $f = f_1 + i f_2 \in H_L^m([0,\infty))$, where f_1 and f_2 are, respectively, the real and imaginary parts of f, it is clear by definition of $H_L^m([0,\infty))$ that $f_1, f_2 \in H_L^m([0,\infty))$. Moreover, if f_1 and f_2 each satisfy (2.1.1), then f also satisfies (2.1.1).

Indeed,

$$\begin{aligned}\int_0^\infty dx\,\big|f^{(m)}(x)\big|^2 &= \int_0^\infty dx\,\big|f_1^{(m)}(x)+if_2^{(m)}(x)\big|^2\\ &= \int_0^\infty dx\,\big(f_1^{(m)}(x)+if_2^{(m)}(x)\big)\big(f_1^{(m)}(x)-if_2^{(m)}(x)\big)\\ &= \int_0^\infty dx\,\big[f_1^{(m)}(x)\big]^2+\int_0^\infty dx\,\big[f_2^{(m)}(x)\big]^2\\ &\geqslant \frac{[(2m-1)!!]^2}{2^{2m}}\int_0^\infty dx\left[\frac{f_1(x)^2}{x^{2m}}+\frac{f_2(x)^2}{x^{2m}}\right]\\ &= \frac{[(2m-1)!!]^2}{2^{2m}}\int_0^\infty dx\,\frac{|f(x)|^2}{x^{2m}}. \end{aligned}\tag{2.4.10}$$

Consequently, to prove that an arbitrary $f \in H_L^m([0,\infty))$ satisfies the inequality in (2.1.1), it suffices to assume that f is real-valued. The same argument applies of course to inequality (2.4.1). $\diamond$

A closer inspection of Glazman's proof readily reveals that it can be extended to the space $H_L^m([0,\infty))$:

Corollary 2.4.3. *Let $m \in \mathbb{N}$ and $f \in H_L^m([0,\infty))$. Then,*

$$\int_0^\infty dx\,\big|f^{(m)}(x)\big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}}\int_0^\infty dx\,\frac{|f(x)|^2}{x^{2m}}.\tag{2.4.11}$$

Moreover, if $f \not\equiv 0$, the inequalities (2.4.11) *are strict.*

Proof. By Remark 2.4.2 it suffices to consider real-valued f. Comparing (2.4.3), (2.4.6), and (2.4.7) with Theorem 2.3.2 (i) legitimizes all steps in (2.4.2)–(2.4.6) (taking $\alpha = 0$) for $f \in H_L^m([0,\infty))$. Finally, also strict inequality holds for $H_L^m([0,\infty)) \ni f \not\equiv 0$ as the powers in (2.4.9) do not lie in $H_L^m([0,\infty))$. $\square$

In our new proof of Birman's inequalities (2.1.1) below, we make repeated use of the elementary inequality

$$2xy \leqslant \varepsilon x^2 + \frac{1}{\varepsilon}y^2, \quad x,y \in \mathbb{R},\ \varepsilon > 0,\tag{2.4.12}$$

following instantly from $\left(\varepsilon^{1/2}x - \varepsilon^{-1/2}y\right)^2 \geqslant 0$. In [109], Schmincke established various one-parameter integral inequalities using this fact; see also [50] where new two-parameter inequalities are given.

We note that the proof of Theorem 2.4.4 below is not shorter than other existing proofs, but we find it interesting as it reduces the sequence of Birman inequalities to repeated use of just the first such inequality, namely, Hardy's inequality (i.e., the case $m = 1$ in (2.4.13)):

Theorem 2.4.4. *Let* $m \in \mathbb{N}$ *and* $f \in H_L^m([0,\infty))$. *Then,*

$$\int_0^\infty dx \left|f^{(m)}(x)\right|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^\infty dx \, \frac{|f(x)|^2}{x^{2m}}. \tag{2.4.13}$$

Moreover, if $f \not\equiv 0$, *the inequalities* (2.4.13) *are strict.*

Proof. Let $\varepsilon > 0$, and $f \in H_L^m([0,\infty))$, $m \in \mathbb{N}$. We first prove

$$\int_0^\infty dx \left|f^{(m)}(x)\right|^2 \geqslant \begin{cases} (-\varepsilon^2 + \varepsilon) \int_0^\infty dx \, \frac{|f(x)|^2}{x^2}, & m = 1, \\ \frac{(2m-3)!!}{2^{2m-2}} (-\varepsilon^2 + (2m-1)\varepsilon) \int_0^\infty dx \, \frac{|f(x)|^2}{x^{2m}}, & m \geqslant 2. \end{cases} \tag{2.4.14}$$

Maximizing over $\varepsilon \in (0,\infty)$ then yields (2.4.13).

We prove (2.4.14) by induction on $m \in \mathbb{N}$. For $m = 1$, let $f \in H_1$ be real-valued on $[0,\infty)$; see Remark 2.4.2. Then

$$\begin{aligned} \int_0^\infty dx \, \frac{[f(x)]^2}{x^2} &= -\int_0^\infty dx \, [f(x)]^2 \left(\frac{1}{x}\right)' \\ &= -\left.\frac{[f(x)]^2}{x}\right|_0^\infty + 2\int_0^\infty dx \, \frac{f(x)f'(x)}{x} \\ &= 2\int_0^\infty dx \, \frac{f(x)f'(x)}{x} \text{ by Theorem 2.3.2 } (ii) \text{ and } (iii) \\ &\leqslant 2\left(\int_0^\infty dx \, \frac{[f(x)]^2}{x^2}\right)^{1/2} \left(\int_0^\infty dx \, [f'(x)]^2\right)^{1/2} \\ &\leqslant \varepsilon \int_0^\infty dx \, \frac{[f(x)]^2}{x^2} + \frac{1}{\varepsilon}\int_0^\infty dx \, [f'(x)]^2 \text{ using (2.4.12).} \end{aligned} \tag{2.4.15}$$

This last inequality can be rewritten as

$$\int_0^\infty dx \, [f'(x)]^2 \geqslant (-\varepsilon^2 + \varepsilon) \int_0^\infty dx \, \frac{[f(x)]^2}{x^2}. \tag{2.4.16}$$

Since the maximum of $\varepsilon \to -\varepsilon^2 + \varepsilon$ occurs at $\varepsilon = 1/2$ with maximum value $1/4$, one concludes that

$$\int_0^\infty dx\, [f'(x)]^2 \geqslant \frac{1}{4} \int_0^\infty dx\, \frac{[f(x)]^2}{x^2} \geqslant (-\varepsilon^2 + \varepsilon) \int_0^\infty dx\, \frac{[f(x)]^2}{x^2}. \tag{2.4.17}$$

The inequalities in (2.4.17) establish both (2.4.13) and (2.4.14) when $m = 1$. Incidentally, this argument also provides a proof of the classical Hardy inequality (2.1.2). We now assume that (2.4.13) holds for $m = 1, \dots, k-1$ for some $k \in \mathbb{N}$. Let $f \in H_k([0,\infty))$; by (2.1.8), $f' \in H_{k-1}([0,\infty))$ and so, from our induction hypothesis,

$$\int_0^\infty dx\, \left[f^{(k)}(x)\right]^2 = \int_0^\infty dx\, \left[[f'(x)]^{(k-1)}\right]^2 \geqslant \frac{[(2k-3)!!]^2}{2^{2k-2}} \int_0^\infty dx\, \frac{[f'(x)]^2}{x^{2(k-1)}}. \tag{2.4.18}$$

On the other hand, assuming f is real-valued, we note, from the definition of $H_k([0,\infty))$ and Theorem 2.3.2 (i) that both f/x^k and f'/x^{k-1} belong to $L^2((0,\infty))$. Moreover,

$$\begin{aligned}
&\int_0^\infty dx\, \frac{[f(x)]^2}{x^{2k}} = -\frac{1}{2k-1} \int_0^\infty dx\, [f(x)]^2 \left(x^{-2k+1}\right)' \\
&\quad = -\frac{1}{2k-1} \left(\frac{[f(x)]^2}{x^{2k-1}} \bigg|_0^\infty - 2 \int_0^\infty dx\, \frac{f(x) f'(x)}{x^{2k-1}} \right) \\
&\quad = \frac{2}{2k-1} \int_0^\infty dx\, \frac{f(x) f'(x)}{x^{2k-1}} \quad \text{by Theorem 2.3.2}\,(ii) \text{ and } (iii) \\
&\quad \leqslant \frac{2}{2k-1} \left(\int_0^\infty dx\, \frac{[f(x)]^2}{x^{2k}} \right)^{1/2} \left(\int_0^\infty dx\, \frac{[f'(x)]^2}{x^{2(k-1)}} \right)^{1/2} \\
&\quad \leqslant \frac{1}{2k-1} \left(\varepsilon \int_0^\infty dx\, \frac{[f(x)]^2}{x^{2k}} + \frac{1}{\varepsilon} \int_0^\infty dx\, \frac{[f'(x)]^2}{x^{2(k-1)}} \right) \quad \text{by (2.4.12)}.
\end{aligned} \tag{2.4.19}$$

Rearranging terms in this last inequality yields

$$\int_0^\infty dx\, \frac{[f'(x)]^2}{x^{2(k-1)}} \geqslant \left(-\varepsilon^2 + (2k-1)\varepsilon\right) \int_0^\infty dx\, \frac{[f(x)]^2}{x^{2k}}. \tag{2.4.20}$$

Combining (2.4.18) and (2.4.20), one obtains

$$\int_0^\infty dx\, \left[f^{(k)}(x)\right]^2 \geqslant \frac{[(2k-3)!!]^2}{2^{2k-2}} (-\varepsilon^2 + (2k-1)\varepsilon) \int_0^\infty dx\, \frac{[f(x)]^2}{x^{2k}}, \tag{2.4.21}$$

implying (2.4.14). The maximum of the function $\varepsilon \to -\varepsilon^2 + (2k-1)\varepsilon$ over $(0,\infty)$ occurs at $\varepsilon = (2k-1)/2$ with the maximum value being $(2k-1)^2/4$. Substituting this value into (2.4.21) yields

$$
\begin{aligned}
\int_0^\infty dx\,\big[f^{(k)}(x)\big]^2 &\geqslant \frac{[(2k-3)!!]^2}{2^{2k-2}}\frac{(2k-1)^2}{4}\int_0^\infty dx\,\frac{[f(x)]^2}{x^{2k}} \\
&= \frac{[(2k-1)!!]^2}{2^{2k}}\int_0^\infty dx\,\frac{[f(x)]^2}{x^{2k}},
\end{aligned} \tag{2.4.22}
$$

completing the proof of (2.4.13). Strict inequality in (2.4.13) is clear from Corollary 2.4.3, alternatively, one can apply the argument following (2.4.7) (with $j=1$, $\alpha=0$) and a similar one involving the final ε-step in (2.4.19). $\square$

Remark 2.4.5. Hardy's work on his celebrated inequality started in 1915, [67] (see also [68]– [70], [71, Sect. 9.8], and the historical comments in [85, Chs. 1, 3, App.]). Higher-order Hardy inequalities, including weight functions, are discussed in [86, Ch. 4] and [101, Sect. 10], however, Birman's sequence of inequalities, [19], is not mentioned in these sources. $\diamond$

Remark 2.4.6. The characterization of functions in $H_L^m([0,\infty))$ in Theorem 2.3.5 provides us with two equivalent ways of expressing Birman's inequalities. Indeed, we have already established that

$$
\int_0^\infty dx\,\big|f^{(m)}(x)\big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}}\int_0^\infty dx\,\frac{|f(x)|^2}{x^{2m}}, \quad f\in H_L^m([0,\infty)). \tag{2.4.23}
$$

Alternatively, via Theorem 2.3.5, one can now express this inequality as

$$
\int_0^\infty dx\,|f(x)|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}}\int_0^\infty dx\,\frac{1}{x^{2m}}\left|\int_0^x dt_1\int_0^{t_1} dt_2\cdots\int_0^{t_{m-1}} dt_m\, f(t_m)\right|^2 \\
f\in L^2((0,\infty)). \tag{2.4.24}
$$

In the case $m=1$, both forms of Hardy's inequality are given in [71] (cf. Theorem 253, p. 175, and Theorem 327, p. 240). $\diamond$

The constants $[(2m-1)!!]^2/2^{2m}$ in Birman's sequence of inequalities (2.4.13) are optimal as shown in the following section.

2.5 Optimality of Constants

The principal purpose of this section is to prove the following result:

Theorem 2.5.1. *The constants* $[(2m-1)!!]^2/2^{2m}$ *in the Birman sequence of Hardy–Rellich-type inequalities*

$$\int_0^\infty dx \left|f^{(m)}(x)\right|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^\infty dx\, \frac{|f(x)|^2}{x^{2m}}, \quad f \in H_L^m([0,\infty)),\ m \in \mathbb{N}, \tag{2.5.1}$$

are optimal in the following sense: The inequality (2.5.1) *ceases to be valid if* $[(2m-1)!!]^2/2^{2m}$ *is replaced by* $[(2m-1)!!]^2/2^{2m} + \varepsilon$ *for any* $\varepsilon > 0$ *on the right-hand side of* (2.5.1).

Proof. We follow the strategy of proof in [11, p. 4] in connection with weighted Hardy inequalities. Fix some $a > 0$ and let $\chi_{(0,a)}$ denote the characteristic function on $(0,a)$. For $\sigma > -1/2$, let $f_\sigma \in L^2((0,\infty))$ be given by

$$f_\sigma(x) := x^\sigma \chi_{(0,a)}(x), \quad x \in (0,\infty), \tag{2.5.2}$$

and define

$$F_{m,\sigma}(x) := \int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m\, f_\sigma(t_m), \quad x \in (0,\infty). \tag{2.5.3}$$

One observes that $F_{m,\sigma} \in D_m([0,\infty)) = H_L^m([0,\infty))$ by Theorem 2.3.5. Since $F_{m,\sigma}^{(m)} = f_\sigma$ a.e. on $[0,\infty)$ one has

$$\int_0^\infty dx \left|F_{m,\sigma}^{(m)}(x)\right|^2 = \int_0^\infty dx\, |f_\sigma(x)|^2 = \int_0^a dx\, x^{2\sigma} = \frac{a^{1+2\sigma}}{1+2\sigma}. \tag{2.5.4}$$

An induction argument then shows that

$$F_{m,\sigma}(x) = \begin{cases} \frac{x^{m+\sigma}}{\prod_{j=1}^m (j+\sigma)}, & 0 \leqslant x \leqslant a, \\ \sum_{k=0}^{m-1} b_k x^k, & x > a, \end{cases} \tag{2.5.5}$$

where

$$b_k := \frac{(-1)^{m-k+1} a^{m-k+\sigma}}{k!(m-k-1)!(m-k+\sigma)}, \quad 0 \leqslant k \leqslant m-1. \tag{2.5.6}$$

A straightforward computation yields

$$
\begin{aligned}
\int_0^\infty dx\, x^{-2m} F_{m,\sigma}^2(x) &= \int_0^a dx\, x^{-2m} F_{m,\sigma}^2(x) + \int_a^\infty dx\, x^{-2m} F_{m,\sigma}^2(x) \\
&= \frac{1}{\prod_{j=1}^m (j+\sigma)^2} \int_0^a dx\, x^{2\sigma} + \int_a^\infty dx\, x^{-2m} \left(\sum_{k=0}^{m-1} b_k x^k \right)^2 \\
&= \frac{a^{1+2\sigma}}{(1+2\sigma) \prod_{j=1}^m (j+\sigma)^2} + C(a), \qquad (2.5.7)
\end{aligned}
$$

where

$$
0 < C(a) := \int_a^\infty dx\, x^{-2m} \left(\sum_{k=0}^{m-1} b_k x^k \right)^2 \underset{a\uparrow\infty}{\longrightarrow} 0. \qquad (2.5.8)
$$

Thus,

$$
\begin{aligned}
\frac{\int_0^\infty f_\sigma^2(x)\, dx}{\int_0^\infty x^{-2n} F_{n,\sigma}^2(x)\, dx} &= \frac{\prod_{j=1}^n (j+\sigma)^2}{1 + (1+2\sigma) C(a) a^{-1-2\sigma} \prod_{j=1}^n (j+\sigma)^2} \\
&= \frac{\prod_{j=1}^n (2j-1)^2}{2^{2n}} + (1+2\sigma) D(a,\sigma) + O\big((1+2\sigma)^2\big),
\end{aligned} \qquad (2.5.9)
$$

where $D(\,\cdot\,,\cdot)$ satisfies $D(a,\sigma) > 0$, $D(a,-1/2) > 0$, if a is chosen sufficiently large, due to the fact that $C(a) \underset{a\uparrow\infty}{\longrightarrow} 0$. Thus, choosing σ sufficiently close to $-1/2$ will undercut any choice of $\varepsilon > 0$ in a replacement of $[(2n-1)!!]^2/2^{2n}$ by $[(2n-1)!!]^2/2^{2n} + \varepsilon$ for *any* $\varepsilon > 0$ on the right-hand side of (2.5.1). $\square$

Without going into further details we note that the argument just presented also works for the weighted extension of the Birman inequalities (2.4.1).

We also note that the constants in (2.5.1) coincide of course with the ones obtained by Yafaev [118] upon specializing his result to the spherically symmetric case.

Remark 2.5.2. Let $m \in \mathbb{N}$. To motivate the choice of the function f_σ, and hence that of $F_{m,\sigma}(x) = c\, x^{m+\sigma} = c\, x^{m-(1/2)+\varepsilon}$, writing $\sigma = -(1/2) + \varepsilon$, $\varepsilon > 0$, near $x = 0$ in the above proof, it suffices to recall that Birman's inequalities,

$$
\int_0^\infty dx\, \big| f^{(m)}(x) \big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^\infty dx\, \frac{|f(x)|^2}{x^{2m}}, \quad f \in H_L^m([0,\infty)), \qquad (2.5.10)
$$

are naturally associated with the differential expression

$$\tau_{2m} := (-1)^n \frac{d^{2m}}{dx^{2m}} - \frac{[(2m-1)!!]^2}{2^{2m}} \frac{1}{|x|^{2m}}, \quad x \in (0,\infty). \tag{2.5.11}$$

According to Birman's inequalities,

$$\tau_{2m}\big|_{C_0^\infty((0,\infty))} \geqslant 0, \tag{2.5.12}$$

and the function $y_m(x) = x^{m-(1/2)}$, $x \geqslant 0$, satisfies

$$\tau_{2m} y_m = 0 \tag{2.5.13}$$

and hence formally saturates the lower bound 0 of τ_{2m}. To ensure membership in $H_L^m([0,\infty))$ one thus regularizes y_m with the help of the parameter $\varepsilon > 0$, yielding $F_{m,\sigma}(x) = c\, x^{m-(1/2)+\varepsilon}$, $x \geqslant 0$. $\diamond$

2.6 The Continuous Cesàro Operator T_1 and its Generalizations T_m

As shown in the proof of Proposition 2.3.1, for any $f \in L^2((0,\infty))$,

$$\int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m f(t_m) \in H_L^m([0,\infty)), \tag{2.6.1}$$

thus, we can introduce for $m \in \mathbb{N}$,

$$\begin{aligned} &(T_m f)(x) := \frac{1}{x^m} \int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m f(t_m), \qquad x \in (0,\infty), \\ &f \in \operatorname{dom}(T_m) = L^2((0,\infty)). \end{aligned} \tag{2.6.2}$$

The operator T_m is patterned after the continuous Cesàro operator,

$$(T_1 f)(x) := \frac{1}{x} \int_0^x dt\, f(t), \quad x \in (0,\infty),\ f \in L^2((0,\infty)). \tag{2.6.3}$$

We now prove the following result.

Theorem 2.6.1. *Let $m \in \mathbb{N}$ and define T_m as in* (2.6.2), (2.6.3). *Then T_m is a bounded linear operator on $L^2((0,\infty))$ with norm*

$$\|T_m\| = \frac{2^m}{(2m-1)!!}. \tag{2.6.4}$$

Proof. We abbreviate the reciprocal of the square root of the Birman constant by

$$A_m := \frac{2^m}{(2m-1)!!}. \tag{2.6.5}$$

Let $f \in L^2((0,\infty))$, and write

$$F(x) = \int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m f(t_m), \quad x \in (0,\infty). \tag{2.6.6}$$

Then $F \in H_L^m([0,\infty))$ and

$$F^{(m)} = f \text{ a.e. on } [0,\infty). \tag{2.6.7}$$

Hence, by (2.4.13) in Theorem 2.4.4, one concludes that

$$\int_0^\infty dx\, \frac{|F(x)|^2}{x^{2m}} \leqslant A_m^2 \int_0^\infty dx\, \left|F^{(m)}(x)\right|^2 = A_m^2 \int_0^\infty dx\, |f(x)|^2. \tag{2.6.8}$$

Since $T_m f = F/x^m$, (2.6.8) implies

$$\|T_m f\|_{L^2((0,\infty))} \leqslant A_m \|f\|_{L^2((0,\infty))}; \tag{2.6.9}$$

in particular, T_m is bounded and $\|T_m\| \leqslant A_m$. To show $\|T_m\| = A_m$, let $0 < K < A_m$ so $K^2 < A_m^2$. Since, by Theorem 2.5.1, the constant A_m^2 is sharp, there exists $G \in H_L^m([0,\infty))$ such that

$$\int_0^\infty dx\, \frac{|G(x)|^2}{x^{2m}} > K^2 \int_0^\infty dx\, \left|G^{(m)}(x)\right|^2. \tag{2.6.10}$$

Let $g := G^{(m)} \in L^2((0,\infty))$ such that

$$G(x) = \int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m g(t_m). \tag{2.6.11}$$

Then $T_m g = G/x^m$ and

$$\|T_m g\|_{L^2((0,\infty))} > K \|g\|_{L^2((0,\infty))}. \tag{2.6.12}$$

Thus,

$$\begin{aligned} A_m &= \inf\left\{C > 0 \,\middle|\, \|T_m f\|_{L^2((0,\infty))} \leqslant C\|f\|_{L^2((0,\infty))} \text{ for all } f \in L^2((0,\infty))\backslash\{0\}\right\} \\ &= \|T_m\|, \end{aligned} \tag{2.6.13}$$

completing the proof. $\square$

It will soon be clear that while T_m is bounded, it is noncompact, see (2.6.77).

Next, we turn to the inverse of T_m and state the following fact.

Lemma 2.6.2. *Let $m \in \mathbb{N}$, then*

$$(T_m^{-1} f)(x) = \frac{d^m}{dx^m} x^m f(x), \quad x \in (0, \infty), \tag{2.6.14}$$

$$f \in \operatorname{dom}\left(T_m^{-1}\right) = \big\{g \in L^2((0,\infty)) \,\big|\, g \in AC_{loc}^{(m-1)}((0,\infty));\ (x^m g)^{(m)} \in L^2((0,\infty)); \\ \lim_{x \downarrow 0} (x^m g(x))^{(j)} = 0,\ j = 0, \dots, m-1\big\}.$$

Proof. For $f \in \operatorname{ran}(T_m) \cap AC_{loc}^{(m-1)}((0,\infty))$, consider the equation

$$(T_m g)(x) = f(x), \tag{2.6.15}$$

or, equivalently,

$$g(x) = (x^m f(x))^{(m)}. \tag{2.6.16}$$

Since T_m is one-to-one, it is clear from (2.6.16) that the form of the inverse of T_m is given by

$$(R_m g)(x) = (x^m g(x))^{(m)}, \quad x \in (0, \infty). \tag{2.6.17}$$

We now seek to find the (largest) domain, $\operatorname{dom}(R_m) \subseteq L^2((0,\infty))$. For any such choice of domain,

$$\operatorname{dom}(R_m) \subseteq \big\{g \in L^2((0,\infty)) \,\big|\, g \in AC_{loc}^{(m-1)}((0,\infty));\ (x^m g(x))^{(m)} \in L^2((0,\infty))\big\}, \tag{2.6.18}$$

and it is clear that

$$(R_m \circ T_m) g(x) = R_m(T_m g)(x) = g(x). \tag{2.6.19}$$

Conversely, we see that if

$$\begin{aligned} g(x) &= (T_m \circ R_m) g(x) = T_m(R_m g)(x) \\ &= \frac{1}{x^m} \int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \, (t_m^m g(t_m))^{(m)}, \end{aligned} \tag{2.6.20}$$

then it is necessary, for $j = 0, 1, \ldots, m-1$, that the limits

$$\lim_{x \to 0^+} (x^m g(x))^{(j)} \tag{2.6.21}$$

all must exist and equal 0. Consequently, if we define

$$\begin{aligned} \operatorname{dom}(R_m) := \big\{ g \in L^2((0,\infty)) \,\big|\, g \in AC_{loc}^{(m-1)}((0,\infty));\ (x^m g(x))^{(m)} \in L^2(0,\infty), \\ \lim_{x \to 0^+} (x^m g(x))^{(j)} \text{ exists and equals } 0,\ j = 0, 1, \ldots m-1 \big\}, \end{aligned} \tag{2.6.22}$$

then, by (2.6.19) and (2.6.20), it follows that operator R_m, defined by (2.6.17) and (2.6.22), is the inverse of T_m in $L^2((0,\infty))$. $\square$

One notes that $\operatorname{dom}(R_m) = \operatorname{ran}(T_m)$ as given in (2.6.22) is dense in $L^2((0,\infty))$. Indeed, one verifies that for any $\alpha > -1$,

$$\big\{ x^{\alpha/2} e^{-x/2} L_k^\alpha(x) \,\big|\, k \in \mathbb{N}_0 \big\} \subset \operatorname{dom}(R_m), \tag{2.6.23}$$

where $\{L_k^\alpha\}_{k=0}^\infty$ is the sequence of Laguerre polynomials which forms a complete orthogonal set in the Hilbert space $L^2((0,\infty))$. Indeed, it is clear that for $j = 0, 1, \ldots, m-1$,

$$\lim_{x \to 0} (x^{m+\alpha/2} e^{-x/2} L_k^\alpha(x))^{(j)} = 0 \ \text{ and } \ (x^{m+\alpha/2} e^{-x/2} L_k^\alpha(x))^{(m)} \in L^2((0,\infty)). \tag{2.6.24}$$

In the following we will show that the m boundary conditions in (2.6.14) can actually be replaced by the $(m-1)$ L^2-conditions

$$x^j g^{(j)} \in L^2((0,\infty)), \quad j = 1, \ldots, m-1. \tag{2.6.25}$$

To prove this we start with the following elementary observations: For $m \in \mathbb{N}$ and $x \in (0,\infty)$,

$$(i)\ f \in AC_{loc}((0,\infty)) \ \text{ if and only if } \ x^m f \in AC_{loc}((0,\infty)). \tag{2.6.26}$$

$$(ii)\ (x^m f(x))^{(k)} = a_k(m,k) x^m f^{(k)}(x) + \cdots + a_0(m,k) x^{m-k} f(x), \quad 0 \leqslant k \leqslant m,$$

$$\text{where } \ a_j(m,k) = \binom{k}{j} \prod_{\ell=0}^{k-j-1} (m-\ell), \quad 0 \leqslant j \leqslant k, \tag{2.6.27}$$

$$(iii)\, x^k(xf(x))^{(k+1)} = x^{k+1}f^{(k+1)}(x) + (k+1)x^k f^{(k)}(x), \quad k \in \mathbb{N}_0, \tag{2.6.28}$$

$$(iv)\, x(x^k f^{(k)}(x))' = x^{k+1}f^{(k+1)}(x) + kx^k f^{(k)}(x), \quad k \in \mathbb{N}_0. \tag{2.6.29}$$

Lemma 2.6.3. *Let $m \in \mathbb{N}$, then*

$$\begin{aligned} &\operatorname{dom}\big(T_1^{-m}\big) \\ &\quad = \big\{g \in L^2((0,\infty)) \,\big|\, g \in AC_{loc}^{(m-1)}((0,\infty));\, x^j g^{(j)} \in L^2((0,\infty)),\, j = 1,\dots,m\big\}. \end{aligned} \tag{2.6.30}$$

Proof. We start with the case $m = 1$ and note that

$$g \in L^2((0,\infty)),\, g \in AC_{loc}((0,\infty)), \ \text{ and } \ xg' \in L^2((0,\infty)) \ \text{ implies } \ \lim_{x\downarrow 0} xg(x) = 0. \tag{2.6.31}$$

Indeed, observing

$$\int_{x_0}^x dt\, tg'(t) = [tg(t)]\big|_{x_0}^x - \int_{x_0}^x dt\, g(t) \tag{2.6.32}$$

shows that $c = \lim_{x\downarrow 0} xg(x)$ exists. If $c \neq 0$, then without loss of generality we can assume that $c > 0$. Then $xg(x) > c/2$, equivalently, $g(x) > c/(2x)$ for sufficiently small $0 < x$, yielding $g \notin L^2((0,\infty))$, a contradiction. Thus, $c = 0$ and (2.6.31) holds.

Since $(xg)' = xg' + g$, this implies

$$\operatorname{dom}\big(T_1^{-1}\big) = \big\{g \in L^2((0,\infty)) \,\big|\, g \in AC_{loc}((0,\infty));\, xg' \in L^2((0,\infty))\big\} \tag{2.6.33}$$

and hence verifies (2.6.30) for $m = 1$.

Next, we use induction on $m \in \mathbb{N}$. Assume (2.6.30) holds for $m \in \mathbb{N}$ fixed. Then for $m+1$, one obtains

$$\begin{aligned} &\operatorname{dom}\big(T_1^{-(m+1)}\big) = \operatorname{dom}\big(T_1^{-m}T_1^{-1}\big) \\ &\quad = \big\{g \in \operatorname{dom}\big(T_1^{-1}\big) \,\big|\, \big(T_1^{-1}g\big) \in \operatorname{dom}\big(T_1^{-m}\big)\big\} \end{aligned}$$

$$= \{g \in \operatorname{dom}(T_1^{-1}) \mid (xg)' \in \operatorname{dom}(T_1^{-m})\}$$

$$= \{g \in L^2((0,\infty)) \mid g \in AC_{loc}((0,\infty));\ xg' \in L^2((0,\infty));\ (xg)' \in L^2((0,\infty)); \\ (xg)' \in AC_{loc}^{(m-1)}((0,\infty));\ x^j(xg)^{(j+1)} \in L^2((0,\infty)),\ j = 1,\dots,m\} \tag{2.6.34}$$

$$= \{g \in L^2((0,\infty)) \mid g \in AC_{loc}((0,\infty));\ (xg)' \in AC_{loc}^{(m-1)}((0,\infty)); \\ x^j(xg)^{(j+1)} \in L^2((0,\infty)),\ j = 0,\dots,m\} \tag{2.6.35}$$

$$= \{g \in L^2((0,\infty)) \mid g \in AC_{loc}^{(m)}((0,\infty));\ x^j g^{(j)} \in L^2((0,\infty)),\ j = 1,\dots,m+1\}, \tag{2.6.36}$$

as desired. Here we used the induction hypothesis in (2.6.34), and again the fact $(xg)' = xg' + g$ and $f \in L^2((0,\infty))$ to conclude that $(xg)' \in L^2((0,\infty))$ if and only if $xg' \in L^2((0,\infty))$ in (2.6.35). To arrive at (2.6.36) one uses (2.6.26) and hence,

$$g \in AC_{loc}((0,\infty)) \text{ and } (xg)' \in AC_{loc}^{(m-1)}((0,\infty)) \text{ if and only if } g \in AC_{loc}^{(m)}((0,\infty)), \tag{2.6.37}$$

as well as (2.6.28) and

$$x^j(xg)^{(j+1)} = x^{j+1}g^{(j+1)} + (j+1)x^j g^{(j)} \in L^2((0,\infty)), \quad j = 0,\dots,m, \tag{2.6.38}$$

which iteratively yields $x^k g^{(k)} \in L^2((0,\infty))$ for $1 \leqslant k \leqslant m+1$. □

Lemma 2.6.4. *Let* $m \in \mathbb{N}$. *Assume* $f \in AC_{loc}^{(m-1)}((0,\infty))$ *and* $x^k f^{(k)} \in L^2((0,\infty))$ *for* $k = 0,1,\dots,m$. *Then*

$$\lim_{x\downarrow 0}(x^m f(x))^{(j)} = 0, \quad j = 0,1,\dots,m-1. \tag{2.6.39}$$

Proof. The case $m = 1$ holds by (2.6.31).

For $m = 2$, assume $f \in AC_{loc}^{(1)}((0,\infty))$ with $f, xf', x^2f'' \in L^2((0,\infty))$. Then for $j = 0$, one has again by (2.6.31),

$$\lim_{x\downarrow 0} xf(x) = 0 \text{ and hence } \lim_{x\downarrow 0} x^2 f(x) = 0. \tag{2.6.40}$$

For $j = 1$, one notes that $xf' \in AC_{loc}((0,\infty))$, see (2.6.26), and $x(xf')' = x^2 f'' + xf' \in L^2((0,\infty))$. Hence, applying (2.6.31) to $g = xf'$ yields

$$\lim_{x \downarrow 0} x(xf'(x)) = \lim_{x \downarrow 0} x^2 f'(x) = 0. \tag{2.6.41}$$

Combining (2.6.40) and (2.6.41) shows

$$\lim_{x \downarrow 0} (x^2 f(x))' = \lim_{x \downarrow 0} \left[x^2 f'(x) + 2x f(x)\right] = 0, \tag{2.6.42}$$

proving (2.6.39) for $m = 2$.

Next, we prove (2.6.39) for general $m \in \mathbb{N}$. Let $m \in \mathbb{N}$ be fixed and assume the hypotheses of the lemma, that is,

$$f \in AC_{loc}^{(m-1)}((0,\infty)) \text{ and } x^k f^{(k)} \in L^2((0,\infty)), \quad k = 0, 1, \dots, m. \tag{2.6.43}$$

One notes that $f \in AC_{loc}^{(m-1)}((0,\infty))$ implies $x^j f^{(j)} \in AC_{loc}((0,\infty))$ for $j = 0, \dots, m-1$, see (2.6.26). Using (2.6.29) and (2.6.43),

$$x(x^j f^{(j)})' = x^{j+1} f^{(j+1)} + j x^j f^{(j)} \in L^2((0,\infty)), \quad j = 0, \dots, m-1. \tag{2.6.44}$$

Thus, applying (2.6.31) iteratively to $g_j := x^j f^{(j)}$ one arrives at

$$\lim_{x \downarrow 0} x g_j(x) = \lim_{x \downarrow 0} x^{j+1} f^{(j)}(x) = 0, \quad j = 0, 1, \dots, m-1. \tag{2.6.45}$$

In particular, for any $m \in \mathbb{N}$ with $m > j$,

$$\lim_{x \downarrow 0} x^m f^{(j)}(x) = 0, \quad j = 0, 1, \dots, m-1. \tag{2.6.46}$$

Thus, by (2.6.27) and (2.6.46), one obtains

$$\begin{aligned} \lim_{x \downarrow 0} (x^m f(x))^{(j)} &= a_j \lim_{x \downarrow 0} x^m f^{(j)}(x) + \cdots + a_0 \lim_{x \downarrow 0} x^{m-j} f(x) \\ &= a_j \cdot 0 + \cdots + a_0 \cdot 0 = 0, \quad j = 0, \dots, m-1. \end{aligned} \tag{2.6.47}$$

$\square$

Introducing

$$p_m(z) = \prod_{k=0}^{m-1}(z+k), \quad z \in \mathbb{C},\ m \in \mathbb{N}, \tag{2.6.48}$$

we are now ready to characterize T_m^{-1} in terms of T_1^{-1}:

Theorem 2.6.5. *Let $m \in \mathbb{N}$, then*

$$T_m^{-1} = p_m\big(T_1^{-1}\big), \tag{2.6.49}$$

$$\begin{aligned}\operatorname{dom}\left(T_m^{-1}\right) = \operatorname{dom}\left(T_1^{-m}\right) = \big\{g \in L^2((0,\infty)) \,\big|\, g \in AC_{loc}^{(m-1)}((0,\infty));\\ x^j g^{(j)} \in L^2((0,\infty)),\ j = 1,\dots,m\big\}.\end{aligned} \tag{2.6.50}$$

Proof. Focusing at first on (2.6.50) suppose $f \in \operatorname{dom}\left(T_m^{-1}\right) = \operatorname{ran}(T_m)$. Then, for some $h \in L^2((0,\infty))$,

$$f(x) = \frac{1}{x^m}\int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m\, h(t_m), \tag{2.6.51}$$

equivalently,

$$x^m f(x) = \int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m\, h(t_m). \tag{2.6.52}$$

Taking the k-th derivative, $0 \leqslant k < m$, of (2.6.52) yields

$$\begin{aligned}a_k x^m f^{(k)}(x) + \cdots + a_0 x^{m-k} f(x) = \int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-k-1}} dt_{m-k-1} h(t_{m-k-1}),\\ k = 0,\dots,m-1,\end{aligned} \tag{2.6.53}$$

where the coefficients $a_j(m,k)$ are given by (2.6.27). Dividing (2.6.53) by x^{m-k} yields

$$a_k x^k f^{(k)} + \cdots + a_1 x f' + a_0 f = T_{m-k}\, h \in L^2((0,\infty)), \tag{2.6.54}$$

which iteratively proves that $x^k f^{(k)} \in L^2((0,\infty))$ for $k = 0,\dots,m-1$. Finally, taking the m-th derivative of (2.6.52) shows

$$a_n x^m f^{(m)} + \cdots + a_1 x f' + a_0 f = h \in L^2((0,\infty)), \tag{2.6.55}$$

proving $x^m f^{(m)} \in L^2((0,\infty))$ and thus $f \in \operatorname{dom}\big(T_1^{-m}\big)$.

Conversely, suppose $f \in \operatorname{dom}\big(T_1^{-m}\big)$. Then $(x^m f)^{(m)} \in L^2((0,\infty))$ since $x^j f^{(j)} \in L^2((0,\infty))$ for $j = 0, \dots, m$ by hypothesis, and

$$(x^m f)^{(m)} = a_n x^m f^{(m)} + \cdots + a_1 x f' + a_0 f. \tag{2.6.56}$$

The condition $\lim_{x\downarrow 0}(x^m f(x))^{(j)} = 0$, $j = 0, \dots, m-1$, follows from Lemma 2.6.4.

Turning to the proof of (2.6.49), one notes that the case $m = 1$ in (2.6.49) obviously holds. Hence, assume (2.6.49) holds for some $m \in \mathbb{N}$ fixed. Then for $m+1$ one computes,

$$\begin{aligned}
p_{m+1}(T_1^{-1})f &= \prod_{k=0}^{m}(T_1^{-1} + k)f = T_m^{-1}(T_1^{-1} + m)f \text{ by hypothesis} \\
&= T_m^{-1}T_1^{-1}f + mT_m^{-1}f \\
&= (x^m(xf' + f))^{(m)} + m(x^m f)^{(m)} \\
&= (x^{m+1}f')^{(m)} + (x^m f)^{(m)} + m(x^m f)^{(m)} \\
&= (x^{m+1}f')^{(m)} + ((m+1)x^m f)^{(m)} \\
&= (x^{m+1}f' + (m+1)x^m f)^{(m)} \\
&= ((x^{m+1}f)')^{(m)} \\
&= (x^{m+1}f)^{(m+1)} \\
&= T_{m+1}^{-1}f, \quad f \in \operatorname{dom}\big(T_{m+1}^{-1}\big) = \operatorname{dom}\big(T_1^{-m-1}\big).
\end{aligned} \tag{2.6.57}$$

□

Given Theorem 2.6.5, spectral analysis of T_m reduces to that of T_1, respectively, T_1^{-1}, via the spectral mapping theorem. Thus, by (2.6.33), we recall that

$$\begin{aligned}
&(T_1^{-1}f)(x) = (xf)'(x) = xf'(x) + f(x), \\
&f \in \operatorname{dom}\big(T_1^{-1}\big) = \big\{g \in L^2((0,\infty)) \,\big|\, g \in AC_{loc}((0,\infty); (xg)' \in L^2((0,\infty))\big\}.
\end{aligned} \tag{2.6.58}$$

Next, we introduce the unitary Mellin transform $\mathcal{M}$ and its inverse, $\mathcal{M}^{-1}$, via the pair of formulas

$$\mathcal{M}\colon \begin{cases} L^2((0,\infty);) \to L^2(\mathbb{R}; d\lambda), \\ f \mapsto (\mathcal{M}f)(\lambda) \equiv f^*(\lambda) := (2\pi)^{-1/2}\, \text{s-lim}_{a\uparrow\infty} \int_{1/a}^{a} dx\, f(x) x^{-(1/2)+i\lambda} \\ \hfill \text{for a.e. } \lambda \in \mathbb{R}, \end{cases} \tag{2.6.59}$$

$$\mathcal{M}^{-1}\colon \begin{cases} L^2(\mathbb{R}; d\lambda) \to L^2((0,\infty);), \\ f^* \mapsto (\mathcal{M}^{-1}f^*)(x) \equiv f(x) := (2\pi)^{-1/2}\, \text{s-lim}_{b\uparrow\infty} \int_{-b}^{b} d\lambda\, f^*(\lambda) x^{-(1/2)-i\lambda} \\ \hfill \text{for a.e. } x \in (0,\infty). \end{cases} \tag{2.6.60}$$

For details on $\mathcal{M}$ (resp., $\mathcal{M}^{-1}$) we refer, for instance, to [115, Sect. 3.17] (see also, [117, Sect. 1.3]).

The fact,

$$i\left(\frac{d}{dx}x - \frac{1}{2}\right) x^{-(1/2)-i\lambda} = \lambda x^{-(1/2)-i\lambda}, \quad x \in (0,\infty),\ \lambda \in \mathbb{R}, \tag{2.6.61}$$

naturally leads to the following definition of the operator S_1 in $L^2((0,\infty);)$,

$$S_1 := i\big(T_1^{-1} - 2^{-1} I_{L^2((0,\infty))}\big), \quad \operatorname{dom}(S_1) = \operatorname{dom}\big(T_1^{-1}\big), \tag{2.6.62}$$

and establishes that S_1 is unitarily equivalent to the operator of multiplication by the independent variable in $L^2(\mathbb{R})$,

$$\begin{gathered} \big(\mathcal{M} S_1 \mathcal{M}^{-1} f^*\big)(\lambda) = \lambda f^*(\lambda) \ \text{ for a.e. } \lambda \in \mathbb{R} \text{ and} \\ \text{for all } f^* \in L^2(\mathbb{R}; d\lambda) \text{ such that } \lambda f^* \in L^2(\mathbb{R}; d\lambda). \end{gathered} \tag{2.6.63}$$

Summarizing, the Mellin transform diagonalizes S_1 and hence T_1. Denoting by $C(z_0; r_0) \subset \mathbb{C}$ the circle of radius $r_0 > 0$ centered at $z_0 \in \mathbb{C}$, one obtains the following result.

Theorem 2.6.6. *Introduce S_1 in $L^2((0,\infty);)$ as in (2.6.62). Then S_1 is self-adjoint and hence T_1 is normal. Moreover, the spectra of S_1 and T_1 are simple and purely absolutely continuous. In particular,*

$$\sigma(S_1) = \sigma_{ac}(S_1) = \mathbb{R}, \quad \sigma_p(S_1) = \sigma_{sc}(S_1) = \emptyset, \tag{2.6.64}$$

$$\sigma(T_1) = \sigma_{ac}(T_1) = C(1;1), \quad \sigma_p(T_1) = \sigma_{sc}(T_1) = \emptyset. \tag{2.6.65}$$

Proof. The claims concerning S_1 follow from (2.6.62), unitarity of $\mathcal{M}$, and the fact that the operator of multiplication by the independent variable in $L^2(\mathbb{R}; d\lambda)$ has purely absolutely continuous spectrum. The pair of maps

$$C(1;1) \ni z \mapsto i\big(z^{-1} - 2^{-1}\big) \in \mathbb{R}, \quad \mathbb{R} \ni \lambda \mapsto \big[-i\lambda + 2^{-1}\big]^{-1} \in C(1;1), \tag{2.6.66}$$

establishes the facts concerning T_1. □

Remark 2.6.7. Introducing the family of operators

$$(T_{1,z}f)(x) := \int_0^1 dt\, s^{-z} f(st) = x^{z-1} \int_0^x du\, u^{-z} f(u), \quad \mathrm{Re}(z) < 1/2, \tag{2.6.67}$$

one verifies that ($\mathrm{Re}(z) < 1/2$)

$$T_{1,0} = T_1, \tag{2.6.68}$$

$$\left(x^z \frac{d}{-} x^{1-z} T_{1,z} f\right)(x) = f(x), \tag{2.6.69}$$

$$x^z \frac{d}{-} x^{1-z} = x\frac{d}{-} + I - zI = \frac{d}{-}x - zI, \tag{2.6.70}$$

$$\begin{aligned} T_{1,z} &= \left(\frac{d}{-}x - zI\right)^{-1} = \big(T_1^{-1} - zI\big)^{-1} = (I - zT_1)^{-1} T_1, \\ &= -z^{-1} I + z^{-2}\big(z^{-1} I - T_1\big)^{-1}, \end{aligned} \tag{2.6.71}$$

$$(T_1 - zI)^{-1} = -z^{-1} I - z^{-2} T_{1,z^{-1}}. \tag{2.6.72}$$

◇

The fact, $\sigma(T_1) = C(1;1)$, as well as the resolvent formula (2.6.78) for T_1 are well-known, we refer, for instance, to [23], and [21] (see also [6], [62], [89], [90], and the references cited therein). What appears to be less well-known is the a.c. nature of the spectrum of T_1 and the spectral representation in terms of the Mellin transform. Much of the work on the spectral theory for T_1 focused on p-dependence of the spectrum in L^p-spaces (on finite intervals and on the half-line), Hardy, Bergman, and Dirichlet spaces, etc. For related classes of integral operators see, for instance, [92], [100], [104], and the references cited therein.

Remark 2.6.8. One notes the curious fact that while the closed, symmetric operator A_1 in $L^2((0,\infty))$, defined by

$$(A_1 f)(x) = i f'(x), \quad f \in \operatorname{dom}(A_1) = \big\{ g \in L^2((0,\infty)) \,\big|\, g \in AC_{loc}([0,\infty)); \\ g(0_+) = 0;\, g' \in L^2((0,\infty)) \big\}, \tag{2.6.73}$$

is the prime example of a symmetric operator with unequal deficiency indices (1 and 0), and hence has no self-adjoint extensions in $L^2((0,\infty))$, the right multiplication of id/dx with x in (2.6.62), followed by the shift $-i/2$, yields the self-adjoint operator S_1 in $L^2((0,\infty))$. $\diamond$

Formula (2.6.49), $T_m^{-1} = p_m\big(T_1^{-1}\big)$, $m \in \mathbb{N}$, together with the spectral theorem applied to T_1, then yield

$$\sigma\big(T_m^{-1}\big) = p_m\big(\sigma\big(T_1^{-1}\big)\big), \quad m \in \mathbb{N}. \tag{2.6.74}$$

Equivalently, introducing the rational function r_m by

$$r_m(z) = z^m \prod_{k=1}^{m-1} (1+kz)^{-1}, \quad z \in \mathbb{C}\backslash\big\{-\ell^{-1}\big\}_{1\leqslant \ell \leqslant m-1},\ m \in \mathbb{N}, \tag{2.6.75}$$

the formula

$$T_m = r_m(T_1), \quad m \in \mathbb{N}, \tag{2.6.76}$$

yields the following facts.

Theorem 2.6.9. *Let $m \in \mathbb{N}$. Then, T_m is normal and*

$$\sigma(T_m) = r_m(\sigma(T_1)) = \big\{ r_m\big(1+e^{i\theta}\big) \,\big|\, \theta \in [0,2\pi] \big\}, \quad m \in \mathbb{N}, \tag{2.6.77}$$

$$\sigma(T_m) = \sigma_{ac}(T_m), \quad \sigma_p(T_m) = \sigma_{sc}(T_m) = \emptyset. \tag{2.6.78}$$

Proof. Normality of T_m is clear from (2.6.76). The facts (2.6.77), (2.6.78) follow from combining Theorem 2.6.6, (2.6.76), and the spectral theorem for normal operators. $\square$

We have not been able to find discussions of T_m in the literature.

The spectrum of T_m, for various values of $m \in \mathbb{N}$, is illustrated next:

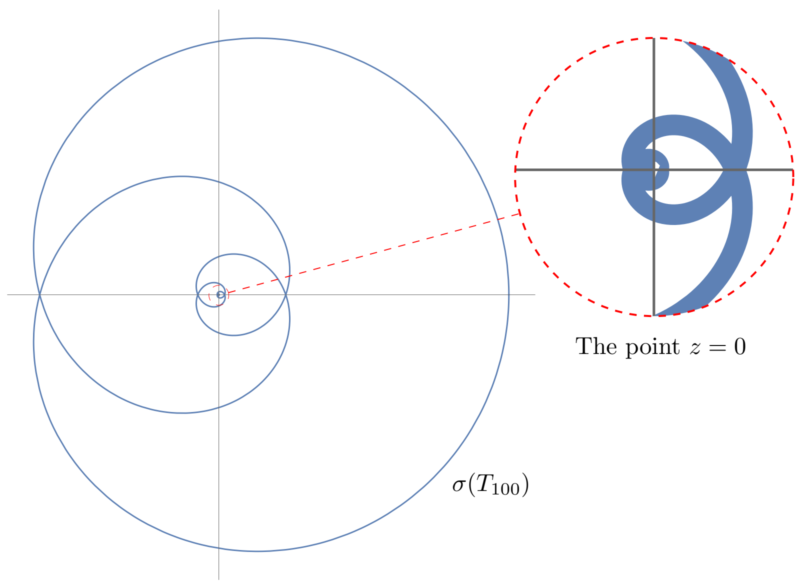

Figure 2.1: 14× Magnification of $\sigma(T_{100})$

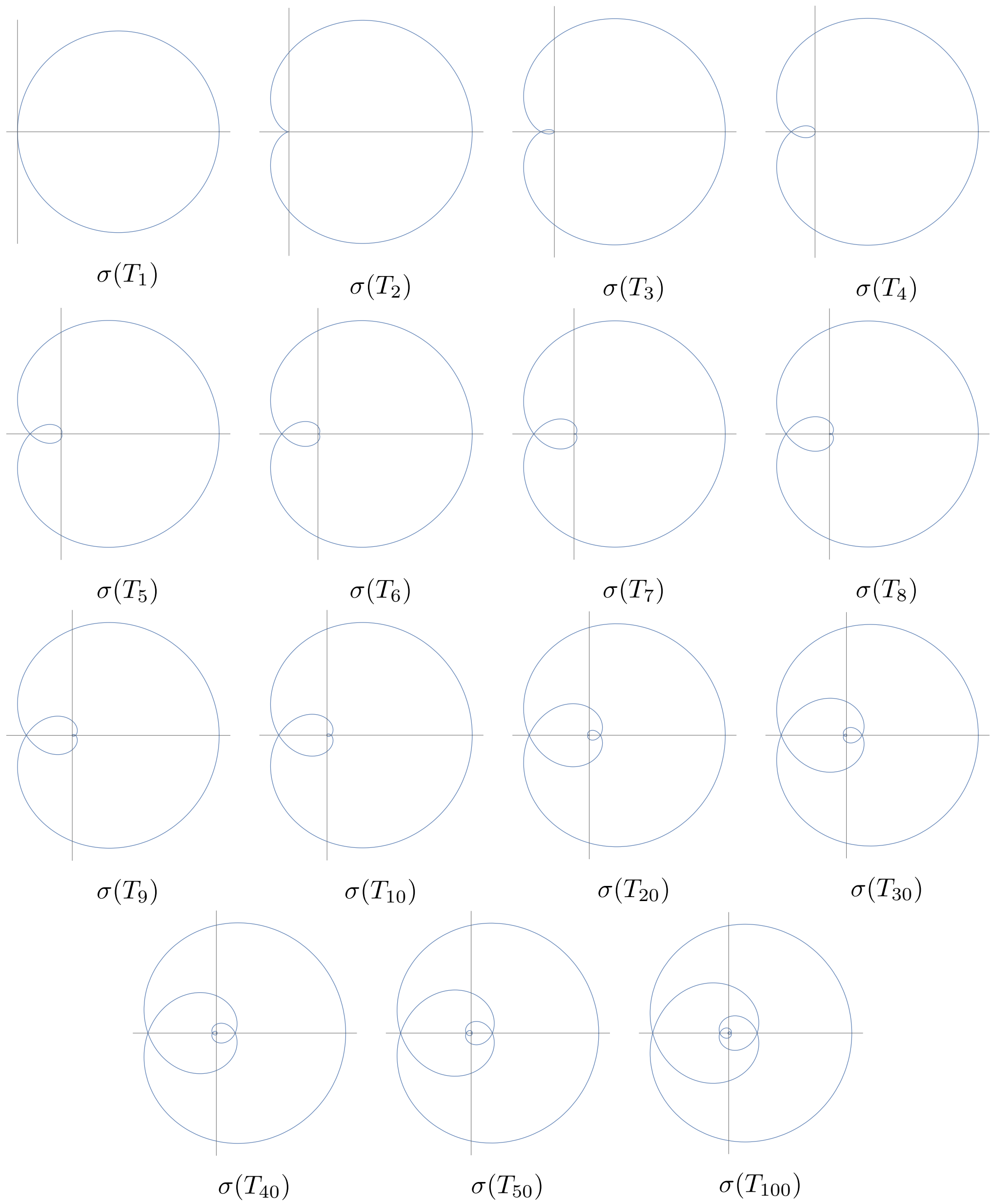

Figure 2.2: The Spectrum of T_m for certain $m \in \mathbb{N}$

2.7 The Birman Inequalities on the Finite Interval $[0,b]$

For fixed $b \in (0,\infty)$ and $m \in \mathbb{N}$, introduce the set[2]

$$
\begin{aligned}
H^m_{LR}([0,b]) := \big\{ f : [0,b] \to \mathbb{C} \,\big|\, f^{(m)} \in L^2((0,b));\ f^{(j)} \in AC([0,b]); \\
f^{(j)}(0) = f^{(j)}(b) = 0,\ j = 0,1,\dots,m-1 \big\},
\end{aligned}
\tag{2.7.1}
$$

with associated inner product $(\,\cdot\,,\,\cdot\,)_{H^m_{LR}([0,b])}$,

$$
(f,g)_{H^m_{LR}([0,b])} := \int_0^b dx\, \overline{f^{(m)}(x)}\, g^{(m)}(x), \quad f,g \in H^m_{LR}([0,b]), \tag{2.7.2}
$$

and norm

$$
\|f\|_{H^m_{LR}([0,b])} = \big\| f^{(m)} \big\|_{L^2((0,b))}, \quad f \in H^m_{LR}([0,b]). \tag{2.7.3}
$$

Next, we state the following result which yields the Birman inequalities on $[0,b]$ as well as sharpness and equality results on $H^m_{LR}([0,b]) = H^m_0((0,b))$, where $H^m_0((0,b))$ denotes the standard Sobolev space on $(0,b)$ obtained upon completion of $C_0^\infty((0,b))$ in the norm of $H^m((0,b))$, that is,

$$
\begin{aligned}
H^m((0,b)) = \big\{ f : [0,b] \to \mathbb{C} \,\big|\, f^{(j)} \in AC([0,b]),\ j = 0,1,\dots,m-1; \\
f^{(k)} \in L^2((0,b)),\ k = 0,1,\dots,m \big\},
\end{aligned}
\tag{2.7.4}
$$

$$
H^m_0((0,b)) = \big\{ f \in H^m((0,b)) \,\big|\, f^{(j)}(0) = f^{(j)}(b) = 0,\ j = 0,1,\dots,m-1 \big\}. \tag{2.7.5}
$$

Theorem 2.7.1. *Let $m \in \mathbb{N}$, then the following items (i)–(iv) hold:*

(i) For each $m \in \mathbb{N}$, and $b \in (0,\infty)$,

$$
H^m_{LR}([0,b]) = H^m_0((0,b)) \tag{2.7.6}
$$

as sets. In particular,

$$
f \in H^m_{LR}([0,b]) \ \textit{implies}\ f^{(j)} \in L^2((0,b)), \quad j = 0,1,\dots,m. \tag{2.7.7}
$$

In addition, the norms in $H^m_{LR}([0,b])$ and $H^m_0((0,b))$ are equivalent.

---

[2] It is possible to replace the boundary conditions at $x = 0$ by $f/x^m \in L^2$ and/or the one at $x = b$ by $f/(b-x)^n \in L^2$, leading to additional spaces $H^m_{LR}(0,b])'$, $H^m_{LR}([0,b))'$, and $H^m_{LR}((0,b))'$ in analogy to (2.3.2), but we omit further details at this point.

(ii) The following hold:

(α) Let $a, c \in [0,\infty)$, $a < c$, $f\colon [a,c] \to \mathbb{C}$, with $f^{(j)} \in AC([a,c])$, $f^{(j)}(a) = 0$, $j = 0,1,\dots,m-1$, and $f^{(m)} \in L^2((a,c))$. Then,

$$\int_a^c dx\,\big|f^{(m)}(x)\big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}} \int_a^c dx\,\frac{|f(x)|^2}{(x-a)^{2m}}. \tag{2.7.8}$$

(β) Let $a, c \in [0,\infty)$, $a < c$, $f : [a,c] \to \mathbb{C}$, with $f^{(j)} \in AC([a,c])$, $f^{(j)}(c) = 0$, $j = 0,1,\dots,m-1$, and $f^{(m)} \in L^2((a,c))$. Then,

$$\int_a^c dx\,\big|f^{(m)}(x)\big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}} \int_a^c dx\,\frac{|f(x)|^2}{(c-x)^{2m}}. \tag{2.7.9}$$

(γ) Introducing the distance of $x \in (0,b)$ to the boundary $\{0,b\}$ of $(0,b)$ by

$$\delta(x) = \min\{x, |b-x|\}, \quad x \in (0,b),\ b \in (0,\infty), \tag{2.7.10}$$

one has

$$\int_0^b dx\,\big|f^{(m)}(x)\big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^b dx\,\frac{|f(x)|^2}{\delta(x)^{2m}}, \quad f \in H_0^n((0,b)). \tag{2.7.11}$$

In all cases (α)–(γ), if $f \not\equiv 0$, the inequalities (2.7.8), (2.7.9), *and* (2.7.11) *are strict.*

(iii) The constant $[(2m-1)!!]^2/2^{2m}$ is sharp in all cases (α)–(γ) in item (ii).

Proof. (*i*) Equality of $H^m_{LR}([0,b])$ with the standard Sobolev space $H_0^m((0,b))$ in (2.7.6) (and hence the fact (2.7.7)) follows from [24, p. 29] (discussing the endpoint behavior of $f \in H^m((0,b))$ at $\{0,b\}$) and especially, from [24, Theorem 2, p. 127 and Corollary 6, p. 128]. Alternatively, one can exploit the boundary conditions $f^{(j)}(0) = f^{(j)}(b) = 0$, $0 \leqslant j \leqslant m$, and combine [43, Corollary V.3.21] and the Friedrichs inequality [43, p. 242],

$$\big\|f^{(j)}\big\|_{L^2((0,b))} \leqslant C\big\|f^{(m)}\big\|_{L^2((0,b))}, \quad f \in H_0^m((0,b)), \tag{2.7.12}$$

with $C = C(j,m,b) \in (0,\infty)$ independent of $f \in H_0^m((0,b))$.

For the rest of the proof we assume that f is real-valued.

To prove item (ii) part (α) one first follows the proof of Theorem 2.4.4, observing that for $k \in \mathbb{N}$,

$$\begin{aligned}\int_a^c dx\, \frac{[f(x)]^2}{(x-a)^{2k}} &= -\frac{1}{2k-1}\left(\frac{[f(x)]^2}{(x-a)^{2k-1}}\bigg|_a^c - 2\int_a^c dx\, \frac{f(x)f'(x)}{(x-a)^{2k-1}} \right) \\ &= -\frac{1}{2k-1}\left(\frac{f(c)^2}{(c-a)^{2k-1}} - 2\int_a^c dx\, \frac{f(x)f'(x)}{(x-a)^{2k-1}} \right) \\ &\leqslant \frac{2}{2k-1}\int_a^c dx\, \frac{f(x)f'(x)}{(x-a)^{2k-1}}, \end{aligned} \tag{2.7.13}$$

and then continues as in (2.4.19).

Part (β) follows from (α) by reflecting about the interval midpoint.

For part (γ), one can follow the argument provided in [32, Corollary 5.3.2] in the context of the Hardy inequality $m = 1$: Splitting the interval $(0, b)$ into $(0, b/2] \cup [b/2, b)$, exploiting the fact

$$\begin{aligned}\int_0^{b/2} dx\, \frac{[f(x)]^2}{x^{2m}} + \int_{b/2}^b dx\, \frac{[f(x)]^2}{(b-x)^{2m}} &= \int_0^{b/2} dx\, \frac{[f(x)]^2}{\delta(x)^{2m}} + \int_{b/2}^b dx\, \frac{[f(x)]^2}{\delta(x)^{2m}} \\ &= \int_0^b dx\, \frac{[f(x)]^2}{\delta(x)^{2m}}, \end{aligned} \tag{2.7.14}$$

and then separately applying parts (α) to $(0, b/2)$, and (β) to $(b/2, b)$, yields (2.7.11).

To prove strict inequality in (2.7.8)–(2.7.11), it suffices to consider part (α) since (β) follows by reflection and either (α) or (β) implies (γ). One infers from (2.4.9) that functions that would yield equality are of the type

$$g_0(x) = c_{m-1}(x-a)^{\lambda+m-1} + c_{m-2}(x-a)^{m-2} + c_{m-3}(x-a)^{m-3} + \cdots + c_1(x-a) + c_0. \tag{2.7.15}$$

The fact $g_0^{(j)}(a) = 0$ for $j = 0, 1, \dots, m-2$ shows

$$c_0 = c_1 = \cdots = c_{m-2} = 0, \tag{2.7.16}$$

so that

$$g_0(x) = c_{m-1}(x-a)^{\lambda+m-1}. \tag{2.7.17}$$

Equation (2.7.17) suggests equality in (2.7.8) holds only for functions of the form

$$g_0(x) = d_0(x-a)^{\mu} \tag{2.7.18}$$

for some $d_0 \in \mathbb{C}$, $\mu \in \mathbb{R}$. To prove $d_0 = 0$, assume otherwise. First, one notes that $g_0^{(m)} \in L^2((a,c))$ implies

$$2(\mu - m) > -1 \ \text{ or, } \ \mu > m - 1/2. \tag{2.7.19}$$

Inductively, one sees that

$$[\mu(\mu-1)\cdots(\mu-m+1)]^2 > \frac{[(2m-1)!!]^2}{2^{2m}}, \quad m \in \mathbb{N}. \tag{2.7.20}$$

Computing the left side of (2.7.8) then yields

$$\int_0^{b/2} dx\, \big[g_0^{(m)}(x)\big]^2 = |d_0|^2[\mu(\mu-1)\cdots(\mu-m+1)]^2 \int_0^{b/2} dx\,(x-a)^{2(\mu-m)} \tag{2.7.21}$$

$$> |d_0|^2 \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^{b/2} dx\,(x-a)^{2(\mu-m)} \tag{2.7.22}$$

$$= |d_0|^2 \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^{b/2} dx\, \frac{[g_0(x)]^2}{(x-a)^{2m}}, \tag{2.7.23}$$

contradicting equality in (2.7.8), and hence $d_0 = 0$.

To prove item (iii) for case (α) one chooses $f_\sigma(x) = (x-a)^\sigma \chi_{(a,c)}$ and then proceeds as in the proof of Theorem 2.5.1. To settle case (β) one uses case (α) combined with reflection about the interval midpoint. If in case (γ), the constant $[(2m-1)!!]^2/2^{2m}$ would not be optimal and a larger constant should exist, then considering the two intervals $(0, b/2)$ and $(b/2, b)$ would lead to a larger constant on at least one of them as well, contradicting cases (α) or (β). $\square$

2.8 The Vector-Valued Case

In this section we indicate that all results described thus far extend to the vector-valued case in which f is not just complex-valued, but actually, $\mathcal{H}$-valued, with $\mathcal{H}$ a separable, complex Hilbert space.

To set the stage we briefly review some facts on Bochner integrability and associated vector-valued L^p- and Sobolev spaces.

Regarding details of the Bochner integral we refer, for instance, to [9, p. 6–21], [42, p. 44–50], [76, p. 71–86], [88, Sect. 4.2], [93, Ch. III], [119, Sect. V.5]. In particular, if $(a,b) \subseteq \mathbb{R}$ is a finite or infinite interval and $\mathcal{B}$ a Banach space, and if $p \geqslant 1$, the symbol $L^p((a,b);dx;\mathcal{B})$, in short, $L^p((a,b);\mathcal{B})$, whenever Lebesgue measure is understood, denotes the set of equivalence classes of strongly measurable $\mathcal{B}$-valued functions which differ at most on sets of Lebesgue measure zero, such that $\|f(\cdot)\|_{\mathcal{B}}^p \in L^1((a,b))$. The corresponding norm in $L^p((a,b);\mathcal{B})$ is given by

$$\|f\|_{L^p((a,b);\mathcal{B})} = \left(\int_{(a,b)} dx\, \|f(x)\|_{\mathcal{B}}^p\right)^{1/p} \tag{2.8.1}$$

and $L^p((a,b);\mathcal{B})$ is a Banach space.

If $\mathcal{H}$ is a separable Hilbert space, then so is $L^2((a,b);\mathcal{H})$ (see, e.g., [18, Subsects. 4.3.1, 4.3.2], [20, Sect. 7.1]).

One recalls that by a result of Pettis [103], if $\mathcal{B}$ is separable, weak measurability of $\mathcal{B}$-valued functions implies their strong measurability.

A map $f : [c,d] \to \mathcal{B}$ (with $[c,d] \subset (a,b)$) is called *absolutely continuous on* $[c,d]$, denoted by $f \in AC([c,d];\mathcal{B})$, if

$$f(x) = f(x_0) + \int_{x_0}^{x} dt\, g(t), \quad x_0, x \in [c,d], \tag{2.8.2}$$

for some $g \in L^1((c,d);\mathcal{B})$. In particular, f is then strongly differentiable a.e. on (c,d) and

$$f'(x) = g(x) \text{ for a.e. } x \in (c,d). \tag{2.8.3}$$

Similarly, $f : [c,d] \to \mathcal{B}$ is called *locally absolutely continuous*, denoted by $f \in AC_{loc}([c,d];\mathcal{B})$, if $f \in AC([c',d'];\mathcal{B})$ on any closed subinterval $[c',d'] \subset (c,d)$.

Sobolev spaces $W^{m,p}((a,b);\mathcal{B})$ for $m \in \mathbb{N}$ and $p \geqslant 1$ are defined as follows: $W^{1,p}((a,b);\mathcal{B})$ is the set of all $f \in L^p((a,b);\mathcal{B})$ such that there exists a $g \in L^p((a,b);\mathcal{B})$ and an $x_0 \in (a,b)$ such that

$$f(x) = f(x_0) + \int_{x_0}^{x} dt\, g(t) \text{ for a.e. } x \in (a,b). \tag{2.8.4}$$

In this case g is the strong derivative of f, $g = f'$. Similarly, $W^{m,p}((a,b);\mathcal{B})$ is the set of all $f \in L^p((a,b);\mathcal{B})$ so that the first m strong derivatives of f are in $L^p((a,b);\mathcal{B})$.

Finally, $W^{m,p}_{\mathrm{loc}}((a,b);\mathcal{B})$ is the set of $\mathcal{B}$-valued functions defined on (a,b) for which the restrictions to any compact interval $[\alpha,\beta]\subset(a,b)$ are in $W^{m,p}((\alpha,\beta);\mathcal{B})$. In particular, this applies to the case $m=0$ and thus defines $L^p_{\mathrm{loc}}((a,b);\mathcal{B})$. If a is finite we may allow $[\alpha,\beta]$ to be a subset of $[a,b)$ and denote the resulting space by $W^{m,p}_{\mathrm{loc}}([a,b);\mathcal{B})$ (and again this applies to the case $m=0$).

Following a frequent practice (cf., e.g., the discussion in [7, Sect. III.1.2]), we will call elements of $W^{1,1}([c,d];\mathcal{B})$, $[c,d]\subset(a,b)$ (resp., $W^{1,1}_{\mathrm{loc}}((a,b);;\mathcal{B})$), strongly absolutely continuous $\mathcal{B}$-valued functions on $[c,d]$ (resp., strongly locally absolutely continuous $\mathcal{B}$-valued functions on (a,b)), but caution the reader that unless $\mathcal{B}$ possesses the Radon–Nikodym (RN) property, this notion differs from the classical definition of $\mathcal{B}$-valued absolutely continuous functions (we refer the interested reader to [42, Sect. VII.6] for an extensive list of conditions equivalent to $\mathcal{B}$ having the RN property). Here we just mention that reflexivity of $\mathcal{B}$ implies the RN property.

In the special case $\mathcal{B}=\mathbb{C}$, we omit $\mathcal{B}$ and just write $L^p_{(loc)}((a,b))$, as usual.

In the following we will typically employ the special case $p=2$ and use a complex, separable Hilbert space $\mathcal{H}$ for $\mathcal{B}$, denoting the corresponding Sobolev spaces by $H^m((a,b);\mathcal{H})$. The inner product in $L^2((a,b);\mathcal{H})$, in obvious notation, then reads

$$(f,g)_{L^2((a,b);\mathcal{H})}=\int_a^b dx\,(f(x),g(x))_{\mathcal{H}},\quad f,g\in L^2((a,b);\mathcal{H}). \tag{2.8.5}$$

In other words, $L^2((a,b);\mathcal{H})$ can be identified with the constant fiber direct integral of Hilbert spaces,

$$L^2((a,b);\mathcal{H})\simeq\int_{(a,b)}^{\oplus} dx\,\mathcal{H}. \tag{2.8.6}$$

For applications of these concepts to Schrödinger operators with operator-valued potentials we refer to [57]; applications to scattering theory for multidimensional Schrödinger operators are studied in great detail in [88, Chs. IV, V]. The latter reference motivated us to add the present section.

Before stating the sequence of Birman inequalities in the $\mathcal{H}$-valued context, we recall a few basic properties of Bochner integrals which illustrate why all results in Sections 2.2–2.7 in the special complex-valued case (i.e., $\mathcal{H} = \mathbb{C}$) carry over verbatim to the vector-valued situation.

As representative examples we mention, for instance,

$$\bigg\| \int_{(a,b)} dx\, f(x) \bigg\|_{\mathcal{H}} \leqslant \int_{(a,b)} dx\, \|f(x)\|_{\mathcal{H}}, \quad f \in L^1((a,b);\mathcal{H}), \tag{2.8.7}$$

$$\|fg\|_{L^1((a,b);\mathcal{H})} \leqslant \|f\|_{L^2((a,b);\mathcal{H})} \|g\|_{L^2((a,b))}, \quad f \in L^2((a,b);\mathcal{H}),\ g \in L^2((a,b)),$$

with equality if and only if for some $(0,0) \neq (\alpha,\beta) \in \mathbb{R}^2$,

$$\alpha \|f(x)\|_{\mathcal{H}}^2 = \beta |g(x)|^2 \text{ for a.e. } x \in (a,b), \tag{2.8.8}$$

$$\|(f(x),g(x))_{\mathcal{H}}\|_{L^1((a,b))} \leqslant \|f\|_{L^2((a,b);\mathcal{H})} \|g\|_{L^2((a,b);\mathcal{H})}, \quad f,g \in L^2((a,b);\mathcal{H}),$$

with equality if and only if for some $(0,0) \neq (\alpha,\beta) \in \mathbb{R}^2$,

$$\alpha \|f(x)\|_{\mathcal{H}}^2 = \beta \|g(x)\|_{\mathcal{H}}^2 \text{ for a.e. } x \in (a,b), \tag{2.8.9}$$

$$\int_c^d dx\, (f'(x), g(x))_{\mathcal{H}} + \int_c^d dx\, (f(x), g'(x))_{\mathcal{H}} = (f(d), g(d))_{\mathcal{H}} - (f(c), g(c))_{\mathcal{H}},$$
$$f,g \in AC([a,b];\mathcal{H}),\ (c,d) \subseteq (a,b). \tag{2.8.10}$$

Given these preliminaries we now introduce the spaces

$$\begin{aligned} H_L^m([0,\infty);\mathcal{H}) := \big\{ f : [0,\infty) \to \mathcal{H} \,\big|\, & f^{(j)} \in AC_{loc}([0,\infty);\mathcal{H});\ f^{(m)} \in L^2((0,\infty);\mathcal{H}); \\ & f^{(j)}(0) = 0,\ j = 0,1,\dots,m-1 \big\}, \quad m \in \mathbb{N}. \end{aligned} \tag{2.8.11}$$

As in the scalar context, the space $H_L^m([0,\infty);\mathcal{H})$ is a Hilbert space when endowed with the inner product

$$(f,g)_{H_L^m([0,\infty);\mathcal{H})} := \int_0^\infty dx\, \big(f^{(m)}(x), g^{(m)}(x)\big)_{\mathcal{H}}, \quad f,g \in H_L^m([0,\infty);\mathcal{H}), \tag{2.8.12}$$

and norm

$$\|f\|_{H_L^m([0,\infty);\mathcal{H})} = \big\|f^{(m)}\big\|_{L^2((0,\infty);\mathcal{H})}, \quad f \in H_L^m([0,\infty);\mathcal{H}). \tag{2.8.13}$$

Similarly, introducing

$$\widehat{H}_L^m((0,\infty);\mathcal{H}) := \big\{f : (0,\infty) \to \mathcal{H} \,\big|\, f^{(j)} \in AC_{loc}((0,\infty);\mathcal{H}),\ j = 0,1,\dots,m-1;$$
$$f^{(m)}, f/x^m \in L^2((0,\infty);\mathcal{H})\big\}, \quad m \in \mathbb{N}, \tag{2.8.14}$$

one proves as in the scalar context that

$$H_L^m([0,\infty);\mathcal{H}) = \widehat{H}_L^m((0,\infty);\mathcal{H}), \quad m \in \mathbb{N}, \tag{2.8.15}$$

and that $C_0^\infty((0,\infty);\mathcal{H})$ is dense in $H_L^m([0,\infty);\mathcal{H})$. For the latter assertion it suffices to replace (2.3.8) by

$$T_{\overline{g_0}}(\varphi) := \int_0^\infty dx\, (g_0(x), \varphi(x))_{\mathcal{H}}, \quad \varphi \in C_0^\infty((0,\infty);\mathcal{H}). \tag{2.8.16}$$

These facts are shown as in the scalar context upon introducing the a.e. nonnegative Lebesgue measurable weight function $w : (a,b) \to \mathbb{R}$ and $\varphi, \psi : (a,b) \to \mathbb{C}$ Lebesgue measurable functions satisfying conditions (i)–(iii) in Theorem 2.2.1.

Define the linear operators $A, B : L^2((a,b);w;\mathcal{H}) \to L^2_{loc}((a,b);w;\mathcal{H})$ by

$$(Af)(x) := \varphi(x) \int_x^b dt\, \psi(t)w(t)f(t), \quad f \in L^2_{loc}((a,b);w;\mathcal{H}), \tag{2.8.17}$$

and

$$(Bf)(x) := \psi(x) \int_a^x dt\, \varphi(t)w(t)f(t), \quad f \in L^2_{loc}((a,b);w;\mathcal{H}), \tag{2.8.18}$$

and the function $K : (a,b) \to \mathbb{R}$ by

$$K(x) := \left(\int_a^x dt\, w(t)\, |\varphi(t)|^2\right)^{1/2} \left(\int_x^b dt\, w(t)\, |\psi(t)|^2\right)^{1/2}. \tag{2.8.19}$$

Then again A and B are bounded linear operators in $L^2((a,b);w;\mathcal{H})$ if and only if

$$K := \sup_{x\in(a,b)} K(x) < \infty. \tag{2.8.20}$$

Moreover, if $K < \infty$, then A and B are adjoints of each other in $L^2((a,b);w;\mathcal{H})$, with

$$\|Af\|_{L^2((a,b);w;\mathcal{H})} = \|Bf\|_{L^2((a,b);w;\mathcal{H})} \leqslant 2K\|f\|_{L^2((a,b);w;\mathcal{H})},$$
$$f \in L^2((a,b);w;\mathcal{H}). \tag{2.8.21}$$

Furthermore, the constant $2K$ in (2.2.7) is best possible; that is,

$$\|A\|_{\mathcal{B}(L^2((a,b);w;\mathcal{H}))} = \|B\|_{\mathcal{B}(L^2((a,b);w;\mathcal{H}))} = 2K. \tag{2.8.22}$$

Also Theorem 2.3.2 extends to the present vector-valued situation in the following form: Let $f \in H_L^m([0,\infty);\mathcal{H})$, then

$$f^{(m-j)}/x^j \in L^2((0,\infty);\mathcal{H}), \quad j = 0,1,\dots,m, \tag{2.8.23}$$

$$\lim_{x\uparrow\infty} \frac{\|f^{(j)}(x)\|_{\mathcal{H}}^2}{x^{2m-2j-1}} = 0, \quad j = 0,1,\dots,m-1, \tag{2.8.24}$$

$$\lim_{x\downarrow 0} \frac{\|f^{(j)}(x)\|_{\mathcal{H}}^2}{x^{2m-2j-1}} = 0, \quad j = 0,1,\dots,m-1. \tag{2.8.25}$$

With these preparations at hand, we can now formulate the vector-valued extension of Birman's sequence of Hardy–Rellich-type inequality:

Theorem 2.8.1. *For $0 \neq f \in H_L^m([0,\infty);\mathcal{H})$, one has*

$$\int_0^\infty dx\, \big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 > \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^\infty dx\, \frac{\|f(x)\|_{\mathcal{H}}^2}{x^{2m}}, \quad m \in \mathbb{N}. \tag{2.8.26}$$

For the proof of Theorem 2.8.1 one can now follow either of the two proofs given in Section 2.4. Optimality of the constants is clear from the special case $\mathcal{H} = \mathbb{C}$.

The special case $m = 1$, that is, the $\mathcal{H}$-valued Hardy inequality (in fact, a weighted version of the latter) appeared in [88, p. 4.50].

It remains to discuss the finite interval case $(0,b)$, $b \in (0,\infty)$. We start by introducing the set[3] (with $m \in \mathbb{N}$),

$$\begin{split} H_{LR}^m([0,b];\mathcal{H}) := \big\{f : [0,b] \to \mathcal{H} \,\big|\, f^{(m)} \in L^2((0,b);\mathcal{H});\ f^{(j)} \in AC([0,b];\mathcal{H}); \\ f^{(j)}(0) = f^{(j)}(b) = 0,\ j = 0,1,\dots,m-1\big\}, \end{split} \tag{2.8.27}$$

[3] Again, as noted at the beginning of Section 2.7, one could introduce analogous spaces with the boundary conditions at $x = 0$ and/or $x = b$ replaced by L^2-conditions for $\|f(\,\cdot\,)\|_{\mathcal{H}}/x^m$ and $\|f(\,\cdot\,)\|_{\mathcal{H}}/(b-x)^n$, respectively, but refrain from doing so at this point.

and the standard $\mathcal{H}$-valued Sobolev spaces,

$$H^m((0,b);\mathcal{H}) = \big\{f : [0,b] \to \mathcal{H} \,\big|\, f^{(j)} \in AC([0,b];\mathcal{H}),\ j = 0,1,\dots,m-1; \\ f^{(k)} \in L^2((0,b);\mathcal{H}),\ k = 0,1,\dots,m\big\}, \tag{2.8.28}$$

$$H_0^m((0,b);\mathcal{H}) = \big\{f \in H^m((0,b);\mathcal{H}) \,\big|\, f^{(j)}(0) = f^{(j)}(b) = 0,\ j = 0,1,\dots,m-1\big\}. \tag{2.8.29}$$

Next, we derive an elementary version of an $\mathcal{H}$-valued Friedrichs inequality as follows: Suppose $f \in H_1([0,b];\mathcal{H})$, then

$$f(x) = \int_0^x dt\, f'(t), \quad x \in [0,b],\ f(0) = 0, \tag{2.8.30}$$

implies

$$\|f(x)\|_{\mathcal{H}} \leqslant \int_0^x dt\, \|f'(t)\|_{\mathcal{H}} \leqslant x^{1/2}\bigg(\int_0^x dt\, \|f'(t)\|_{\mathcal{H}}^2\bigg)^{1/2} \leqslant b^{1/2}\|f'\|_{L^2((0,b);\mathcal{H})}, \tag{2.8.31}$$

and hence $\|f(\,\cdot\,)\|_{\mathcal{H}} \in L^2((0,b))$. Thus, squaring and then integrating (2.8.31) with respect to x from 0 to b yields

$$\|f\|_{L^2((0,b);\mathcal{H})} \leqslant b\|f'\|_{L^2((0,b);\mathcal{H})}, \quad f \in H_1([0,b];\mathcal{H}). \tag{2.8.32}$$

Consequently,

$$H_1([0,b];\mathcal{H}) = H_0^1((0,b);\mathcal{H}), \tag{2.8.33}$$

and iterating this process finally yields

$$H_{LR}^m([0,b];\mathcal{H}) = H_0^m((0,b);\mathcal{H}), \quad m \in \mathbb{N}. \tag{2.8.34}$$

Theorem 2.8.2. *Let $m \in \mathbb{N}$, $b \in (0,\infty)$, and define $H_{LR}^m([0,b];\mathcal{H})$ and $H_0^m((0,b);\mathcal{H})$ as above. Then the following items (i)–(iii) hold:*

(i) For each $m \in \mathbb{N}$,

$$H_{LR}^m([0,b];\mathcal{H}) = H_0^m((0,b);\mathcal{H}) \tag{2.8.35}$$

as sets. In particular,

$$f \in H^m_{LR}([0,b];\mathcal{H}) \ \textit{implies} \ f^{(j)} \in L^2((0,b);\mathcal{H}), \quad j = 0,1,\dots,m. \tag{2.8.36}$$

In addition, the norms in $H^m_{LR}([0,b];\mathcal{H})$ *(cf.* (2.8.13)*) and* $H^m_0((0,b);\mathcal{H})$ *are equivalent.*

(ii) Recalling $\delta(x) = \min\{x, |b-x|\}$, $x \in (0,b)$, *one has*

$$\int_0^b dx \, \big\|f^{(m)}(x)\big\|^2_{\mathcal{H}} > \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^b dx \, \frac{\|f(x)\|^2_{\mathcal{H}}}{\delta(x)^{2m}}, \quad f \in H^m_0((0,b);\mathcal{H})\backslash\{0\}. \tag{2.8.37}$$

(iii) The constant $[(2m-1)!!]^2/2^{2m}$ *is sharp.*

One can follow the special scalar case treated in the proof of Theorem 2.7.1 line by line.

CHAPTER THREE

On Weighted Hardy-Type Inequalities

The content of this chapter relies on (but is not identical to) the paper published as: C. Y. Chuah, F. Gesztesy, L. Littlejohn, T. Mei, I. Michael, and M. M. H. Pang, *On Weighted Hardy-Type Inequalities*, Math. Ineq. & App. (to appear).

3.1 Introduction

To put the results derived in this paper into some perspective, we very briefly recall some of the history of Hardy's celebrated inequality. We will exclusively focus on the continuous case even though Hardy originally started to investigate the discrete case (i.e., sums instead of integrals).

Hardy's inequality, in its primordial version, is of the form

$$\int_0^\infty dx\, |f'(x)|^2 \geqslant 4^{-1} \int_0^\infty dx\, x^{-2}|f(x)|^2, \quad f \in C_0^\infty((0,\infty)), \tag{3.1.1}$$

with the constant 4^{-1} being optimal and the inequality being a strict one for $f \neq 0$. (This extends to all $f \in AC([0,R])$ for all $R > 0$, $f' \in L^2((0,\infty); dx)$, with $f(0_+) = 0$, but we will not dwell on this improvement right now.) Hardy's work on his celebrated inequality started in 1915, [67], see also [68]– [70], and the historical comments in [85, Chs. 1, 3, App.]. Soon afterwards, Hardy also proved a weighted Hardy inequality (with power weights) of the form (cf. [66], [71, Sect. 9.8]),

$$\begin{gathered}\int_0^\infty dx\, x^\alpha |f'(x)|^p \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^\infty dx\, x^{\alpha-p}|f(x)|^p, \\ p \in [1,\infty),\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0,\infty)).\end{gathered} \tag{3.1.2}$$

Again, the constant $(|\alpha - p + 1|/p)^p$ is optimal and the inequality is strict for $f \neq 0$.

Equation (3.1.2) represents just the tip of an iceberg of weighted inequalities of Hardy-type. More generally, modern treatments of this subject are devoted to weighted inequalities of the form

$$\left(\int_a^b dx\, v(x)|f'(x)|^p\right)^{1/p} \geqslant C_{p,q}\left(\int_a^b dx\, w(x)|f(x)|^q\right)^{1/q}, \quad f \in C_0^\infty((a,b)), \tag{3.1.3}$$

for appropriate $a, b \in \mathbb{R} \cup \{\pm\infty\}$, $a < b$, $p, q \in [1,\infty) \cup \{\infty\}$, and appropriate weight functions $0 \leqslant v, w \in L^1_{loc}((a,b); dx)$. Again, this extends to certain optimal spaces for f, far beyond $f \in C_0^\infty((a,b))$. We refer to [85, Chs. 2–5], [87, Chs. 1,2], [101, Ch. 1], and the extensive literature cited therein. In particular, we mention the following integral versions of the two-weighted Hardy-type inequality (3.1.3) (the former is sometimes referred to as the differential version),

$$\begin{gathered}\left(\int_a^b dx\, v(x)|F(x)|^p\right)^{1/p} \geqslant C_{p,q}\left(\int_a^b dx\, w(x)\left|\int_a^x dx'\, F(x')\right|^q\right)^{1/q}, \\ F \in C_0^\infty((a,b)),\end{gathered} \tag{3.1.4}$$

and its companion (or "dual") version

$$\begin{gathered}\left(\int_a^b dx\, v(x)|F(x)|^p\right)^{1/p} \geqslant C_{p,q}\left(\int_a^b dx\, w(x)\left|\int_x^b dx'\, F(x')\right|^q\right)^{1/q}, \\ F \in C_0^\infty((a,b)).\end{gathered} \tag{3.1.5}$$

We note that many authors make the additional assumption $F \geqslant 0$ in (3.1.4), (3.1.5).

Before describing the results obtained in this paper in some detail, we pause for a moment to introduce our notation: We start by briefly summarizing essentials on Bochner integrability and associated vector-valued L^p-spaces. Regarding details of the Bochner integral we refer, for instance, to [9, p. 6–21], [26, Ch. 1], [42, p. 44–50], [76, p. 71–86], [88, Sect. 4.2], [93, Ch. III], [119, Sect. V.5]. In particular, if $p \geqslant 1$, $(a,b) \subseteq \mathbb{R}$ is a finite or infinite interval, $0 \leqslant w \in L^1_{loc}((a,b); dx)$ is a weight function, and $\mathcal{B}$ a Banach space, the symbol $L^p((a,b); wdx; \mathcal{B})$ denotes the set of equivalence classes of strongly measurable $\mathcal{B}$-valued functions which differ at most on sets of Lebesgue measure zero, such that $\|f(\,\cdot\,)\|_{\mathcal{B}}^p \in L^1((a,b); wdx)$. The corresponding norm in $L^p((a,b); wdx; \mathcal{B})$ is given by

$$\|f\|_{L^p((a,b);wdx;\mathcal{B})} = \left(\int_{(a,b)} w(x)dx\, \|f(x)\|_{\mathcal{B}}^p\right)^{1/p} \tag{3.1.6}$$

and $L^p((a,b);wdx;\mathcal{B})$ is a Banach space. If $\mathcal{H}$ is a separable Hilbert space, then so is $L^2((a,b);wdx;\mathcal{H})$ (see, e.g., [18, Subsects. 3.3.1, 3.3.2], [20, Sect. 7.1]).

One recalls that by a result of Pettis [103], if $\mathcal{B}$ is separable, weak measurability of $\mathcal{B}$-valued functions implies their strong measurability.

A map $f : [c,d] \to \mathcal{B}$ (with $[c,d] \subset (a,b)$) is called *absolutely continuous on* $[c,d]$, denoted by $f \in AC([c,d];\mathcal{B})$, if

$$f(x) = f(x_0) + \int_{x_0}^{x} dt\, g(t), \quad x_0, x \in [c,d], \tag{3.1.7}$$

for some $g \in L^1((c,d);dx;\mathcal{B})$. In particular, f is then strongly differentiable a.e. on (c,d) and

$$f'(x) = g(x) \text{ for a.e. } x \in (c,d). \tag{3.1.8}$$

Similarly, $f : [c,d] \to \mathcal{B}$ is called *locally absolutely continuous*, denoted by $f \in AC_{loc}([c,d];\mathcal{B})$, if $f \in AC([c',d'];\mathcal{B})$ on any closed subinterval $[c',d'] \subset (c,d)$.

In the special case $\mathcal{B} = \mathbb{C}$, we omit $\mathbb{C}$ and just write $L^p((a,b);wdx)$, respectively, $L^p_{loc}((a,b);wdx)$, as usual.

For $p \in [1,\infty)$, its Hölder conjugate index p' is given in a standard manner by $p' = p/(p-1) \in (1,\infty) \cup \{\infty\}$.

If $\mathcal{H}$ represents a complex, separable Hilbert space, then $\mathcal{B}(\mathcal{H})$ denotes the Banach space (the C^*-algebra) of bounded, linear operators defined on all of $\mathcal{H}$, and $\mathcal{B}_p(\mathcal{H})$ denote the ℓ^p-based Schatten–von Neumann trace ideals, $p \in [1,\infty)$, with $\operatorname{tr}_{\mathcal{H}}(T)$ abbreviating the trace of a trace class operator $T \in \mathcal{B}_1(\mathcal{H})$.

Finally, we are in a position to briefly describe the principal result of our paper in Section 3.2. Assume that $-\infty \leqslant a < b \leqslant \infty$, $p \in [1,\infty)$, and suppose that $0 \leqslant w_1 \in AC_{loc}((a,b))$, $0 \leqslant [-w_1']$ a.e. on (a,b), $0 \leqslant w_2 \in L^1_{loc}((a,b);dx)$, and $[-w_1']^{1-p} w_2^p \in L^1_{loc}((a,b);dx)$. If $F \in C_0((a,b);\mathcal{B})$, then we prove that

$$\begin{aligned} &\int_a^b dx\, w_1(x)^p [-w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p \\ &\quad \geqslant p^{-p} \int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p. \end{aligned} \tag{3.1.9}$$

Moreover, we prove the companion result with $\int_a^x dx' \ldots$ replaced by $\int_x^b dx' \ldots$. As an important special case of (3.1.9) one recovers the classical form of the power weighted Hardy inequality

$$\int_0^b dx\ x^\alpha\, \|F(x)\|_{\mathcal{B}}^p \geqslant \left(\frac{|\alpha+1-p|}{p}\right)^p \int_0^b dx\, x^{\alpha-p}\left(\int_0^x dx'\, \|F(x')\|_{\mathcal{B}}\right)^p, \tag{3.1.10}$$
$$0<b\leqslant\infty,\ p\in[1,\infty),\ \alpha<p-1.$$

As alluded to earlier, the constant $[(|\alpha-p+1|)/p]^p$ on the right-hand side of (3.1.10) is best possible, and equality holds if and only if $F=0$ a.e. on $(0,b)$. After describing appropriate iterations of (3.1.10) (again, including the companion results with $\int_a^x dx' \ldots$ replaced by $\int_x^b dx' \ldots$), we also recover as a special case the entire infinite sequence of the power weighted Birman–Hardy–Rellich-type inequalities (cf. [19, p. 48], [53], [60, pp. 83–84]) at the end of Section 3.2, namely,

$$\int_0^b dx\, x^\alpha |f^{(m)}(x)|^p \geqslant \prod_{j=1}^k \left(\frac{|\alpha-jp+1|}{p}\right)^p \int_0^b dx\, x^{\alpha-kp}|f^{(m-k)}(x)|^p, \tag{3.1.11}$$
$$0<b\leqslant\infty,\ p\in[1,\infty),\ 1\leqslant k\leqslant m,\ m\in\mathbb{N},\ \alpha\in\mathbb{R},\ f\in C_0^\infty((0,b)).$$

Replacing the restrictive hypothesis $F\in C_0((a,b);\mathcal{B})$ by the finiteness condition of the left-hand side in (3.1.9), and a detailed discussion of best possible constants in these inequalities are the principal subjects of Section 3.3.

Finally, in Section 3.4 we consider extensions of (3.1.10) and of the infinite sequence of Birman–Hardy–Rellich-type inequalities to the operator-valued context, extending some results of Hansen [64]. More specifically, assuming $F:(0,b)\to\mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $\|F(\,\cdot\,)\|_{\mathcal{B}_p(\mathcal{H})}\in L^p((0,b);x^\alpha dx)$, we derive the inequality

$$\operatorname{tr}_{\mathcal{H}}\left(\int_0^b dx\ x^\alpha\, |F(x)|^p\right) \geqslant \left(\frac{|\alpha-p+1|}{p}\right)^p \operatorname{tr}_{\mathcal{H}}\left(\int_0^b dx\, x^{\alpha-p}\left|\int_0^x dx'\, F(x')\right|^p\right),$$
$$0<b\leqslant\infty,\ p\in[1,\infty),\ \alpha<p-1. \tag{3.1.12}$$

Again, the constant $[(|\alpha-p+1|)/p]^p$ on the right-hand side of (3.1.12) is best possible, and equality holds if and only if $F=0$ a.e. on $(0,b)$.

Moreover, for $p \in [1,2]$, we remove the trace in inequality (3.1.12) as follows: Suppose that $F : (0,\infty) \to \mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $F(\,\cdot\,) \geqslant 0$ a.e. on $(0,\infty)$, and $\int_0^\infty dx\, x^\alpha F(x)^p \in \mathcal{B}(\mathcal{H})$, then we derive the operator-valued inequality

$$\begin{aligned}\int_0^\infty dx\; x^\alpha\, F(x)^p \geqslant \bigg(\frac{|\alpha-p+1|}{p}\bigg)^p \int_0^\infty dx\, x^{\alpha-p}\bigg(\int_0^x dx'\, F(x')\bigg)^p&, \\ p\in[1,2],\ \alpha<p-1&.\end{aligned} \tag{3.1.13}$$

Once again, the constant $[(|\alpha-p+1|)/p]^p$ on the right-hand sides of (3.1.13) is best possible, and equality holds if and only if $F=0$ a.e. on $(0,\infty)$. We also derive the corresponding companion results with $\int_a^x dx' \ldots$ replaced by $\int_x^b dx' \ldots$.

We emphasize that (3.1.12) and (3.1.13) with $\alpha = 0$ (and hence $p>1$) were proved by Hansen in [64].

3.2 Weighted Hardy-Type Inequalities Employing an Ad Hoc Approach

In this section we derive weighted Hardy inequalities employing an elementary ad hoc approach.

We begin by deriving a weighted Hardy inequality for $\mathcal{B}$-valued functions and hence make the following assumptions.

Hypothesis 3.2.1. *Let $-\infty \leqslant a < b \leqslant \infty$, $p \in [1,\infty)$, and $0 \leqslant w_2 \in L^1_{loc}((a,b);dx)$.*

(i) Suppose that $0 \leqslant w_1 \in AC_{loc}((a,b))$, $0 \leqslant [-w_1']$ a.e. on (a,b), $[-w_1']^{1-p}w_2^p \in L^1_{loc}((a,b);dx)$.

(ii) Suppose that $0 \leqslant w_1 \in AC_{loc}((a,b))$, $0 \leqslant w_1'$ a.e. on (a,b), $[w_1']^{1-p}w_2^p \in L^1_{loc}((a,b);dx)$.

The principal result of this section then reads as follows:

Theorem 3.2.2. *Let $p \in [1,\infty)$, and suppose that $F \in C_0((a,b);\mathcal{B})$.*

(*i*) *Assume Hypothesis* 3.2.1 (*i*), *then*

$$\begin{aligned}
&\int_a^b dx\, w_1(x)^p[-w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p \\
&\quad \geqslant p^{-p} \int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p.
\end{aligned} \tag{3.2.1}$$

(*ii*) *Assume Hypothesis* 3.2.1 (*ii*), *then*

$$\begin{aligned}
&\int_a^b dx\, w_1(x)^p[w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p \\
&\quad \geqslant p^{-p} \int_a^b dx\, w_1'(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p.
\end{aligned} \tag{3.2.2}$$

Proof. It suffices to prove (3.2.1) and then hint at the analogous proof of (3.2.2). Since

$$\begin{aligned}
&\frac{d}{dx}\bigg(w_1(x) \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \bigg) = w_1'(x) \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \\
&\quad + p w_1(x) \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^{p-1} w_2(x) \|F(x)\|_{\mathcal{B}},
\end{aligned} \tag{3.2.3}$$

one obtains

$$\begin{aligned}
&\int_a^b dx\, \frac{d}{dx}\bigg(w_1(x) \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \bigg) \\
&\quad = w_1(x) \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \bigg|_{x=a}^b \\
&\quad = w_1(b) \bigg(\int_a^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \\
&\quad = \int_a^b dx\, w_1'(x) \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \\
&\qquad + p \int_a^b dx\, w_1(x) \bigg(\int_a^x dx' w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^{p-1} w_2(x) \|F(x)\|_{\mathcal{B}}.
\end{aligned} \tag{3.2.4}$$

Here we used that by hypothesis, $0 \leqslant w_1$ is monotonically decreasing, and that the right-hand side of (3.2.4) exists employing $F \in C_0((a,b);\mathcal{B})$. Thus, with $p^{-1} + [p']^{-1} = 1$, an application of Hölder's inequality yields

$$w_1(b) \bigg(\int_a^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p + \int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p$$

$$= p \int_a^b dx\, w_1(x) \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^{p-1} w_2(x) \|F(x)\|_{\mathcal{B}}$$

$$\leqslant p \bigg[\int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \bigg]^{1/p'}$$

$$\times \bigg[\int_a^b dx\, w_1(x)^p [-w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p \bigg]^{1/p}. \tag{3.2.5}$$

In particular,

$$\int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p$$

$$\leqslant p \bigg[\int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x') \|F(x')\| \bigg)^p \bigg]^{1/p'}$$

$$\times \bigg[\int_a^b dx\, w_1(x)^p [-w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p \bigg]^{1/p}, \tag{3.2.6}$$

and hence

$$\begin{aligned} &\bigg[\int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \bigg]^{1/p} \\ &\quad \leqslant p \bigg[\int_a^b dx\, w_1(x)^p [-w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p \bigg]^{1/p}, \end{aligned} \tag{3.2.7}$$

completing the proof of item (i).

For the proof of item (ii) one notes the identity

$$\frac{d}{dx} \bigg(w_1(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \bigg) = w_1'(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p$$

$$- p w_1(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^{p-1} w_2(x) \|F(x)\|_{\mathcal{B}}, \tag{3.2.8}$$

and then obtains upon integrating (3.2.8) with respect to x from a to b,

$$w_1(a) \bigg(\int_a^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p + \int_a^b dx\, w_1'(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p$$

$$= p \int_a^b dx\, w_1(x) w_2(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^{p-1} \|F(x)\|_{\mathcal{B}}. \tag{3.2.9}$$

In particular,

$$\begin{aligned} &\int_a^b dx\, w_1'(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^p \\ &\quad \leqslant p \int_a^b dx\, w_1(x) w_2(x) \bigg(\int_x^b dx'\, w_2(x') \|F(x')\|_{\mathcal{B}} \bigg)^{p-1} \|F(x)\|_{\mathcal{B}}, \end{aligned} \tag{3.2.10}$$

and now one can repeat the Hölder inequality argument as in item (i). (Alternatively, if $b < \infty$, one can also prove item (ii) by the change of variable $x \mapsto a + (b - x)$, i.e., by reflecting the interval (a, b) at its midpoint). $\square$

We illustrate our general result with the following well-known special case, the power-weighted Hardy inequality. For pertinent references on inequalities (3.2.11), (3.2.12) below, we recall, for instance, [11, Theorem 1.2.1], [66], [71, p. 245–246], [85, p. 23, 43], [87, p. 9–11], [101, Lemma 1.3], and the references therein.

Example 3.2.3. *Let* $p \in [1, \infty)$, $a = 0$, $b \in (0, \infty) \cup \{\infty\}$, $w_2(x) = 1$, *and suppose that the map* $F : (0, b) \to \mathcal{B}$ *is weakly measurable satisfying* $\|F(\,\cdot\,)\|_{\mathcal{B}} \in L^p((0, b); x^\alpha dx)$.

(i) *If* $w_1(x) = |\alpha - p + 1|^{-1} x^{-|\alpha - p + 1|}$, $\alpha < p - 1$, *then* (3.2.1) *reduces to the classical form*

$$\int_0^b dx\ x^\alpha\,\|F(x)\|_{\mathcal{B}}^p \geqslant \bigg(\frac{|\alpha - p + 1|}{p}\bigg)^p \int_0^b dx\, x^{\alpha - p}\bigg(\int_0^x dx'\,\|F(x')\|_{\mathcal{B}}\bigg)^p. \qquad (3.2.11)$$

(ii) *If* $w_1(x) = [|\alpha - p + 1|]^{-1} x^{|\alpha - p + 1|}$, $\alpha > p - 1$, *then* (3.2.2) *reduces to the complementary classical form*

$$\int_0^b dx\ x^\alpha\,\|F(x)\|_{\mathcal{B}}^p \geqslant \bigg(\frac{|\alpha - p + 1|}{p}\bigg)^p \int_0^b dx\, x^{\alpha - p}\bigg(\int_x^b dx'\,\|F(x')\|_{\mathcal{B}}\bigg)^p. \qquad (3.2.12)$$

In both cases (i) *and* (ii), *the constant* $[(|\alpha - p + 1|)/p]^p$ *is best possible and equality holds if and only if* $F = 0$ *a.e. on* $(0, b)$.

The case $F \in C_0((0, b); \mathcal{B})$ in Example 3.2.3 is a corollary of Theorem 3.2.2 and optimality of the constants on the right-hand sides of (3.2.11), (3.2.12), and the fact that equality is only attained in the trivial case $F = 0$ a.e. on $(0, b)$, is a classical result (see, e.g., [11, Theorem 1.2.1]). The extension of Example 3.2.3 to the case $F \in L^p((0, b); x^\alpha dx; \mathcal{B})$, $p \in [1, \infty)$, follows along the lines in [101, Theorem 1.14, Sects. 1.3, 1.5]. We will briefly return to this issue after Theorem 3.3.4

Iterating the weighted Hardy inequality yields the sequence of vector-valued Birman inequalities as follows. Consider the iterated Hardy-type operators,

$$\begin{aligned}
(H_{-,1}F)(x) &= \int_a^x dt_1\, F(t_1), \\
(H_{-,m}F)(x) &= H_{-,1}\bigg(\int_a^{\bullet} dt_2 \cdots \int_a^{t_{m-1}} dt_m\, F(t_m)\bigg)(x) \\
&= \int_a^x dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{m-1}} dt_m\, F(t_m) \\
&= [(m-1)!]^{-1} \int_a^x dt\, (x-t)^{m-1} F(t), \quad m \in \mathbb{N},\ m \geqslant 2, \qquad (3.2.13) \\
&\qquad\qquad F \in L^p((a,c);dx) \ \text{ for all } c \in (a,b), \\
(H_{+,1}F)(x) &= \int_x^b dt_1\, F(t_1), \\
(H_{+,m}F)(x) &= H_{+,1}\bigg(\int_{\bullet}^b dt_2 \cdots \int_{t_{m-1}}^b dt_m\, F(t_m)\bigg)(x) \\
&= \int_x^b dt_1 \int_{t_1}^b dt_2 \cdots \int_{t_{m-1}}^b dt_m\, F(t_m) \\
&= [(m-1)!]^{-1} \int_x^b dt\, (x-t)^{m-1} F(t), \quad m \in \mathbb{N},\ m \geqslant 2, \qquad (3.2.14) \\
&\qquad\qquad F \in L^p((c,b);dx) \ \text{ for all } c \in (a,b).
\end{aligned}$$

Applying (3.2.11) and (3.2.12) iteratively in the form (with $a = 0$)

$$\begin{aligned}
&\int_0^b dx\, x^{\alpha-p}[(H_{\mp,1}(G_m(\,\cdot\,))(x)]^p \\
&\quad \leqslant \bigg(\frac{p}{|\alpha-p+1|}\bigg)^p \int_0^b dx\, x^{\alpha} G_m(x)^p, \quad p \in [1,\infty),\ \alpha \lesseqgtr p-1,
\end{aligned} \qquad (3.2.15)$$

for appropriate $0 \leqslant G_m \in L^p((0,b);x^{\alpha}dx)$, $p \in [1,\infty)$, then yields for $F : (0,\infty) \to \mathcal{B}$ a weakly measurable map satisfying $\|F(\,\cdot\,)\|_{\mathcal{B}} \in L^p((0,b);x^{\alpha}dx)$

$$\begin{aligned}
&\int_0^b dx\, x^{\alpha}\|F(x)\|_{\mathcal{B}}^p \\
&\quad \geqslant \prod_{k=1}^{m}\bigg(\frac{|\alpha-kp+1|}{p}\bigg)^p \int_0^b dx\, x^{\alpha-mp}[(H_{\mp,m}\|F(\,\cdot\,)\|_{\mathcal{B}})(x)]^p, \qquad (3.2.16) \\
&\qquad\qquad 0 < b \leqslant \infty,\ p \in [1,\infty),\ \alpha \lesseqgtr \begin{cases} p-1, \\ mp-1, \end{cases} \quad m \in \mathbb{N}.
\end{aligned}$$

It is well-known that the constants in (3.2.11), (3.2.12) and (3.2.16) are all optimal and that, in fact, these inequalities are all strict unless $F = 0$ on $(0, b)$.

Turning to the differential form of the iterated (integral) Hardy inequalities (3.2.16), and adding appropriate boundary conditions for F at both endpoints a, b, permits one to avoid the gap $(p-1, mp-1)$ for α in (3.2.16) as follows: Assuming $F \in C_0^\infty((a,b);\mathcal{B})$ for simplicity, and introducing

$$\begin{aligned} \widetilde{f}(x) &= \int_0^x dx' \|F(x')\|_{\mathcal{B}},\ x \in (0,b), \quad \widetilde{f}^{(m)}(a) = 0,\ m \in \mathbb{N}, \\ \widetilde{g}(x) &= \int_x^b dx' \|F(x')\|_{\mathcal{B}},\ x \in (0,b), \quad \widetilde{g}^{(n)}(b) = 0,\ m \in \mathbb{N}, \end{aligned} \tag{3.2.17}$$

inequalities (3.2.11) and (3.2.12) become

$$\int_0^b dx\, x^\alpha \big[\widetilde{f}'(x)\big]^p \geqslant \bigg(\frac{|\alpha - p + 1|}{p}\bigg)^p \int_0^b dx\, x^{\alpha - p} \widetilde{f}(x)^p, \quad \alpha < p - 1, \tag{3.2.18}$$

$$\int_0^b dx\, x^\alpha \big[-\widetilde{g}'(x)\big]^p \geqslant \bigg(\frac{|\alpha - p + 1|}{p}\bigg)^p \int_0^b dx\, x^{\alpha - p} \widetilde{g}(x)^p, \quad \alpha > p - 1. \tag{3.2.19}$$

As a special case one obtains

$$\begin{gathered} \int_0^b dx\, x^\alpha |f'(x)|^p \geqslant \bigg(\frac{|\alpha - p + 1|}{p}\bigg)^p \int_0^b dx\, x^{\alpha - p} |f(x)|^p, \\ 0 < b \leqslant \infty,\ p \in [1,\infty),\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0,b)). \end{gathered} \tag{3.2.20}$$

Iterating (3.2.20) yields the well-known result

$$\begin{gathered} \int_0^b dx\, x^\alpha |f^{(m)}(x)|^p \geqslant \prod_{j=1}^k \bigg(\frac{|\alpha - jp + 1|}{p}\bigg)^p \int_0^b dx\, x^{\alpha - kp} |f^{(m-k)}(x)|^p, \\ 0 < b \leqslant \infty,\ p \in [1,\infty),\ 1 \leqslant k \leqslant m,\ m \in \mathbb{N},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0,b)). \end{gathered} \tag{3.2.21}$$

For additional results on higher-order (overdetermined) Hardy-type inequalities see also [87, Ch. 4], [97], [98], [99].

3.3 More on Weighted Hardy-Type Inequalities

To remove the assumption $F \in C_0((0,b);\mathcal{B})$ in Theorem 3.2.2 and to take a closer look at the issue of best possible constants in the inequality, we next recall (a generalization of) a celebrated 1969 result due to Talenti [113], Tomaselli [116], and shortly afterwards by Chisholm and Everitt [27] and Muckenhoupt [95], followed by Chisholm, Everitt, and Littlejohn [28]. For exhaustive textbook presentations we refer, for instance, to [11, Sect. 1.2], [32, Sect. 5.3], [44, Sect. 2.2], [71, Sects. 9.8, 9.9], [85, Chs. 3, 4], [87, Chs. 1, 3, 4], [101, Sects. 1.1–1.3, 1.5, 1.6, 1.10].

In addition to $H_{\mp,1}$ in (3.2.13), (3.2.14), we now also introduce the generalized (weighted) Hardy operators as follows.

Hypothesis 3.3.1. *Let* $-\infty \leqslant a < b \leqslant \infty$ *and* $p \in [1, \infty)$.

(i) *Assume that* v *and* w *are weight functions satisfying* v, w *measurable on* (a, b), $v > 0$, $w > 0$ *a.e. on* (a, b).

(ii) *Suppose that* $\phi_\mp, \psi_\mp$ *satisfy for all* $c \in (a, b)$,

$$\begin{aligned} &0 < \phi_\mp \text{ a.e. on } (a,b),\ 0 < \psi_\mp \text{ a.e. on } (a,b), \\ &\phi_- \in L^p((c,b); v dx), \quad \psi_- \in L^{p'}\big((a,c); w^{-p'/p} dx\big), \\ &\phi_+ \in L^p((a,c); v dx), \quad \psi_+ \in L^{p'}\big((c,b); w^{-p'/p} dx\big). \end{aligned} \tag{3.3.1}$$

Given Hypothesis 3.3.1 we introduce

$$(H_{-,\phi_-,\psi_-} F)(x) = \phi_-(x) \int_a^x dx'\, \psi_-(x') F(x'), \quad x \in (a,b), \tag{3.3.2}$$
$$F \in L^p((a,c); w dx) \text{ for all } c \in (a,b),$$
$$(H_{+,\phi_+,\psi_+} F)(x) = \phi_+(x) \int_x^b dx'\, \psi_+(x') F(x'), \quad x \in (a,b), \tag{3.3.3}$$
$$F \in L^p((c,b); w dx) \text{ for all } c \in (a,b).$$

In particular, $H_{\mp,1,1} = H_{\mp,1}$.

The following result, Theorem 3.3.2, is well-known and a special case of more general situations recorded in the literature. For instance, we refer to [22], [61], [85, p. 38–40], [87, Theorem 2.3] (after specializing to the case $\varphi_1 = \psi_1 = 1$), and [101, Theorem 1.14, Lemma 5.4 in Ch. 1] (choosing $q = p$ in their results).

Theorem 3.3.2. *Assume Hypothesis* 3.3.1 (i).

(i) *There exists a constant* $C_- \in (0,\infty)$ *such that*

$$C_-\bigg(\int_a^b dx\, w(x)F(x)^p\bigg)^{1/p} \geqslant \bigg(\int_a^b dx\, v(x)[(H_{-,1}F)(x)]^p\bigg)^{1/p}, \tag{3.3.4}$$

for all F *measurable on* (a,b) *and* $F \geqslant 0$ *a.e. on* (a,b), *if and only if*

$$A_- := \sup_{c\in(a,b)} \bigg(\int_c^b dx\, v(x)\bigg)^{1/p}\bigg(\int_a^c dx\, w(x)^{-p'/p}\bigg)^{1/p'} < \infty. \tag{3.3.5}$$

(*If* $p=1$ *and hence* $p'=\infty$, *the second factor in the right-hand side of* (3.3.5) *is interpreted as* $\|1/w\|_{L^\infty((a,c);dx)}$.) *Moreover, the smallest constant* $C_{0,-} \in (0,\infty)$ *in* (3.3.4) *satisfies*

$$\begin{aligned} &A_- \leqslant C_{0,-} \leqslant p^{1/p}(p')^{1/p'}A_-, \quad p\in(1,\infty),\\ &C_{0,-} = A_-, \quad p=1. \end{aligned} \tag{3.3.6}$$

(ii) *There exists a constant* $C_+ \in (0,\infty)$ *such that*

$$C_+\bigg(\int_a^b dx\, w(x)F(x)^p\bigg)^{1/p} \geqslant \bigg(\int_a^b dx\, v(x)[(H_{+,1}F)(x)]^p\bigg)^{1/p}, \tag{3.3.7}$$

for all F *measurable on* (a,b) *and* $F \geqslant 0$ *a.e. on* (a,b), *if and only if*

$$A_+ := \sup_{c\in(a,b)} \bigg(\int_a^c dx\, v(x)\bigg)^{1/p}\bigg(\int_c^b dx\, w(x)^{-p'/p}\bigg)^{1/p'} < \infty. \tag{3.3.8}$$

(*If* $p=1$ *and hence* $p'=\infty$, *the second factor in the right-hand side of* (3.3.8) *is interpreted as* $\|1/w\|_{L^\infty((c,b);dx)}$.) *Moreover, the smallest constant* $C_{0,+} \in (0,\infty)$ *in* (3.3.7) *satisfies*

$$\begin{aligned} &A_+ \leqslant C_{0,+} \leqslant p^{1/p}(p')^{1/p'}A_+, \quad p\in(1,\infty),\\ &C_{0,+} = A_+ \quad p=1. \end{aligned} \tag{3.3.9}$$

We emphasize that items (i) and (ii) in Theorem 3.3.2 do not exclude the trivial case where the left-hand sides of (3.3.4) and (3.3.7) are infinite.

We also note that Theorem 3.3.2 naturally extends to $p=\infty$, but as we will not use this in this note we omit further details (cf. [101, Sect. 1.5]). Moreover, [101, Sects. 1.3, 1.5] actually discuss the more general case with p replaced by $q \in [1,\infty) \cup \{\infty\}$ on the right-hand sides of (3.3.5), (3.3.8).

To extend the considerations in Theorem 3.3.2 to the case where $H_{\mp,1}$ is replaced by the weighted Hardy operator $H_{\mp,\phi_\mp,\psi_\mp}$ one recalls the following elementary fact, still assuming $F \geqslant 0$ a.e. on (a,b).

$$\begin{aligned}
\|H_{-,\phi_-,\psi_-}F\|_{L^p((a,b);vdx)} &= \|H_{-,1}(\psi_- F)\|_{L^p((a,b);v\phi_-^p dx)} \\
&= \bigg(\int_a^b dx\, v(x)\phi_-(x)^p \bigg|\int_a^x dx'\, \psi_-(x')F(x')\bigg|^p\bigg)^{1/p} \\
&\leqslant \widetilde{C}_- \bigg(\int_a^b dx\, w(x)\psi_-(x)^{-p}[\psi_-(x)F(x)]^p\bigg)^{1/p} \\
&= \widetilde{C}_- \|F\|_{L^p((a,b);wdx)} \\
&= \widetilde{C}_- \|\psi_- F\|_{L^p((a,b);w\psi_-^{-p}dx)},
\end{aligned} \tag{3.3.10}$$

as well as

$$\begin{aligned}
\|H_{+,\phi_+,\psi_+}F\|_{L^p((a,b);vdx)} &= \|H_{+,1}(\psi_+ F)\|_{L^p((a,b);v\phi_+^p dx)} \\
&= \bigg(\int_a^b dx\, v(x)\phi_+(x)^p \bigg|\int_x^b dx'\, \psi_+(x')F(x')\bigg|^p\bigg)^{1/p} \\
&\leqslant \widetilde{C}_+ \bigg(\int_a^b dx\, w(x)\psi_+(x)^{-p}[\psi_+(x)F(x)]^p\bigg)^{1/p} \\
&= \widetilde{C}_+ \|F\|_{L^p((a,b);wdx)} \\
&= \widetilde{C}_+ \|\psi_+ F\|_{L^p((a,b);w\psi_+^{-p}dx)},
\end{aligned} \tag{3.3.11}$$

are equivalent to

$$\big\|H_{-,1}\widetilde{F}_-\big\|_{L^p((a,b);v\phi_-^p dx)} \leqslant \widetilde{C}_- \big\|\widetilde{F}_-\big\|_{L^p((a,b);w\psi_-^{-p}dx)}, \tag{3.3.12}$$

$$\big\|H_{+,1}\widetilde{F}_+\big\|_{L^p((a,b);v\phi_+^p dx)} \leqslant \widetilde{C}_- \big\|\widetilde{F}_+\big\|_{L^p((a,b);w\psi_+^{-p}dx)}, \tag{3.3.13}$$

upon identifying $\widetilde{F}_\mp = \psi_\mp F \geqslant 0$ and replacing the original weights v and w by $\widetilde{v} = v\phi_\mp^p$ and $\widetilde{w} = w\psi_\mp^{-p}$, respectively.

Thus, one obtains the following consequence of Theorem 3.3.2, (3.3.10)–(3.3.13) (see also [87, Theorem 2.3]):

Corollary 3.3.3. *Assume Hypothesis* 3.3.1.

(*i*) *There exists a constant* $\widetilde{C}_- \in (0,\infty)$ *such that*

$$\big(\widetilde{C}_-\big)^p \int_a^b dx\, w(x) G(x)^p \geqslant \int_a^b dx\, v(x) [(H_{-,\phi_-,\psi_-} G)(x)]^p, \tag{3.3.14}$$

for all G *measurable on* (a,b) *and* $G \geqslant 0$ *a.e. on* (a,b), *if and only if*

$$\widetilde{A}_- := \sup_{c\in(a,b)} \bigg(\int_c^b dx\, v(x)\phi_-(x)^p\bigg)^{1/p} \bigg(\int_a^c dx\, w(x)^{-p'/p}\psi_-(x)^{p'}\bigg)^{1/p'} < \infty. \tag{3.3.15}$$

(*If* $p = 1$ *and hence* $p' = \infty$, *the second factor in the right-hand side of* (3.3.15) *is interpreted as* $\|\psi_-/w\|_{L^\infty((a,c);dx)}$.) *Moreover, the smallest constant* $\widetilde{C}_{0,-} \in (0,\infty)$ *in* (3.3.14) *satisfies*

$$\begin{aligned} &\widetilde{A}_- \leqslant \widetilde{C}_{0,-} \leqslant p^{1/p}(p')^{1/p'}\widetilde{A}_-, \quad p\in(1,\infty), \\ &\widetilde{C}_{0,-} = \widetilde{A}_-, \quad p = 1. \end{aligned} \tag{3.3.16}$$

(*ii*) *There exists a constant* $\widetilde{C}_+ \in (0,\infty)$ *such that*

$$\big(\widetilde{C}_+\big)^p \int_a^b dx\, w(x) G(x)^p \geqslant \int_a^b dx\, v(x) [(H_{+,\phi_+,\psi_+} G)(x)]^p, \tag{3.3.17}$$

for all measurable G *on* (a,b) *and* $G \geqslant 0$ *a.e. on* (a,b), *if and only if*

$$\widetilde{A}_+ := \sup_{c\in(a,b)} \bigg(\int_a^c dx\, v(x)\phi_+(x)^p\bigg)^{1/p} \bigg(\int_c^b dx\, w(x)^{-p'/p}\psi_+(x)^{p'}\bigg)^{1/p'} < \infty. \tag{3.3.18}$$

(*If* $p = 1$ *and hence* $p' = \infty$, *the second factor in the right-hand side of* (3.3.18) *is interpreted as* $\|\psi_+/w\|_{L^\infty((c,b);dx)}$.) *Moreover, the smallest constant* $\widetilde{C}_{0,+} \in (0,\infty)$ *in* (3.3.17) *satisfies*

$$\begin{aligned} &\widetilde{A}_+ \leqslant \widetilde{C}_{0,+} \leqslant p^{1/p}(p')^{1/p'}\widetilde{A}_+, \quad p\in(1,\infty), \\ &\widetilde{C}_{0,+} = \widetilde{A}_+, \quad p = 1. \end{aligned} \tag{3.3.19}$$

An application of Corollary 3.3.3 then permits one to remove the hypothesis $F \in C_0((0,b);\mathcal{B})$ in Theorem 3.2.2, Example 3.2.3, and (3.2.16) as follows.

Theorem 3.3.4. *Let* $p \in [1,\infty)$.

(i) *In addition to Hypothesis* 3.2.1 (i), *assume that* $w_j > 0$ *a.e. on* (a,b), $j = 1,2$, *and that*

$$F \in L^p\big((a,b); w_1^p[-w_1']^{1-p} w_2^p\, dx; \mathcal{B}\big). \tag{3.3.20}$$

Then

$$\begin{aligned}
&\int_a^b dx\, w_1(x)^p[-w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p \\
&\quad \geqslant p^{-p} \int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x')\|F(x')\|_{\mathcal{B}}\bigg)^p.
\end{aligned} \tag{3.3.21}$$

(ii) *In addition to Hypothesis* 3.2.1 (ii), *assume that* $w_j > 0$ *a.e. on* (a,b), $j = 1,2$, *and that*

$$F \in L^p\big((a,b); w_1^p[w_1']^{1-p} w_2^p\, dx; \mathcal{B}\big). \tag{3.3.22}$$

Then

$$\begin{aligned}
&\int_a^b dx\, w_1(x)^p[w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p \\
&\quad \geqslant p^{-p} \int_a^b dx\, w_1'(x) \bigg(\int_x^b dx'\, w_2(x')\|F(x')\|_{\mathcal{B}}\bigg)^p.
\end{aligned} \tag{3.3.23}$$

Proof. It suffices to consider item (i) as item (ii) is proved analogously. Identifying

$$G(\,\cdot\,) = \|F(\,\cdot\,)\|_{\mathcal{B}}, \quad w = w_1^p[-w_1']^{1-p}w_2^p, \quad v = [-w_1'], \quad \phi_- = 1, \quad \psi_- = w_2, \tag{3.3.24}$$

in Corollary 3.3.3 (i), the estimate (3.3.14) proves boundedness of the weighted Hardy operator

$$H_{-,1,w_2} \in \mathcal{B}\big(L^p\big((a,b); w_1^p[-w_1']^{1-p}w_2^p\, dx\big), L^p\big((a,b); [-w_1']\, dx\big)\big). \tag{3.3.25}$$

if and only if

$$\begin{aligned}
\widetilde{A}_- = \sup_{c\in(a,b)} \Bigg[&\bigg(\int_c^b dx\, [-w_1'(x)]\bigg)^{1/p} \\
&\times \bigg(\int_a^c dx\, \big\{w_1(x)^p\big[-w_1'(x)\big]^{1-p} w_2(x)^p\big\}^{-p'/p} w_2(x)^{p'}\bigg)^{1/p'}\Bigg]
\end{aligned}$$

$$= \sup_{c\in(a,b)} \left[\left(\int_c^b dx\, [-w_1'(x)] \right)^{1/p} \left(\int_a^c dx\, w_1(x)^{-p'} [-w_1'(x)] \right)^{1/p'} \right] < \infty, \quad (3.3.26)$$

employing $-p'/p = 1 - p'$, $-(1-p)p'/p = 1$, temporarily assuming $p \in (1,\infty)$. The constant $\widetilde{A}_-$ is easily estimated and one obtains

$$\begin{aligned}
\widetilde{A}_- &= \sup_{c\in(a,b)} \left[[w_1(c) - w_1(b)]^{1/p} \left[\frac{w_1(c)^{1-p'} - w_1(a)^{1-p'}}{p'-1} \right]^{1/p'} \right] \\
&\leqslant (p'-1)^{-1/p'} \sup_{c\in(a,b)} \left[w_1(c)^{(1/p)+[(1-p')/p']} \right] \\
&= (p'-1)^{-1/p'} = (p/p')^{1/p'} < \infty, \quad p \in (1,\infty). \qquad (3.3.27)
\end{aligned}$$

Thus, $\widetilde{C}_- \in (0,\infty)$ as in (3.3.14) exists, implying (3.3.25). Given the estimate (3.3.27), The smallest constant $\widetilde{C}_{0,-}$ as in (3.3.14), (3.3.16) satisfies

$$\widetilde{C}_{0,-} \leqslant p^{1/p}(p')^{1/p'} \widetilde{A}_- \leqslant p^{1/p}(p')^{1/p'}(p/p')^{1/p'} = p, \qquad (3.3.28)$$

proving the estimate (3.3.21).

In the case $p = 1$, $p' = \infty$, the analog of (3.3.26) becomes

$$\begin{aligned}
\widetilde{A}_- &= \sup_{c\in(a,b)} \left[\left(\int_c^b dx\, [-w_1'(x)] \right) \|1/w_1\|_{L^\infty((a,c);dx)} \right] \\
&= \sup_{c\in(a,b)} \left[[w_1(c) - w_1(b)] \|1/w_1\|_{L^\infty((a,c);dx)} \right] \\
&= \sup_{c\in(a,b)} \left[[w_1(c) - w_1(b)] w_1(c)^{-1} \right] \\
&= \sup_{c\in(a,b)} \left[1 - [w_1(b)/w_1(c)] \right] \\
&= \left[1 - [w_1(b)/w_1(a)] \right] \leqslant 1, \quad p = 1, \qquad (3.3.29)
\end{aligned}$$

and hence the fact $\widetilde{C}_{0,-} = \widetilde{A}_-$, according to (3.3.15), also yields (3.3.21) for $p = 1$. $\square$

In particular, we now removed the hypothesis $F \in C_0((0,b);\mathcal{B})$ in Theorem 3.2.2 and replaced it by (3.3.20), (3.3.22). Consequently, this illustrates that Example 3.2.3 and (3.2.16) now extend from $F \in C_0((0,b);\mathcal{B})$ to $F \in L^p((a,b); x^\alpha dx; \mathcal{B})$.

Due to the fundamental importance of the constants $\widetilde{A}_{\mp}$ in connection with smallest constants $\widetilde{C}_{0,\mp}$ in Hardy-type inequalities (as detailed in (3.3.16), (3.3.19)), we now take a second look at them.

Lemma 3.3.5. *Let $p \in [1,\infty)$.*

(i) Assume Hypothesis 3.2.1 *(i), then*

$$\widetilde{A}_{-} = \begin{cases} (p/p')^{1/p'}\Big[1 - \big[w_1(b)^{1/p}/w_1(a)^{1/p}\big]\Big], & p \in (1,\infty), \\ \big[1 - [w_1(b)/w_1(a)]\big], & p = 1. \end{cases} \tag{3.3.30}$$

In particular, if $w_1(b) = 0$, or $1/w_1(a) = 0$, then

$$\widetilde{A}_{-} = \begin{cases} (p/p')^{1/p'}, & p \in (1,\infty), \\ 1, & p = 1. \end{cases} \tag{3.3.31}$$

(ii) Assume Hypothesis 3.2.1 *(ii), then*

$$\widetilde{A}_{+} = \begin{cases} (p/p')^{1/p'}\Big[1 - \big[w_1(a)^{1/p}/w_1(b)^{1/p}\big]\Big], & p \in (1,\infty), \\ \big[1 - [w_1(a)/w_1(b)]\big], & p = 1. \end{cases} \tag{3.3.32}$$

In particular, if $w_1(a) = 0$, or $1/w_1(b) = 0$, then

$$\widetilde{A}_{+} = \begin{cases} (p/p')^{1/p'}, & p \in (1,\infty), \\ 1, & p = 1. \end{cases} \tag{3.3.33}$$

Proof. Again, we prove item (i) only. Starting with the case $p \in (1,\infty)$, we first prove (3.3.31) directly (even though that is not necessary). Suppose that $w_1(b) = 0$, then

$$\begin{aligned} \widetilde{A}_{-} &= \sup_{c\in(a,b)} w_1(c)^{p'/(pp')}\left[\frac{w_1(c)^{1-p'} - w_1(a)^{1-p'}}{p'-1}\right]^{1/p'} \\ &= (p'-1)^{-1/p'} \sup_{c\in(a,b)} \left[1 - \frac{w_1(c)^{p'-1}}{w_1(a)^{p'-1}}\right]^{1/p'} \end{aligned}$$

$$= (p'-1)^{-1/p'} = (p/p')^{1/p'}, \tag{3.3.34}$$

as the supremum is attained for $c=b$. Similarly, if $1/w_1(a)=0$, then

$$\begin{aligned}
\widetilde{A}_- &= \sup_{c\in(a,b)} [w_1(c)-w_1(b)]^{1/p} \left[\frac{w_1(c)^{1-p'}}{p'-1}\right]^{1/p'} \\
&= (p'-1)^{-1/p'} \sup_{c\in(a,b)} \left[1-\frac{w_1(b)}{w_1(c)}\right]^{1/p} \\
&= (p'-1)^{-1/p'} = (p/p')^{1/p'}, \qquad (3.3.35)
\end{aligned}$$

as the supremum is attained for $c=a$. To deal with the general case (3.3.30) (which of course, directly yields (3.3.34), (3.3.35)) one can proceed as follows.

$$\begin{aligned}
\widetilde{A}_- &= \sup_{c\in(a,b)} \left\{ [w_1(c)-w_1(b)]^{1/p} \left[\frac{w_1(c)^{1-p'}-w_1(a)^{1-p'}}{p'-1}\right]^{1/p'} \right\} \\
&= (p'-1)^{-1/p'} \sup_{c\in(a,b)} \left\{ \left[1-\frac{w_1(b)}{w_1(c)}\right]^{1/p} \left[1-\frac{w_1(c)^{p'-1}}{w_1(a)^{p'-1}}\right]^{1/p'} \right\}. \qquad (3.3.36)
\end{aligned}$$

To maximize the right-hand side of (3.3.36), we introduce the absolutely continuous function

$$\eta(c) := \left[1-\frac{w_1(b)}{w_1(c)}\right]^{1/p} \left[1-\frac{w_1(c)^{p'-1}}{w_1(a)^{p'-1}}\right]^{1/p'}, \quad c\in(a,b), \tag{3.3.37}$$

and note that $\eta'(c)=0$ is equivalent to

$$w_1(c) = w_1(a)^{1/p} w_1(b)^{1/p'}. \tag{3.3.38}$$

Relation (3.3.38) yields a maximum of η on (a,b) ($c\in\{a,b\}$ being excluded as a maximum since $\eta(a)=\eta(b)=0$ if $p\in(1,\infty)$) and insertion of (3.3.38) into the right-hand side of (3.3.36) then yields (3.3.30) for $p\in(1,\infty)$.

The case $p=1$ (and hence, $p'=\infty$) follows from

$$\begin{aligned}
\widetilde{A}_- &= \sup_{c\in(a,b)} \left\{ [w_1(c)-w_1(b)] \|1/w_1\|_{L^\infty((a,c);dx)} \right\} \\
&= \sup_{c\in(a,b)} \left\{ [w_1(c)-w_1(b)]/w_1(c) \right\}
\end{aligned}$$

$$= \left[1 - \frac{w_1(b)}{w_1(a)}\right], \tag{3.3.39}$$

as the supremum is attained at $c = a$. $\square$

Remark 3.3.6. (i) One observes that the first lines on the right-hand sides of (3.3.30)–(3.3.33) indeed converge to the second lines on the right-hand sides of (3.3.30)–(3.3.33) as $p \downarrow 1$ and $p' \uparrow \infty$.

(ii) If $w_1(b) \neq 0$ and $1/w_1(a) \neq 0$ (resp., if $w_1(a) \neq 0$ and $1/w_1(b) \neq 0$), then (3.3.30) (resp., (3.3.32)) proves in conjunction with (3.3.16) (resp., (3.3.19)) that inequality (3.3.21) (resp., (3.3.23)), and hence our ad hoc inequality (3.2.1) (resp., (3.2.2)) is not optimal, that is, the constant p^{-p} in (3.3.21) and (3.3.23) is not optimal.

3.4 Some Applications to the Operator-Valued Case

The principal purpose of this section is to extend Example 3.2.3 to the operator-valued situation.

We start with a few preparations. Given a separable, complex Hilbert space $\mathcal{H}$, we recall that we denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of linear, bounded operators $T\colon \mathcal{H} \to \mathcal{H}$ defined on all of $\mathcal{H}$. Similarly, $\mathcal{B}_p(\mathcal{H})$ denote the ℓ^p-based Schatten–von Neumann trace ideals, $p \in [1, \infty)$.

The eigenvalues of a bounded linear operator $B \in \mathcal{B}(\mathcal{H})$ are abbreviated by $\lambda_j(B)$, $j \in \mathcal{J}$, with $\mathcal{J} \subseteq \mathbb{N}$ an appropriate index set, and the trace of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$ is denoted by $\operatorname{tr}_{\mathcal{H}}(A)$ and computed via Lidskii's theorem as

$$\operatorname{tr}_{\mathcal{H}}(A) = \sum_{j \in \mathcal{J}} \lambda_j(A). \tag{3.4.1}$$

In particular, if $T \in \mathcal{B}_p(\mathcal{H})$ for some $p \in [1, \infty)$, and $|T|$ is defined by $|T| := (T^*T)^{1/2}$, one recalls the fact,

$$\|T\|_{\mathcal{B}_p(\mathcal{H})}^p = \operatorname{tr}_{\mathcal{H}}\big(|T|^p\big). \tag{3.4.2}$$

Moreover, if $A : (0,\infty) \to \mathcal{B}(\mathcal{H})$ is weakly measurable, $0 \leqslant A(\,\cdot\,) \in \mathcal{B}_1(\mathcal{H})$ a.e. on $(0,\infty)$, and $\operatorname{tr}_{\mathcal{H}}(A(\,\cdot\,)) \in L^1((a,b);dt)$, then by an application of the monotone convergence theorem,

$$\begin{aligned}
\bigg\| \int_a^b dt\, A(t) \bigg\|_{\mathcal{B}_1(\mathcal{H})} &= \operatorname{tr}_{\mathcal{H}}\bigg(\int_a^b dt\, A(t) \bigg) \\
&= \sum_{n\in\mathcal{N}} \int_a^b dt\, (e_n, A(t)e_n)_{\mathcal{H}} = \int_a^b dt\, \sum_{n\in\mathcal{N}} (e_n, A(t)e_n)_{\mathcal{H}} \\
&= \int_a^b dt\, \operatorname{tr}_{\mathcal{H}}(A(t)) = \int_a^b dt\, \|A(t)\|_{\mathcal{B}_1(\mathcal{H})},
\end{aligned} \tag{3.4.3}$$

where $\{e_n\}_{n\in\mathcal{N}}$ represents a complete orthonormal system in $\mathcal{H}$, with $\mathcal{N} \subseteq \mathbb{N}$ an appropriate index set. In this context we also recall the well-known fact,

$$\bigg\| \int_a^b dt\, A(t) \bigg\|_{\mathcal{B}_p(\mathcal{H})} \leqslant \int_a^b dt\, \|A(t)\|_{\mathcal{B}_p(\mathcal{H})}, \quad p \in [1,\infty), \tag{3.4.4}$$

and similarly with $\mathcal{B}_p(\mathcal{H})$ replaced by $\mathcal{B}(\mathcal{H})$.

Given these preparations, one can restate Example 3.2.3 in the case where $\mathcal{B} = \mathcal{B}_p(\mathcal{H})$ as follows.

Corollary 3.4.1. *Let $p \in [1,\infty)$, $b \in (0,\infty)\cup\{\infty\}$, and suppose that $F : (0,b) \to \mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $\|F(\,\cdot\,)\|_{\mathcal{B}_p(\mathcal{H})} \in L^p((0,b);x^\alpha dx)$, with $\alpha \in \mathbb{R}$ chosen according to items (i) and (ii) below:*

(i) If $\alpha < p-1$, then (3.2.11) *implies*

$$\operatorname{tr}_{\mathcal{H}}\bigg(\int_0^b dx\; x^\alpha\, |F(x)|^p \bigg) \geqslant \bigg(\frac{|\alpha - p + 1|}{p} \bigg)^p \operatorname{tr}_{\mathcal{H}}\bigg(\int_0^b dx\, x^{\alpha-p} \bigg| \int_0^x dx'\, F(x') \bigg|^p \bigg). \tag{3.4.5}$$

(ii) If $\alpha > p-1$, then (3.2.12) *implies*

$$\operatorname{tr}_{\mathcal{H}}\bigg(\int_0^b dx\; x^\alpha\, |F(x)|^p \bigg) \geqslant \bigg(\frac{|\alpha - p + 1|}{p} \bigg)^p \operatorname{tr}_{\mathcal{H}}\bigg(\int_0^b dx\, x^{\alpha-p} \bigg| \int_x^b dx'\, F(x') \bigg|^p \bigg). \tag{3.4.6}$$

In both cases (i) and (ii), the constant $[(|\alpha-p+1|)/p]^p$ is best possible and equality holds if and only if $F = 0$ a.e. on $(0,b)$.

Proof. It suffices to consider item (i). Then an application of (3.4.1)–(3.4.4) yields

$$\begin{aligned}
&\bigg(\frac{|\alpha-p+1|}{p}\bigg)^p \operatorname{tr}_{\mathcal{H}}\bigg(\int_0^b dx\, x^{\alpha-p}\bigg|\int_0^x dx'\, F(x')\bigg|^p\bigg) \\
&\quad = \bigg(\frac{|\alpha-p+1|}{p}\bigg)^p \int_0^b dx\, x^{\alpha-p}\operatorname{tr}_{\mathcal{H}}\bigg(\bigg|\int_0^x dx'\, F(x')\bigg|^p\bigg) \quad \text{(by (3.4.3))} \\
&\quad = \bigg(\frac{|\alpha-p+1|}{p}\bigg)^p \int_0^b dx\, x^{\alpha-p}\bigg\|\int_0^x dx'\, F(x')\bigg\|_{\mathcal{B}_p(\mathcal{H})}^p \quad \text{(by (3.4.2))} \\
&\quad \leqslant \bigg(\frac{|\alpha-p+1|}{p}\bigg)^p \int_0^b dx\, x^{\alpha-p}\bigg(\int_0^x dx'\, \|F(x')\|_{\mathcal{B}_p(\mathcal{H})}\bigg)^p \quad \text{(by (3.4.4))} \\
&\quad \leqslant \int_0^b dx\, x^{\alpha}\|F(x)\|_{\mathcal{B}_p(\mathcal{H})}^p \quad \text{(by (3.2.11))} \\
&\quad = \int_0^b dx\, x^{\alpha}\operatorname{tr}_{\mathcal{H}}\big(|F(x)|^p\big) \quad \text{(by (3.4.2))} \\
&\quad = \operatorname{tr}_{\mathcal{H}}\bigg(\int_0^b dx\, x^{\alpha}|F(x)|^p\bigg) \quad \text{(by (3.4.3))}.
\end{aligned} \tag{3.4.7}$$

The final part about optimality of the constant on the right-hand side in (3.4.5) and (3.4.6), and the equality part, then follow as in Example 3.2.3. □

We note that the case $\alpha = 0$, $b = \infty$, $F(\,\cdot\,) \geqslant 0$ a.e. on $(0, \infty)$ in (3.4.5) was proved by Hansen [64, Theorem 2.4] on the basis of a convexity argument (see also [65], [82]). Our strategy of proof is different and based on that in Theorem 3.2.2.

Next, following Hansen [64], we will remove the trace in Corollary 3.4.1 in the case where $p \in [1, 2]$.

We start by recalling [64, Lemma 2.1]:

Lemma 3.4.2. *Let $p \in [1, 2]$, and suppose that $F : (0, \infty) \to \mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $F(\,\cdot\,) \geqslant 0$ a.e. on $(0, \infty)$, and $\int_0^\infty dx\, x^{-1}F(x)^p \in \mathcal{B}(\mathcal{H})$. Then,*

$$\int_0^\infty dx\, x^{-1}F(x)^p \geqslant \int_0^\infty dx\, x^{-1-p}\bigg(\int_0^x dx'\, F(x')\bigg)^p. \tag{3.4.8}$$

The constant 1 *on the right-hand side of the inequality* (3.4.8) *is best possible.*

Employing Lemma 3.4.2 we can prove the principal result of this section.

Theorem 3.4.3. *Let $p \in [1,2]$, and suppose that $F : (0,\infty) \to \mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $F(\,\cdot\,) \geqslant 0$ a.e. on $(0,\infty)$, and $\int_0^\infty dx\, x^\alpha F(x)^p \in \mathcal{B}(\mathcal{H})$, with $\alpha \in \mathbb{R}$ chosen according to items (i) and (ii) below:*

(i) If $\alpha < p-1$, then

$$\int_0^\infty dx\ x^\alpha\, F(x)^p \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^\infty dx\, x^{\alpha-p}\bigg(\int_0^x dx'\, F(x')\bigg)^p. \tag{3.4.9}$$

(ii) If $\alpha > p-1$, then

$$\int_0^\infty dx\ x^\alpha\, F(x)^p \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^\infty dx\, x^{\alpha-p}\bigg(\int_x^\infty dx'\, F(x')\bigg)^p. \tag{3.4.10}$$

In both cases (i) and (ii), the constant $[(|\alpha+1-p|)/p]^p$ is best possible and equality holds if and only if $F = 0$ a.e. on $(0,\infty)$.

Proof. We start by proving item (i). Closely following the strategy of proof in [64, Theorem 2.3], we introduce

$$G(x) = F\big(x^{p/|\alpha-p+1|}\big)x^{(1+\alpha)/|\alpha-p+1|}, \quad x > 0, \tag{3.4.11}$$

and the change of variables

$$y = x^{p/|\alpha-p+1|}, \quad dy = [p/|\alpha - p + 1|]x^{(1+\alpha)/|\alpha-p+1|}dx. \tag{3.4.12}$$

Then Lemma 3.4.2 applied to G yields

$$\begin{aligned}
\int_0^\infty dx\, x^{-1}G(x)^p &= \int_0^\infty dx\, x^{-1}F\big(x^{p/|\alpha-p+1|}\big)^p x^{p(1+\alpha)/|\alpha-p+1|} \\
&\geqslant \int_0^\infty dx\, x^{-1-p}\bigg(\int_0^x dx' G(x')\bigg)^p \quad \text{(by (3.4.8))} \\
&= \int_0^\infty dx\, x^{-1-p}\bigg(\int_0^x dt\, F\big(t^{p/|\alpha-p+1|}\big)t^{(1+\alpha)/|\alpha-p+1|}\bigg)^p \\
&= \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^\infty dx\, x^{-1-p}\bigg(\int_0^{x^{p/|\alpha-p+1|}} dy\, F(y)\bigg)^p.
\end{aligned} \tag{3.4.13}$$

Introducing another change of variables

$$w = x^{p/|\alpha-p+1|}, \quad dw\, w^{-1} = [p/|\alpha - p + 1|] dx\, x^{-1}, \tag{3.4.14}$$

then implies

$$\begin{aligned}
&\left(\frac{|\alpha-p+1|}{p}\right) \int_0^\infty dw\, w^\alpha F(w)^p \\
&\quad = \left(\frac{|\alpha-p+1|}{p}\right) \int_0^\infty dw\, w^{-1} F(w)^p w^{1+\alpha} \\
&\quad = \int_0^\infty dx\, x^{-1} F\big(x^{p/|\alpha-p+1|}\big)^p x^{p(1+\alpha)/|\alpha-p+1|} \\
&\quad \geqslant \left(\frac{|\alpha-p+1|}{p}\right)^p \int_0^\infty dx\, x^{-1-p} \bigg(\int_0^{x^{p/|\alpha-p+1|}} dy\, F(y)\bigg)^p \quad \text{(by (3.4.13))} \\
&\quad = \left(\frac{|\alpha-p+1|}{p}\right)^{p+1} \int_0^\infty dw\, w^{-1-(p-1-\alpha)} \bigg(\int_0^w dy\, F(y)\bigg)^p \quad \text{(by (3.4.14))} \\
&\quad = \left(\frac{|\alpha-p+1|}{p}\right)^{p+1} \int_0^\infty dw\, w^{\alpha-p} \bigg(\int_0^w dy\, F(y)\bigg)^p,
\end{aligned} \tag{3.4.15}$$

proving (3.4.9).

While $\Phi(F) = x^{-1} \int_0^x dx'\, F(x')$ represents a positive, unital map (i.e., $\Phi(F) \geqslant 0$ if $F \geqslant 0$ and $\Phi(I_{\mathcal{H}}) = I_{\mathcal{H}}$), $\int_x^\infty dx'\, F(x')$ cannot possibly be of this type and hence one cannot simply follow the proof of [64, Theorem 2.3] to derive (3.4.10). Fortunately, the following elementary alternative approach applies. Introducing the change of variables,

$$y = 1/x, \quad G(y) = F(1/y) y^{-2}, \tag{3.4.16}$$

in (3.4.9) (w.r.t. x on either side in (3.4.9) and, especially, w.r.t. x' on the right-hand side of (3.4.9)) results in

$$\int_0^\infty dy\, y^\beta G(y)^p \geqslant \left(\frac{|\beta-p+1|}{p}\right)^p \int_0^\infty dx\, x^{\beta-p} \bigg(\int_x^\infty dy\, G(y)\bigg)^p, \tag{3.4.17}$$

where $\beta = 2p - 2 - \alpha$, and hence $\alpha < p - 1$ is equivalent to $\beta > p - 1$.

The final part about optimality of the constant on the right-hand side in (3.4.9) and (3.4.10), and the equality part, then follow as in Corollary 3.4.1 from Example 3.2.3 upon taking the trace on either side of (3.4.9) and (3.4.10). □

Again we note that the case $\alpha = 0$ in (3.4.9) was proved by Hansen [64, Theorem 2.3]; he also proved that Theorem 3.4.3 does not extend to $p > 2$.

While we focused on the underlying interval $(0,\infty)$ in Theorem 3.4.3, the analogous case $(0,b)$, $b\in(0,\infty)$ follows upon employing the variable transformations discussed in [87, p. 36–38].

Extending the definition of $(H_{\mp,m}F)(x)$, $x\in(0,\infty)$, $m\in\mathbb{N}$, in (3.2.13), (3.2.14) to the operator-valued context where $F:(0,\infty)\to\mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $F(\,\cdot\,)\geqslant 0$ a.e. on $(0,\infty)$, and for all $c\in(0,\infty)$, $\int_0^c dx\,F(x)^p\in\mathcal{B}(\mathcal{H})$ in connection with $H_{-,m}$ and $\int_c^\infty dx\,F(x)^p\in\mathcal{B}(\mathcal{H})$ in connection with $H_{+,m}$, the facts (3.4.9), (3.4.10) can be rewritten as

$$\int_0^\infty dx\,x^{\alpha-p}[H_{\mp,1}(F(\,\cdot\,))(x)]^p\leqslant\left(\frac{p}{|\alpha-p+1|}\right)^p\int_0^\infty dx\,x^\alpha F(x)^p, \tag{3.4.18}$$

for $p\in[1,\infty)$, $\alpha\lesseqgtr p-1$. Thus one obtains the following result.

Corollary 3.4.4. *Let $p\in[1,2]$, and suppose that $F:(0,\infty)\to\mathcal{B}(\mathcal{H})$ is a weakly measurable map satisfying $F(\,\cdot\,)\geqslant 0$ a.e. on $(0,\infty)$, and $\int_0^\infty dx\,x^\alpha F(x)^p\in\mathcal{B}(\mathcal{H})$, with $\alpha\in\mathbb{R}$ chosen according to* (3.4.19) *below. Then*

$$\int_0^\infty dx\,x^\alpha F(x)^p\geqslant\prod_{k=1}^m\left(\frac{|\alpha-kp+1|}{p}\right)^p\int_0^\infty dx\,x^{\alpha-mp}[H_{\mp,m}(F(\,\cdot\,))(x)]^p, \tag{3.4.19}$$

$$\alpha\lesseqgtr\begin{cases}p-1,\\ mp-1,\end{cases}\quad m\in\mathbb{N}.$$

Proof. Iterate (3.4.18) by applying it to appropriate $F=F_m$ as in (3.2.16). □

Replacing $F\geqslant 0$ by $|F|=(F^*F)^{1/2}$ and mimicking the differential version of the Hardy inequalities at the end of Section 3.2 yields

$$\begin{gathered}\int_0^\infty dx\,x^\alpha|f'(x)|^p\geqslant\left(\frac{|\alpha-p+1|}{p}\right)^p\int_0^\infty dx\,x^{\alpha-p}|f(x)|^p,\\ p\in[1,2],\ \alpha\in\mathbb{R},\ f\in C_0^\infty((0,\infty);\mathcal{B}(\mathcal{H})).\end{gathered} \tag{3.4.20}$$

Iterating (3.4.20) the yields as in (3.2.21)

$$\begin{gathered}\int_0^\infty dx\,x^\alpha|f^{(m)}(x)|^p\geqslant\prod_{j=1}^k\left(\frac{|\alpha-jp+1|}{p}\right)^p\int_0^\infty dx\,x^{\alpha-kp}|f^{(m-k)}(x)|^p,\\ p\in[1,2],\ 1\leqslant k\leqslant m,\ m\in\mathbb{N},\ \alpha\in\mathbb{R},\ f\in C_0^\infty((0,\infty);\mathcal{B}(\mathcal{H})).\end{gathered} \tag{3.4.21}$$

CHAPTER FOUR

On Power Weighted Birman–Hardy–Rellich-type Inequalities with Logarithmic Refinements via Hartman–Müeller-Pfeiffer Transformations

4.1 Introduction

To be able to describe the content of this paper we start by recalling Birman's infinite sequence of integral inequalities [19], the sequence of Birman–Hardy–Rellich inequalities of the form

$$\begin{gathered} \int_a^b dx\, \big|f^{(m)}(x)\big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}} \int_a^b dx\, x^{-2m}|f(x)|^2, \\ f \in C_0^m((a,b)),\ m \in \mathbb{N}, \quad 0 \leqslant a < b \leqslant \infty, \end{gathered} \tag{4.1.1}$$

which appeared in 1961, and in English translation in 1966 (see also [60, pp. 83–84]). The case $m = 1$ in (4.1.1) represents Hardy's celebrated inequality [70], [71, Sect. 9.8] (see also [85, Chs. 1, 3, App.]), the case $m = 2$ is due to Rellich [107, Sect. II.7] (actually, in the multi-dimensional context). The inequalities (4.1.1) are known to be optimal (i.e., the constant $[2m-1)!!]^2/2^{2m}$ is best possible) and strict (i.e., equality holds in (4.1.1) if and only if $f = 0$ on (a, b)) (see, e.g., [11, p. 4], [13, 34, 53, 70, 71, 102, 107]). We also note that higher-order Hardy inequalities, including weight functions, are discussed in [87, Ch. 4] and [101, Sect. 10], however, Birman's sequence of inequalities is not mentioned in these sources.

The primary aim in this paper is to prove optimal inequalities of the type (4.1.1) with additional weights (of power-type on either side of (4.1.1)) and logarithmic refinements (i.e., additional, only logarithmically weaker, singularities on the right-hand side of (4.1.1)).

To describe these inequalities in detail we need some preparations and introduce the iterated logarithms $\ln_j(\,\cdot\,)$, $j \in \mathbb{N}$ (cf. [72], [74, pp. 324–325])), given by

$$\ln_1(\,\cdot\,) = \ln(\,\cdot\,), \quad \ln_{j+1}(\,\cdot\,) = \ln(\ln_j(\,\cdot\,)), \quad j \in \mathbb{N}, \tag{4.1.2}$$

and also normalized iterated logarithms $L_j(\,\cdot\,)$, $j \in \mathbb{N}$ (see, e.g., [15]),

$$L_1(\,\cdot\,) = \big(1 - \ln(\,\cdot\,)\big)^{-1}, \quad L_{j+1}(\,\cdot\,) = L_1(L_j(\,\cdot\,)), \quad j \in \mathbb{N}. \tag{4.1.3}$$

In addition, we introduce iterated exponentials in the form,

$$e_0 = 0, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{4.1.4}$$

Moreover, for $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, we introduce the constants

$$A(m,\alpha) = \prod_{j=1}^{m} \left(\frac{2j-1-\alpha}{2}\right)^2, \tag{4.1.5}$$

$$B(m,\alpha) = \frac{1}{4^m} \sum_{k=1}^{m} \prod_{\substack{j=1\\ j\neq k}}^{m} (2j-1-\alpha)^2. \tag{4.1.6}$$

One observes that

$$B(m,\alpha) = A(m,\alpha) \sum_{j=1}^{m} \frac{1}{(2j-1-\alpha)^2}, \quad m \in \mathbb{N},\ \alpha \in \mathbb{R}\backslash\{2j-1\}_{j=1}^{m}, \tag{4.1.7}$$

$$A(m,0) = \frac{[(2m-1)!!]^2}{2^{2m}}, \quad m \in \mathbb{N}. \tag{4.1.8}$$

In particular, $A(m,0)$ coincides with the constant in (4.1.1).

The improved Birman inequalities contain additional constants $c_\ell(m,\alpha)$, $\ell = 0,1,\dots,2m$, which are defined in terms of the polynomial

$$P_{m,\alpha}(\lambda) = \sum_{\ell=0}^{2m} c_\ell(m,\alpha)\lambda^\ell = \prod_{j=1}^{m} \left(\lambda^2 - \frac{(2j-1-\alpha)^2}{4}\right), \quad m \in \mathbb{N},\ \alpha \in \mathbb{R}. \tag{4.1.9}$$

Given the notation introduced in (4.1.2)–(4.1.9), we can now describe the principal results proved in this note: Let $m, N \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\rho, \gamma \in (0,\infty)$, $\gamma \geqslant e_N \rho$, and $f \in C_0^\infty((0,\rho))$. Then the power-weighted Birman–Hardy–Rellich sequence with logarithmic refinements on the interior interval $(0,\rho)$ are of the form

$$\begin{aligned}\int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 &\geqslant A(m,\alpha) \int_0^\rho dx\, x^{\alpha-2m} |f(x)|^2 \\ &\quad + B(m,\alpha) \sum_{k=1}^{N} \int_\rho^\infty dx\, x^{\alpha-2m} \prod_{\ell=1}^{k} [\ln_\ell(x/\gamma)]^{-2} |f(x)|^2 \end{aligned} \tag{4.1.10}$$

$$
+\sum_{j=2}^{m}|c_{2j}(m,\alpha)|A(j,0)\int_0^\rho dx\, x^{\alpha-2m}[\ln(x/\gamma)]^{-2j}|f(x)|^2
$$
$$
+\sum_{j=2}^{m}|c_{2j}(m,\alpha)|B(j,0)\sum_{k=1}^{N-1}\int_0^\rho dx\, x^{\alpha-2m}[\ln(x/\gamma)]^{-2j}\prod_{\ell=1}^{k}[\ln_{\ell+1}(x/\gamma)]^{-2}|f(x)|^2.
$$

Moreover, we prove the same sequence of inequalities on the interior interval $(0,\rho)$ for $f\in C_0^\infty((0,\rho))$ and finally both sets of inequalities (exterior and interior) also with the iterated logarithms $\ln_j(\,\cdot\,)$ replaced by the normalized logarithms $L_j(\,\cdot\,)$, $j\in\mathbb{N}$. In the latter case an infinite series of logarithmic terms (i.e., the case $N=\infty$ in the analog of (4.1.10)) will be permitted. Furthermore, we show that all equalities are strict, that is, equality holds if and only if $f=0$ on (ρ,∞) (resp., $(0,\rho)$). In addition, all inequalities are generalized by replacing f on the right-hand side with intermediate derivatives $f^{(m-\ell)}$ for $\ell=1,\dots,m$. For brevity, a careful comparison of our result with the existing ones in the literature is postponed to Remark 4.3.3.

In Section 4.2 we introduce our principal tool, a combined Hartman–Müller-Pfeiffer transformation, our principal results are then proved in Section 4.3. Finally, in Section 4.4 we derive the sequence of power-weighted Birman–Hardy–Rellich inequalities with logarithmic refinements in the vector-valued case, replacing complex-valued $f(\,\cdot\,)$ by $f(\,\cdot\,)\in\mathcal{H}$, with $\mathcal{H}$ a complex, separable Hilbert space.

4.2 The Combined Hartman–Müeller-Pfeiffer Transformation

In this section we introduce an elementary, yet extremely useful, variable transformation, an appropriate combination of special cases of transformations considered by Hartman [72] (see also [74, p. 324–325]) and Müller-Pfeiffer [96, p. 200–207]. We now introduce an extension of these transformations by Hartman and Müller-Pfeiffer applicable to power weights and higher-order derivatives. This will be crucial in proving the power-weighted Birman–Hardy–Rellich inequalities with logarithmic refinements under most general conditions in our principal Section 4.3.

Let $m, N \in \mathbb{N}$ and suppose that

$$\alpha \in \mathbb{R}\backslash\{1, 2, \dots, 2m-1\}. \tag{4.2.1}$$

Given $f \in C_0^\infty((e_N, \infty))$, the transformation

$$x = e^t, \ x \in (e_N, \infty), \quad dx = e^t dt, \qquad t \in (e_{N-1}, \infty), \tag{4.2.2}$$

$$f(x) \equiv f(e^t) = e^{(m-\frac{1+\alpha}{2})t} w(t), \quad w \in C_0^\infty((e_{N-1}, \infty)), \tag{4.2.3}$$

yields

$$\big(x^\alpha f^{(m)}(x)\big)^{(m)} = e^{-(m+\frac{1-\alpha}{2})t} \sum_{\ell=0}^{2m} c_\ell(m, \alpha) w^{(\ell)}(t), \tag{4.2.4}$$

for appropriate constants $c_\ell(m, \alpha)$, $\ell = 0, 1, \dots, 2m$ to be determined next.

The solutions of the differential equation

$$\big(x^\alpha f^{(m)}(x)\big)^{(m)} = 0, \tag{4.2.5}$$

are linear combinations of the following powers of x:

$$\begin{cases} x^j, & j = 0, 1, \dots, m-1, \\ x^{k-\alpha}, & k = m, \dots, 2m-1. \end{cases} \tag{4.2.6}$$

One notes that the solutions (4.2.6) are linearly independent due to (4.2.1).

Thus, recalling (4.2.2)–(4.2.4), it follows that the solutions of

$$\sum_{\ell=0}^{2m} c_\ell(m, \alpha) w^{(\ell)}(t) = 0, \tag{4.2.7}$$

are the functions

$$e^{(\frac{1+\alpha}{2}-m)t} x^j = e^{(j+\frac{1+\alpha}{2}-m)t}, \quad j = 0, 1, \dots, m-1, \tag{4.2.8}$$

and

$$e^{(\frac{1+\alpha}{2}-m)t} x^{k-\alpha} = e^{(k+\frac{1-\alpha}{2}-m)t} \quad k = m, \dots, 2m-1. \tag{4.2.9}$$

Observe that for $j = 0$ and $k = 2m - 1$,

$$\begin{aligned} e^{(j+\frac{1+\alpha}{2}-m)t} &= e^{(\frac{1+\alpha}{2}-m)t} \\ e^{(k+\frac{1-\alpha}{2}-m)t} &= e^{-(\frac{1+\alpha}{2}-m)t}. \end{aligned} \tag{4.2.10}$$

For $j = 1$ and $k = 2m - 2$,

$$\begin{aligned} e^{(j+\frac{1+\alpha}{2}-m)t} &= e^{(\frac{3+\alpha}{2}-m)t} \\ e^{(k+\frac{1-\alpha}{2}-m)t} &= e^{-(\frac{3+\alpha}{2}-m)t}. \end{aligned} \tag{4.2.11}$$

Continuing iteratively, we see that the linearly independent solutions of (4.2.7) are of the form

$$e^{\pm\frac{1}{2}(2j+1-2m+\alpha)t}, \quad j = 0, 1, \ldots, m-1, \tag{4.2.12}$$

By a simple relabeling, (4.2.12) is equivalently

$$e^{\pm\frac{1}{2}(2j-1-\alpha)t}, \quad j = 1, \ldots, m. \tag{4.2.13}$$

The zeros of the characteristic polynomial of (4.2.7) are thus the constant factors in the exponents of (4.2.13). Hence, the characteristic polynomial is given by

$$\begin{aligned} P_{m,\alpha}(\lambda) &= \sum_{\ell=0}^{2m} c_\ell(m,\alpha)\lambda^\ell \\ &= \left(\lambda^2 - \frac{(1-\alpha)^2}{4}\right)\left(\lambda^2 - \frac{(3-\alpha)^2}{4}\right)\cdots\left(\lambda^2 - \frac{(2m-1-\alpha)^2}{4}\right) \\ &= \prod_{j=1}^{m}\left(\lambda^2 - \frac{(2j-1-\alpha)^2}{4}\right). \end{aligned} \tag{4.2.14}$$

Thus, the coefficients $c_\ell(m,\alpha)$, $\ell = 0, 1, \ldots, 2m$, satisfy the following properties:

$$\begin{aligned} &(i)\ c_{2j-1}(m,\alpha) = 0, \quad j = 1, \ldots, m; \\ &(ii)\ c_{2j}(m,\alpha) = (-1)^{m-j}|c_{2j}(m,\alpha)|, \quad j = 0, 1, \ldots, m; \\ &(iii)\ |c_0(m,\alpha)| = A(m,\alpha); \\ &(iv)\ |c_2(m,\alpha)| = 4B(m,\alpha); \end{aligned} \tag{4.2.15}$$

(v) $c_{2m}(m,\alpha) = 1$.

Turning our attention to the iterated logarithms, given $N \in \mathbb{N}$, the transformation (4.2.2) yields

$$\sum_{k=1}^{N} \prod_{j=1}^{k} [\ln_j(x)]^{-2} = t^{-2} + t^{-2} \sum_{k=1}^{N-1} \prod_{j=1}^{k} [\ln_j(t)]^{-2}, \tag{4.2.16}$$

interpreting $\sum_{k=1}^{0}(\,\cdot\,) = 0$.

Remark 4.2.1. If $\alpha \in \{1, 2, \dots, 2m-1\}$, then the solutions of (4.2.5) are linear combinations of the following:

$$\begin{cases} x^j, & j = 0, 1, \dots, j_{m,\alpha}, \\ x^{-k}, & k = 0, 1, \dots, k_{m,\alpha}, \\ x^{\ell}\ln(x), & \ell = 0, 1, \dots, \ell_{m,\alpha}, \end{cases} \tag{4.2.17}$$

for some $j_{m,\alpha}, k_{m,\alpha}, \ell_{m,\alpha} \in \mathbb{N}_0$. However, as we will see in the next section, the improved power-weighted Birman-type inequalities need only be proven for the weight parameters $\alpha \in \mathbb{R}$, $\alpha \neq 1, 2, \dots, 2m-1$, since the case $\alpha \in \{1, 2, \dots, 2m-1\}$ follows by taking the limits $\alpha \to k$ for $k = 1, \dots, 2m-1$. $\diamond$

4.3 Power-Weighted Birman–Hardy–Rellich-type Inequalities with Logarithmic Refinements

In this section we now establish several improvements of existing power-weighted Birman–Hardy–Rellich inequalities in the literature by employing the combined Hartman–Müeller-Pfeiffer variable transformation from section 4.2 in a crucial manner. These weighted inequalities are proven for both types of iterated logarithms $\ln_j(\,\cdot\,)$, $j \in \mathbb{N}$ and $L_j(\,\cdot\,)$, $j \in \mathbb{N}$, and are given on both the exterior interval (ρ, ∞) and interior interval $(0, \rho)$ for any $\rho \in (0, \infty)$.

The principal result of this paper then reads as follows:

Theorem 4.3.1. *Let $\ell, m, N \in \mathbb{N}, \alpha \in \mathbb{R}$, and $\rho, \gamma, \tau \in (0, \infty)$. The following hold:*

(i) If $\rho \geqslant e_N \gamma$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((\rho, \infty))$,

$$\begin{aligned}
\int_\rho^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 &\geqslant A(\ell,\alpha) \int_\rho^\infty dx\, x^{\alpha-2\ell} |f^{(m-\ell)}(x)|^2 \\
&+ B(\ell,\alpha) \sum_{k=1}^N \int_\rho^\infty dx\, x^{\alpha-2\ell} \prod_{p=1}^k [\ln_p(x/\gamma)]^{-2} |f^{(m-\ell)}(x)|^2 \\
&+ \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)| A(j,0) \int_\rho^\infty dx\, x^{\alpha-2\ell} [\ln(x/\gamma)]^{-2j} |f^{(m-\ell)}(x)|^2 \\
&+ \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)| B(j,0) \\
&\times \sum_{k=1}^{N-1} \int_\rho^\infty dx\, x^{\alpha-2\ell} [\ln(x/\gamma)]^{-2j} \prod_{p=1}^k [\ln_{p+1}(x/\gamma)]^{-2} |f^{(m-\ell)}(x)|^2.
\end{aligned} \tag{4.3.1}$$

(ii) If $\rho \geqslant \tau$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((\rho, \infty))$,

$$\begin{aligned}
\int_\rho^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 &\geqslant A(\ell,\alpha) \int_\rho^\infty dx\, x^{\alpha-2\ell} |f^{(m-\ell)}(x)|^2 \\
&+ B(\ell,\alpha) \sum_{k=1}^N \int_\rho^\infty dx\, x^{\alpha-2\ell} \prod_{p=1}^k L_p^2(\tau/x) |f^{(m-\ell)}(x)|^2 \\
&+ \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)| A(j,0) \int_\rho^\infty dx\, x^{\alpha-2\ell} L_1^{2j}(\tau/x) |f^{(m-\ell)}(x)|^2 \\
&+ \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_\rho^\infty dx\, x^{\alpha-2\ell} L_1^{2j}(\tau/x) \prod_{p=1}^k L_{p+1}^2(\tau/x) |f^{(m-\ell)}(x)|^2.
\end{aligned} \tag{4.3.2}$$

(iii) If $\gamma \geqslant e_N \rho$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((0, \rho))$,

$$\begin{aligned}
\int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 &\geqslant A(\ell,\alpha) \int_0^\rho dx\, x^{\alpha-2\ell} |f^{(m-\ell)}(x)|^2 \\
&+ B(\ell,\alpha) \sum_{k=1}^N \int_0^\rho dx\, x^{\alpha-2\ell} \prod_{p=1}^k [\ln_p(\gamma/x)]^{-2} |f^{(m-\ell)}(x)|^2 \\
&+ \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)| A(j,0) \int_0^\rho dx\, x^{\alpha-2\ell} [\ln(\gamma/x)]^{-2j} |f^{(m-\ell)}(x)|^2 \\
&+ \sum_{j=2}^\ell |c_{2j}(\ell,\alpha)| B(j,0)
\end{aligned} \tag{4.3.3}$$

$$\times \sum_{k=1}^{N-1} \int_0^\rho dx\, x^{\alpha-2\ell} [\ln(\gamma/x)]^{-2j} \prod_{p=1}^{k} [\ln_{p+1}(\gamma/x)]^{-2} |f^{(m-\ell)}(x)|^2.$$

(iv) If $\tau \geqslant \rho$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((0,\rho))$,

$$\begin{aligned} \int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 &\geqslant A(\ell,\alpha) \int_0^\rho dx\, x^{\alpha-2\ell} |f^{(m-\ell)}(x)|^2 \\ &+ B(\ell,\alpha) \sum_{k=1}^{N} \int_0^\rho dx\, x^{\alpha-2\ell} \prod_{p=1}^{k} L_p^2(x/\tau) |f^{(m-\ell)}(x)|^2 \\ &+ \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| A(j,0) \int_0^\rho dx\, x^{\alpha-2\ell} L_1^{2j}(x/\tau) |f^{(m-\ell)}(x)|^2 \\ &+ \sum_{j=2}^{\ell} |c_{2j}(\ell,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_0^\rho dx\, x^{\alpha-2\ell} L_1^{2j}(x/\tau) \prod_{p=1}^{k} L_{p+1}^2(x/\tau) |f^{(m-\ell)}(x)|^2. \end{aligned} \tag{4.3.4}$$

Moreover, inequalities (4.3.1)–(4.3.4) *are strict for $f \not\equiv 0$ on (ρ,∞), respectively, $(0,\rho)$.*

We break up the proof of Theorem 4.3.1 into four parts. For simplicity we present the proof in the special case $\ell = m$; the general case follows upon replacing f by $f^{(m-\ell)}$ for $\ell = 1,\dots,m$.

Proof of Theorem 4.3.1 *(i)*. Let $\rho \geqslant e_N\gamma$, pick any $f \in C_0^\infty((\rho,\infty))$, and assume that $\alpha \in \mathbb{R}$ satisfies (4.2.1). The scaling

$$x = \gamma y, \quad dx = \gamma dy, \quad g(y) = f(\gamma y), \quad y \in (\rho/\gamma, \infty), \tag{4.3.5}$$

and using zero extensions, implies

$$g \in C_0^\infty((\rho/\gamma,\infty)) \subseteq C_0^\infty((e_N,\infty)). \tag{4.3.6}$$

Applying the transformation (4.2.2), (4.2.3) to g then yields

$$\big(y^\alpha g^{(m)}(y)\big)^{(m)} = e^{-(m+\frac{1-\alpha}{2})t} \sum_{j=0}^{m} (-1)^{m-j} |c_{2j}(m,\alpha)| w^{(2j)}(t), \tag{4.3.7}$$

for $t \in (e_{N-1},\infty)$, $w \in C_0^\infty((e_{N-1},\infty))$, and $c_{2j}(m,\alpha)$ as in (4.2.15). Thus,

$$(-1)^m \big(y^\alpha g^{(m)}(y)\big)^{(m)} \overline{g(y)} = e^{-t} \sum_{j=0}^{m} (-1)^{2m-j} |c_{2j}(m,\alpha)| w^{(2j)}(t) \overline{w(t)}. \tag{4.3.8}$$

Furthermore, (4.2.2), (4.2.3), and (4.2.16) yield

$$
\begin{aligned}
&y^{\alpha-2m}|g(y)|^2 = e^{-t}|w(t)|^2, && (4.3.9)\\
&y^{\alpha-2m}\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(y)]^{-2}|g(y)|^2 = e^{-t}\bigg\{t^{-2}|w(t)|^2 + t^{-2}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(t)]^{-2}|w(t)|^2\bigg\},
\end{aligned}
$$

and for $j = 2, \dots, m$,

$$
\begin{aligned}
&y^{\alpha-2m}[\ln(y)]^{-2j}|g(y)|^2 = e^{-t}t^{-2j}|w(t)|^2, && (4.3.10)\\
&y^{\alpha-2m}[\ln(y)]^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_{p+1}(y)]^{-2}|g(y)|^2 = e^{-t}t^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(t)]^{-2}|w(t)|^2.
\end{aligned}
$$

Employing the elementary identity,

$$
\begin{gathered}
\int_a^b dx\, x^\alpha\big|f^{(m)}(x)\big|^2 = (-1)^m\int_a^b dx\,\big(x^\alpha f^{(m)}(x)\big)^{(m)}\overline{f(x)},\\
m\in\mathbb{N},\ \alpha\in\mathbb{R},\ f\in C_0^\infty((a,b)),\ 0\leqslant a<b\leqslant\infty,
\end{gathered} \tag{4.3.11}
$$

and items (iii), (iv) of (4.2.15), it follows from (4.3.5)–(4.3.10) that

$$
\begin{aligned}
&\int_\rho^\infty dx\,\bigg\{x^\alpha\big|f^{(m)}(x)\big|^2 - A(m,\alpha)x^{\alpha-2m}|f(x)|^2\\
&\quad - B(m,\alpha)x^{\alpha-2m}\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(x/\gamma)]^{-2}|f(x)|^2\\
&\quad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|A(j,0)x^{\alpha-2m}[\ln(x/\gamma)]^{-2j}|f(x)|^2\\
&\quad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|B(j,0)x^{\alpha-2m}[\ln(x/\gamma)]^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_{p+1}(x/\gamma)]^{-2}|f(x)|^2\bigg\}\\
&= \gamma^{\alpha-2m+1}\int_{e_N}^\infty dy\,\bigg\{y^\alpha\big|g^{(m)}(y)\big|^2 - A(m,\alpha)y^{\alpha-2m}|g(y)|^2\\
&\quad - B(m,\alpha)y^{\alpha-2m}\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(y)]^{-2}|g(y)|^2\\
&\quad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|A(j,0)y^{\alpha-2m}[\ln(y)]^{-2j}|g(y)|^2\\
&\quad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|B(j,0)y^{\alpha-2m}[\ln(y)]^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_{p+1}(y)]^{-2}|g(y)|^2\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma^{\alpha-2m+1}\bigg\{ \sum_{j=0}^{m} |c_{2j}(m,\alpha)| \int_{e_{N-1}}^{\infty} dt\, \big|w^{(j)}(t)\big|^2 - A(m,\alpha) \int_{e_{N-1}}^{\infty} dt\, |w(t)|^2 \\
&\quad - B(m,\alpha) \int_{e_{N-1}}^{\infty} dt\, t^{-2} |w(t)|^2 - B(m,\alpha) \sum_{k=1}^{N-1} \int_{e_{N-1}}^{\infty} dt\, t^{-2} \prod_{p=1}^{k} [\ln_p(t)]^{-2} |w(t)|^2 \\
&\quad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| A(j,0) \int_{e_{N-1}}^{\infty} dt\, t^{-2j} |w(t)|^2 \\
&\quad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_{e_{N-1}}^{\infty} dt\, t^{-2j} \prod_{p=1}^{k} [\ln_p(t)]^{-2} |w(t)|^2 \bigg\} \\
&= \gamma^{\alpha-2m+1}\bigg\{ \sum_{j=1}^{m} |c_{2j}(m,\alpha)| \int_{e_{N-1}}^{\infty} dt\, \big|w^{(j)}(t)\big|^2 \\
&\quad - \sum_{j=1}^{m} |c_{2j}(m,\alpha)| A(j,0) \int_{e_{N-1}}^{\infty} dt\, t^{-2j} |w(t)|^2 \\
&\quad - \sum_{j=1}^{m} |c_{2j}(m,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_{e_{N-1}}^{\infty} dt\, t^{-2j} \prod_{p=1}^{k} [\ln_p(t)]^{-2} |w(t)|^2 \bigg\} \\
&= \gamma^{\alpha-2m+1} \sum_{j=1}^{m} |c_{2j}(m,\alpha)| \bigg\{ \int_{e_{N-1}}^{\infty} dt\, \big|w^{(j)}(t)\big|^2 - A(j,0) \int_{e_{N-1}}^{\infty} dt\, t^{-2j} |w(t)|^2 \\
&\quad - B(j,0) \sum_{k=1}^{N-1} \int_{e_{N-1}}^{\infty} dt\, t^{-2j} \prod_{p=1}^{k} [\ln_p(t)]^{-2} |w(t)|^2 \bigg\}, \qquad (4.3.12)
\end{aligned}
$$

interpreting $\sum_{k=1}^{0}(\,\cdot\,) = 0$. Hence, part (i), for $\alpha \in \mathbb{R}\backslash\{1,2,\dots,2m-1\}$, follows via induction over $N \in \mathbb{N}$. Indeed, for $N = 1$ the equality (4.3.12) yields

$$
\begin{aligned}
&\int_{\rho}^{\infty} dx\, \bigg\{ x^{\alpha} \big|f^{(m)}(x)\big|^2 - A(m,\alpha) x^{\alpha-2m} |f(x)|^2 - B(m,\alpha) x^{\alpha-2m} [\ln(x/\gamma)]^{-2} |f(x)|^2 \\
&\qquad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| A(j,0) x^{\alpha-2m} [\ln(x/\gamma)]^{-2j} |f(x)|^2 \bigg\} \\
&= \gamma^{\alpha-2m+1} \sum_{j=1}^{m} |c_{2j}(m,\alpha)| \bigg\{ \int_{0}^{\infty} dt\, \big|w^{(j)}(t)\big|^2 - A(j,0) \int_{0}^{\infty} dt\, t^{-2j} |w(t)|^2 \bigg\} \\
&\geqslant 0, \qquad (4.3.13)
\end{aligned}
$$

by (4.1.1) as a sum of unweighted Birman–Hardy–Rellich-type inequalities. Assuming (4.3.1) holds for $N-1 \in \mathbb{N}$ then reapplying (4.3.12) proves (4.3.1) for $N \in \mathbb{N}$. Strictness also follows by induction over $N \in \mathbb{N}$ since $f \not\equiv 0$ implies $w \not\equiv 0$ by (4.2.2), (4.2.3) so that (4.3.13), and by induction (4.3.12), is strictly positive. The case $\alpha \in \{1, 2, \ldots, 2m-1\}$ then follows by taking the limits $\alpha \to k$ for $k = 1, \ldots, 2m-1$, noting that $A(m, \alpha)$, $B(m, \alpha)$, and $c_{2j}(m, \alpha)$ are continuous as polynomials in $\alpha \in \mathbb{R}$. This completes the proof for part (i). $\square$

Proof of Theorem 4.3.1 (ii). By taking limits as in part (i), it suffices once more to consider $\alpha \in \mathbb{R}\backslash\{1, 2, \ldots, 2m-1\}$. Let $\rho \geqslant \tau$ and pick any $f \in C_0^\infty((\rho, \infty))$. The scaling

$$x = \tau y, \quad dx = \tau dy, \quad g(y) = f(\tau y), \quad y \in (\rho/\tau, \infty), \tag{4.3.14}$$

yields $g \in C_0^\infty((\rho/\tau, \infty)) \subseteq C_0^\infty((1, \infty))$. Modifying the transformation (4.2.2), (4.2.3) applied to g by

$$\begin{aligned} &y = e^{t-1}, \quad dy = e^{t-1}dt, \quad t \in (1, \infty), \\ &g(y) \equiv g(e^{t-1}) = e^{(m - \frac{1+\alpha}{2})(t-1)} v(t), \quad v \in C_0^\infty((1, \infty)). \end{aligned} \tag{4.3.15}$$

Here v is given by

$$v(t) := w(t-1), \quad t \in (1, \infty), \tag{4.3.16}$$

with $w \in C_0^\infty((0, \infty))$. Setting

$$s = t - 1, \quad ds = dt, \tag{4.3.17}$$

and noting

$$\frac{d}{dt} v(t) = \frac{d}{ds} w(s), \tag{4.3.18}$$

yields, similarly to (4.3.7),

$$\begin{aligned} \big(y^\alpha g^{(m)}(y)\big)^{(m)} &= e^{-(m + \frac{1-\alpha}{2})s} \sum_{\ell=0}^{2m} c_\ell(m, \alpha) w^{(\ell)}(s) \\ &= e^{-(m + \frac{1-\alpha}{2})(t-1)} \sum_{\ell=0}^{2m} c_\ell(m, \alpha) v^{(\ell)}(t). \end{aligned} \tag{4.3.19}$$

Hence, an analogous argument as in section 4.2 shows the constants $c_\ell(m, \alpha)$ satisfy (i)–(v) in (4.2.15) as before. Therefore by (4.3.19),

$$(-1)^m \big(y^\alpha g^{(m)}(y)\big)^{(m)} \overline{g(y)} = e^{1-t} \sum_{j=0}^{m} (-1)^{2m-j} |c_{2j}(m, \alpha)| v^{(2j)}(t) \overline{v(t)}. \tag{4.3.20}$$

Now, (4.3.15) yields

$$L_1(1/y) = \big(1 - \ln(1/y)\big)^{-1} = \big(1 - \ln(e^{1-t})\big)^{-1} = t^{-1}, \tag{4.3.21}$$

and

$$L_2(1/y) = L_1(L_1(1/y)) = \big(1 - \ln(L_1(1/y))\big)^{-1} = \big(1 - \ln(1/t)\big)^{-1} = L_1(1/t). \tag{4.3.22}$$

Inductively, we see that

$$L_1(1/y) = t^{-1}, \quad L_j(1/y) = L_{j-1}(1/t), \quad j = 2, 3, \dots \tag{4.3.23}$$

Hence,

$$\begin{aligned} & y^{\alpha-2m}|g(y)|^2 = e^{1-t}|v(t)|^2, \\ & y^{\alpha-2m}\sum_{k=1}^{N}\prod_{p=1}^{k} L_p^2(1/y)|g(y)|^2 = e^{1-t}\Big\{t^{-2}|v(t)|^2 + t^{-2}\sum_{k=1}^{N-1}\prod_{p=1}^{k} L_p^2(1/t)|v(t)|^2\Big\}, \end{aligned} \tag{4.3.24}$$

and for $j = 2, \dots, m$,

$$\begin{aligned} & y^{\alpha-2m} L_1^{2j}(1/y)|g(y)|^2 = e^{1-t}t^{-2j}|v(t)|^2, \\ & y^{\alpha-2m} L_1^{2j}(1/y)\sum_{k=1}^{N-1}\prod_{p=1}^{k} L_{p+1}^2(1/y)|g(y)|^2 = e^{1-t}t^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k} L_p^2(1/t)|v(t)|^2. \end{aligned} \tag{4.3.25}$$

Again recalling (4.3.11) and (iii)–(iv) of (4.2.15), (4.3.20), (4.3.24), and (4.3.25) yield

$$\begin{aligned} & \int_\rho^\infty dx\,\Big\{x^\alpha\big|f^{(m)}(x)\big|^2 - A(m,\alpha)x^{\alpha-2m}|f(x)|^2 \\ & \quad - B(m,\alpha)x^{\alpha-2m}\sum_{k=1}^{N}\prod_{p=1}^{k} L_p^2(\tau/x)|f(x)|^2 \\ & \quad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|A(j,0)x^{\alpha-2m}L_1^{2j}(\tau/x)|f(x)|^2 \\ & \quad - \sum_{j=2}^{m}|c_{2j}(m,\alpha)|B(j,0)x^{\alpha-2m}L_1^{2j}(\tau/x)\sum_{k=1}^{N-1}\prod_{p=1}^{k} L_{p+1}^2(\tau/x)|f(x)|^2\Big\} \\ & = \tau^{\alpha-2m+1}\sum_{j=1}^{m}|c_{2j}(m,\alpha)|\Big\{\int_1^\infty dt\,\big|v^{(j)}(t)\big|^2 - A(j,0)\int_1^\infty dt\,t^{-2j}|v(t)|^2 \end{aligned} \tag{4.3.26}$$

$$- B(j,0) \sum_{k=1}^{N-1} \int_1^\infty dt\, t^{-2j} \prod_{p=1}^{k} L_p^2(1/t) |v(t)|^2 \Big\},$$

and the proof again follows by induction over $N \in \mathbb{N}$. □

Proof of Theorem 4.3.1 *(iii).* By taking limits as before, we need only consider $\alpha \in \mathbb{R}, \alpha \neq 1, 2, \ldots 2m-1$.

For part (i), let $\gamma \geqslant e_N \rho$ and pick any $f \in C_0^\infty((0,\rho))$. The scaling

$$\begin{aligned} &x = \gamma y, \quad dx = \gamma dy, \quad y \in (0, \rho/\gamma), \\ &g(y) = f(\gamma y), \end{aligned} \tag{4.3.27}$$

yields

$$g \in C_0^\infty((0,\rho/\gamma)) \subseteq C_0^\infty((0, 1/e_N)). \tag{4.3.28}$$

Slightly modify the transformation (4.2.2), (4.2.3) applied to g as follows:

$$\begin{aligned} &y = e^{-t}, \quad dy = -e^{-t} dt, \quad t \in (e_{N-1}, \infty), \\ &g(y) \equiv g(e^{-t}) = e^{-(m - \frac{1+\alpha}{2})t} v(t), \quad v \in C_0^\infty((e_{N-1}, \infty)). \end{aligned} \tag{4.3.29}$$

Here v is given by

$$v(t) := w(-t), \quad t \in (e_{N-1}, \infty), \tag{4.3.30}$$

with $w \in C_0^\infty((-\infty, -e_{N-1}))$. Setting

$$s = -t, \quad ds = -dt, \tag{4.3.31}$$

we have

$$\begin{aligned} \big(y^\alpha g^{(m)}(y)\big)^{(m)} &= e^{-(m + \frac{1-\alpha}{2})s} \sum_{\ell=0}^{2m} c_\ell(m,\alpha) w^{(\ell)}(s) \\ &= e^{(m + \frac{1-\alpha}{2})t} \sum_{\ell=0}^{2m} \widetilde{c}_\ell(m,\alpha) v^{(\ell)}(t), \end{aligned} \tag{4.3.32}$$

where

$$\widetilde{c}_\ell(m,\alpha) = (-1)^\ell c_\ell(m,\alpha), \quad \ell = 0, 1, \ldots, 2m, \tag{4.3.33}$$

so clearly (i)–(v) in (4.2.15) still hold. Hence,

$$(-1)^m \big(y^\alpha g^{(m)}(y)\big)^{(m)} \overline{g(y)} = e^t \sum_{j=0}^{m} (-1)^{2m-j} |c_{2j}(m,\alpha)| v^{(2j)}(t) \overline{v(t)}. \tag{4.3.34}$$

Furthermore,

$$\begin{aligned} &y^{\alpha-2m} |g(y)|^2 = e^t |v(t)|^2, \qquad\qquad (4.3.35)\\ &y^{\alpha-2m} \sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(1/y)]^{-2} |g(y)|^2 = e^t \bigg\{ t^{-2} |v(t)|^2 + t^{-2} \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_p(t)]^{-2} |v(t)|^2 \bigg\}, \end{aligned}$$

and for $j = 2, \dots, m$,

$$\begin{aligned} &y^{\alpha-2m} [\ln(1/y)]^{-2j} |g(y)|^2 = e^t t^{-2j} |v(t)|^2, \qquad\qquad (4.3.36)\\ &y^{\alpha-2m} [\ln(1/y)]^{-2j} \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_{p+1}(1/y)]^{-2} |g(y)|^2 = e^t t^{-2j} \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_p(t)]^{-2} |v(t)|^2. \end{aligned}$$

Applying (4.3.34)–(4.3.36) yields

$$\begin{aligned} &\int_0^\rho dx \,\bigg\{ x^\alpha \big|f^{(m)}(x)\big|^2 - A(m,\alpha) x^{\alpha-2m} |f(x)|^2 \\ &\qquad - B(m,\alpha) x^{\alpha-2m} \sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\gamma/x)]^{-2} |f(x)|^2 \\ &\qquad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| A(j,0) x^{\alpha-2m} [\ln(\gamma/x)]^{-2j} |f(x)|^2 \\ &\qquad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| B(j,0) x^{\alpha-2m} [\ln(\gamma/x)]^{-2j} \sum_{k=1}^{N-1} \prod_{p=1}^{k} [\ln_{p+1}(\gamma/x)]^{-2} |f(x)|^2 \bigg\} \\ &\quad = \gamma^{\alpha-2m+1} \sum_{j=1}^{m} |c_{2j}(m,\alpha)| \bigg\{ \int_{e_{N-1}}^\infty dt \,\big|v^{(j)}(t)\big|^2 - A(j,0) \int_{e_{N-1}}^\infty dt\, t^{-2j} |v(t)|^2 \\ &\qquad - B(j,0) \sum_{k=1}^{N-1} \int_{e_{N-1}}^\infty dt\, t^{-2j} \prod_{p=1}^{k} [\ln_p(t)]^{-2} |v(t)|^2 \bigg\}, \end{aligned} \tag{4.3.37}$$

and the proof follows by induction over $N \in \mathbb{N}$, as before. □

Proof of Theorem 4.3.1 (iv). Let $\tau \geqslant \rho$ and $f \in C_0^\infty((0,\rho))$, scaling

$$\begin{aligned} &x = \tau y, \quad dx = \tau dy, \quad y \in (0, \rho/\tau), \\ &g(y) = f(\tau y), \end{aligned} \tag{4.3.38}$$

so that $g \in C_0^\infty((0,\rho/\tau)) \subseteq C_0^\infty((0,1))$, and applying the modified transformation

$$
\begin{aligned}
&y = e^{-t+1}, \quad dy = -e^{-t+1}dt, \quad t \in (1,\infty), \\
&g(y) \equiv g(e^{-t+1}) = e^{(m-\frac{1+\alpha}{2})(1-t)}v(t), \quad v \in C_0^\infty((1,\infty)),
\end{aligned}
\tag{4.3.39}
$$

where v is given by

$$
v(t) := w(1-t), \quad t \in (1,\infty), \tag{4.3.40}
$$

with $w \in C_0^\infty((-\infty,0))$. Therefore

$$
(-1)^m \big(y^\alpha g^{(m)}(y)\big)^{(m)} \overline{g(y)} = e^{t-1} \sum_{j=0}^{m} (-1)^{2m-j} |c_{2j}(m,\alpha)| v^{(2j)}(t) \overline{v(t)}. \tag{4.3.41}
$$

Also,

$$
\begin{aligned}
&y^{\alpha-2m}|g(y)|^2 = e^{t-1}|v(t)|^2, \qquad (4.3.42)\\
&y^{\alpha-2m} \sum_{k=1}^{N} \prod_{p=1}^{k} L_p^2(y)|g(y)|^2 = e^{t-1}\Big\{t^{-2}|v(t)|^2 + t^{-2} \sum_{k=1}^{N-1} \prod_{p=1}^{k} L_p^2(1/t)|v(t)|^2\Big\},
\end{aligned}
$$

and for $j = 2,\dots,m$,

$$
\begin{aligned}
&y^{\alpha-2m} L_1^{2j}(y)|g(y)|^2 = e^{t-1}t^{-2j}|v(t)|^2, \qquad (4.3.43)\\
&y^{\alpha-2m} L_1^{2j}(y) \sum_{k=1}^{N-1} \prod_{p=1}^{k} L_{p+1}^2(1/y)|g(y)|^2 = e^{t-1}t^{-2j} \sum_{k=1}^{N-1} \prod_{p=1}^{k} L_p^2(1/t)|v(t)|^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_0^\rho dx \,\Big\{ x^\alpha \big|f^{(m)}(x)\big|^2 - A(m,\alpha)x^{\alpha-2m}|f(x)|^2 \\
&\quad - B(m,\alpha)x^{\alpha-2m} \sum_{k=1}^{N} \prod_{p=1}^{k} L_p^2(x/\tau)|f(x)|^2 \\
&\quad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| A(j,0) x^{\alpha-2m} L_1^{2j}(x/\tau)|f(x)|^2 \\
&\quad - \sum_{j=2}^{m} |c_{2j}(m,\alpha)| B(j,0) x^{\alpha-2m} L_1^{2j}(x/\tau) \sum_{k=1}^{N-1} \prod_{p=1}^{k} L_{p+1}^2(x/\tau)|f(x)|^2 \Big\} \\
&= \tau^{\alpha-2m+1} \sum_{j=1}^{m} |c_{2j}(m,\alpha)| \Big\{ \int_1^\infty dt \,\big|v^{(j)}(t)\big|^2 - A(j,0) \int_1^\infty dt \, t^{-2j}|v(t)|^2
\end{aligned}
$$

$$- B(j,0) \sum_{k=1}^{N-1} \int_1^\infty dt\, t^{-2j} \prod_{p=1}^{k} L_p^2(1/t) |v(t)|^2 \Big\}, \tag{4.3.44}$$

and the proof follows by induction over $N \in \mathbb{N}$. $\square$

Theorem 4.3.1 $(ii), (iv)$ can be further improved by replacing the N-th sum with an infinite series. See, for example, [12–15, 59, 114] for similar results and discussions of the convergence of the series $\sum_{k=1}^{\infty} \prod_{j=1}^{k} L_j^2(s)$ for $s \in (0,1)$.

Corollary 4.3.2. *Let $\ell, m \in \mathbb{N}, \alpha \in \mathbb{R}$, and $\rho, \tau \in (0,\infty)$. Then* (4.3.2) *and* (4.3.4) *extend to $N = \infty$.*

Proof. It suffices to discuss the proof of (4.3.2). Given $f \in C_0^\infty((\rho,\infty))$, Theorem 4.3.1 (ii) implies that (4.3.2) holds for any $N \in \mathbb{N}$. Thus, by taking $N \uparrow \infty$ and recalling that increasing sequences bounded above are convergent, (4.3.2) holds with $N = \infty$. $\square$

To put our results in perspective and to compare with existing results in the literature we conclude this section with a few comments.

Remark 4.3.3. For $m \geqslant 2$ these inequalities are new in the following sense: The weight parameter $\alpha \in \mathbb{R}$ is now unrestricted, as opposed to prior results obtained in [2,3,5,8,12–17,25,35–41,47,58,59,77,94,112,114]; the conditions on the logarithmic parameters γ and τ are sharp (and were not previously discussed in the literature); the two integral terms containing $c_{2j}(m,\alpha)$ are new; the inequalities are proven for both iterated logarithms $\ln_j(\,\cdot\,)$ and $L_j(\,\cdot\,)$, $j \in \mathbb{N}$; finally, they are proven on both the exterior interval (ρ,∞) and interior interval $(0,\rho)$ for any $\rho \in (0,\infty)$. $\diamond$

Remark 4.3.4. As discussed in section 4.1, the constant $A(m,\alpha)$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, is known to be sharp for $\alpha = 0$, though not officially proven for general $\alpha \in \mathbb{R}$. Optimality of the constant $B(m,\alpha)$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, has not yet been verified in full generality (i.e. for $\alpha \in \mathbb{R}$ unrestricted, both logarithms $\ln_j(\,\cdot\,)$, $L_j(\,\cdot\,)$, $j \in \mathbb{N}$, and on both intervals $(0,\rho)$, (ρ,∞), $\rho \in (0,\infty)$), but has been proven in several special cases (but within the more general multidimensional setting after reducing the problem to one dimension using radial arguments). For the unweighted case $\alpha = 0$, see e.g. [2, 3, 15–17, 37, 47, 58, 59, 77, 94] for $m = 1$, [14, 25, 94] for $m = 2$, and [5, 114] for higher-order $m \in \mathbb{N}$. The weighted case $\alpha \in (mp-k, 0]$, $k \in \{1, \dots, n\}$, $p \in ((13+\sqrt{105})/4, \infty)$ for higher-order $m \in \mathbb{N}$ was investigated in [13]. $\diamond$

4.4 The Vector-Valued Case

In our final section, we establish that all previous inequalities extend line by line to the vector-valued case in which f is $\mathcal{H}$-valued, with $\mathcal{H}$ a separable, complex Hilbert space. The relevance of such a generalization is briefly mentioned at the end of this section.

We start by stating a power-weighted extension of (4.1.1) for vector-valued functions, which is derived from the more general Hardy result [29, Example 1] by simple iteration (see also [53, Theorem 8.1] for the special case $\alpha = 0$, $a = 0$, $b = \infty$). Inequality (4.4.1) will replace (4.1.1) in the base step of each induction proof.

Lemma 4.4.1. *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $0 \leqslant a < b \leqslant \infty$. Then for all $f \in C_0^\infty((a,b);\mathcal{H})$,*

$$\int_a^b dx\, x^\alpha \big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 \geqslant A(m,\alpha) \int_a^b dx\, x^{\alpha-2m} \|f(x)\|_{\mathcal{H}}^2. \tag{4.4.1}$$

The constant $A(m,\alpha)$ is sharp and equality holds if and only if $f = 0$ on (a,b).

In addition, the combined Hartman–Müeller-Pfeiffer transformation extends to the $\mathcal{H}$-valued context. Indeed, given $m, N \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\alpha \neq 1, \dots, 2m-1$, and $f \in C_0^\infty((e_N,\infty);\mathcal{H})$, one sets

$$\begin{aligned} &x = e^t, \quad dx = e^t dt, \quad t \in (e_{N-1},\infty), \\ &f(x) \equiv f(e^t) = e^{(m-\frac{1+\alpha}{2})t} w(t), \quad w \in C_0^\infty((e_{N-1},\infty);\mathcal{H}), \end{aligned} \tag{4.4.2}$$

so that

$$\big(x^\alpha f^{(m)}(x)\big)^{(m)} = e^{-(m+\frac{1-\alpha}{2})t} \sum_{\ell=0}^{2m} c_\ell(m,\alpha) w^{(\ell)}(t). \tag{4.4.3}$$

Combining (4.4.2) and (4.4.3) yields

$$(-1)^m\Big(\big(x^\alpha f^{(m)}(x)\big)^{(m)}, f(x)\Big)_{\mathcal{H}} = e^{-t}\sum_{j=0}^{m}(-1)^{2m-j}|c_{2j}(m,\alpha)|\big(w^{(2j)}(t), w(t)\big)_{\mathcal{H}}. \quad (4.4.4)$$

Furthermore,

$$\begin{aligned} &x^{\alpha-2m}\|f(x)\|_{\mathcal{H}}^2 = e^{-t}\|w(t)\|_{\mathcal{H}}^2, \\ &x^{\alpha-2m}\sum_{k=1}^{N}\prod_{p=1}^{k}[\ln_p(x)]^{-2}\|f(x)\|_{\mathcal{H}}^2 \\ &\quad = e^{-t}\bigg\{t^{-2}\|w(t)\|_{\mathcal{H}}^2 + t^{-2}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(t)]^{-2}\|w(t)\|_{\mathcal{H}}^2\bigg\}, \end{aligned} \quad (4.4.5)$$

and for $j = 2, \dots, m$,

$$\begin{aligned} &x^{\alpha-2m}[\ln(x)]^{-2j}\|f(x)\|_{\mathcal{H}}^2 = e^{-t}t^{-2j}\|w(t)\|_{\mathcal{H}}^2, \\ &x^{\alpha-2m}[\ln(x)]^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_{p+1}(x)]^{-2}\|f(x)\|_{\mathcal{H}}^2 \\ &\quad = e^{-t}t^{-2j}\sum_{k=1}^{N-1}\prod_{p=1}^{k}[\ln_p(t)]^{-2}\|w(t)\|_{\mathcal{H}}^2. \end{aligned} \quad (4.4.6)$$

The modified variable transformations (4.3.15), (4.3.29), (4.3.39), generalize analogously.

Finally, we note that (4.3.11) extends to the vector-valued situation in the form

$$\int_a^b dx\, x^\alpha \big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 = (-1)^m \int_a^b dx\, \Big(\big(x^\alpha f^{(m)}(x)\big)^{(m)}, f(x)\Big)_{\mathcal{H}}, \quad (4.4.7)$$

for $f \in C_0^\infty((a,b);\mathcal{H})$, where $0 \leqslant a < b \leqslant \infty, m \in \mathbb{N}, \alpha \in \mathbb{R}$.

Given these preliminaries, the vector-valued case becomes completely analogous to the scalar situation treated in Section 4.3:

Theorem 4.4.2. *Let $\ell, m, N \in \mathbb{N}, \alpha \in \mathbb{R}$, and $\rho, \gamma, \tau \in (0,\infty)$. The following hold:*

(i) If $\rho \geqslant e_N\gamma$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((\rho,\infty);\mathcal{H})$,

$$\int_\rho^\infty dx\, x^\alpha \big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 \geqslant A(\ell,\alpha)\int_\rho^\infty dx\, x^{\alpha-2\ell}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2$$

$$+ B(\ell,\alpha)\sum_{k=1}^{N}\int_\rho^\infty dx\, x^{\alpha-2\ell}\prod_{p=1}^{k}[\ln_p(x/\gamma)]^{-2}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \tag{4.4.8}$$

$$+ \sum_{j=2}^{\ell}|c_{2j}(\ell,\alpha)|A(j,0)\int_\rho^\infty dx\, x^{\alpha-2\ell}[\ln(x/\gamma)]^{-2j}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2$$

$$+ \sum_{j=2}^{\ell}|c_{2j}(\ell,\alpha)|B(j,0)$$

$$\times\sum_{k=1}^{N-1}\int_\rho^\infty dx\, x^{\alpha-2\ell}[\ln(x/\gamma)]^{-2j}\prod_{p=1}^{k}[\ln_{p+1}(x/\gamma)]^{-2}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2.$$

(ii) If $\rho \geqslant \tau$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((\rho,\infty);\mathcal{H})$,

$$\int_\rho^\infty dx\, x^\alpha\big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 \geqslant A(\ell,\alpha)\int_\rho^\infty dx\, x^{\alpha-2\ell}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2$$

$$+ B(\ell,\alpha)\sum_{k=1}^{N}\int_\rho^\infty dx\, x^{\alpha-2\ell}\prod_{p=1}^{k}L_p^2(\tau/x)\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \tag{4.4.9}$$

$$+ \sum_{j=2}^{\ell}|c_{2j}(\ell,\alpha)|A(j,0)\int_\rho^\infty dx\, x^{\alpha-2\ell}L_1^{2j}(\tau/x)\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2$$

$$+ \sum_{j=2}^{\ell}|c_{2j}(\ell,\alpha)|B(j,0)\sum_{k=1}^{N-1}\int_\rho^\infty dx\, x^{\alpha-2\ell}L_1^{2j}(\tau/x)\prod_{p=1}^{k}L_{p+1}^2(\tau/x)\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2.$$

(iii) If $\gamma \geqslant e_N\rho$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((0,\rho);\mathcal{H})$,

$$\int_0^\rho dx\, x^\alpha\big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 \geqslant A(\ell,\alpha)\int_0^\rho dx\, x^{\alpha-2\ell}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2$$

$$+ B(\ell,\alpha)\sum_{k=1}^{N}\int_0^\rho dx\, x^{\alpha-2\ell}\prod_{p=1}^{k}[\ln_p(\gamma/x)]^{-2}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \tag{4.4.10}$$

$$+ \sum_{j=2}^{\ell}|c_{2j}(\ell,\alpha)|A(j,0)\int_0^\rho dx\, x^{\alpha-2\ell}[\ln(\gamma/x)]^{-2j}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2$$

$$+ \sum_{j=2}^{\ell}|c_{2j}(\ell,\alpha)|B(j,0)$$

$$\times\sum_{k=1}^{N-1}\int_0^\rho dx\, x^{\alpha-2\ell}[\ln(\gamma/x)]^{-2j}\prod_{p=1}^{k}[\ln_{p+1}(\gamma/x)]^{-2}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2.$$

(iv) If $\tau \geqslant \rho$ and $1 \leqslant \ell \leqslant m$, then for all $f \in C_0^\infty((0,\rho);\mathcal{H})$,

$$\int_0^\rho dx\, x^\alpha\big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 \geqslant A(\ell,\alpha)\int_0^\rho dx\, x^{\alpha-2\ell}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2$$

$$+ B(\ell,\alpha)\sum_{k=1}^{N}\int_0^\rho dx\, x^{\alpha-2\ell}\prod_{p=1}^{k}L_p^2(x/\tau)\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \tag{4.4.11}$$
$$+\sum_{j=2}^{\ell}|c_{2j}(\ell,\alpha)|A(j,0)\int_0^\rho dx\, x^{\alpha-2\ell}L_1^{2j}(x/\tau)\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2$$
$$+\sum_{j=2}^{\ell}|c_{2j}(\ell,\alpha)|B(j,0)\sum_{k=1}^{N-1}\int_0^\rho dx\, x^{\alpha-2\ell}L_1^{2j}(x/\tau)\prod_{p=1}^{k}L_{p+1}^2(x/\tau)\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2.$$

Moreover, inequalities (4.4.8)–(4.4.11) *are strict for* $f \not\equiv 0$ *on* (ρ,∞)*, respectively,* $(0,\rho)$.

Corollary 4.4.3. *Let* $\ell, m \in \mathbb{N}, \alpha \in \mathbb{R}$*, and* $\rho, \tau \in (0,\infty)$*. Then* (4.4.9) *and* (4.4.11) *extend to* $N = \infty$.

Using Lemma 4.4.1 and identity (4.4.7) for the base step in the induction proof over $N \in \mathbb{N}$, one can follow the special scalar case treated in the proof of Theorem 4.3.1, and Corollary 4.3.2 line by line.

We conclude with the observation that the vector-valued Hardy case (i.e., $m = 1$) without logarithmic refinements (i.e., $N = 0$), played an important role in the spectral theory of n-dimensional Schrödinger operators ($n \in \mathbb{N}$, $n \geqslant 2$) as detailed, for instance in [88, Chs. IV, V]. In this context one employs polar coordinates and $\mathcal{H}$ is then naturally identified with $L^2(S^{n-1}; d^{n-1}\omega)$. This aspect will also play a crucial role in the multi-dimensional generalizations of the results presented in this note, see [52].

CHAPTER FIVE

On the Multidimensional Power Weighted Hardy Inequality with Radial and Logarithmic Refinements

The content of this chapter relies on (but is not identical to) the paper published as: F. Gesztesy, L. L. Littlejohn, I. Michael, and M. M. H. Pang, *Radial and Logarithmic Refinements of Hardy's Inequality*, Algebra i Analiz, **30(3)**, 55–65 (2018) (Russian), St. Petersburg Math. J., St. **30**, 429–436 (2019) (English).

5.1 Introduction

To describe the principal aim of this note, we start by recalling the classical Hardy inequality

$$\int_{\Omega} d^n x \, |(\nabla f)(x)|^2 \geqslant \frac{(n-2)^2}{4} \int_{\Omega} d^n x \, |x|^{-2} |f(x)|^2, \tag{5.1.1}$$

valid for $f \in C_0^\infty(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ open, $n \in \mathbb{N}$, $n \geqslant 2$ (interpreting the right-hand side of (5.1.1) as zero if $n = 2$, and hence rendering it trivial in that case). The following extension of Hardy's inequality (in the special case where Ω equals $B_n(x_0; \rho)$, the open ball in $\mathbb{R}^n$ of radius $\rho > 0$ centered at $x_0 \in \mathbb{R}^n$), involving logarithmic refinements, was derived in [50],

$$\begin{aligned} &\int_{\Omega} d^n x \, |(\nabla f)(x)|^2 \\ &\quad \geqslant \int_{\Omega} d^n x |x - x_0|^{-2} |f(x)|^2 \left\{ \frac{(n-2)^2}{4} + \frac{1}{4} \sum_{k=1}^{N} \prod_{j=1}^{k} [\ln_j(\gamma/|x - x_0|)]^{-2} \right\}, \end{aligned} \tag{5.1.2}$$

valid for $f \in C_0^\infty(\Omega)$, assuming $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 2$, Ω is open and bounded with $x_0 \in \Omega$, $N \in \mathbb{N}$, and the logarithmic terms $\ln_j(\,\cdot\,), j \in \mathbb{N}$, are recursively given by

$$\begin{aligned} &\ln_1(\,\cdot\,) := \ln(\,\cdot\,), \\ &\ln_{j+1}(\,\cdot\,) := \ln(\ln_j(\,\cdot\,)), \quad j \in \mathbb{N}, \end{aligned} \tag{5.1.3}$$

for $\gamma > 0$, $x \in \mathbb{R}^n \backslash \{x_0\}$, with $0 < |x - x_0| < \operatorname{diam}(\Omega) < \gamma/e_N$, where

$$e_1 := 1, \quad e_{j+1} := e^{e_j}, \quad j \in \mathbb{N}. \tag{5.1.4}$$

We denote $\sum_{k=1}^{0}(\,\cdot\,) := 0$ and $\prod_{j=1}^{0}(\,\cdot\,) := 1$, so when $N = 0$, $x_0 = 0$, (5.1.2) formally agrees with (5.1.1).

Due to the incredible amount of work on the classical Hardy inequality, we cannot possibly do justice to the existing literature and hence only refer to some of the standard monographs on the subject such as [11, 85, 86], and [101]. In addition, we note that factorizations in the context of Hardy's inequality in balls with optimal constants and logarithmic correction terms were already studied in [49, 55], based on prior work in [79, 80], and [81], although this appears to have gone unnoticed in the recent literature on this subject. Higher-order logarithmic refinements of the multidimensional Hardy–Rellich-type inequality appeared in [4, Theorem 2.1], and a sequence of such multidimensional Hardy–Rellich-type inequalities, with additional generalizations, appeared in [114, Theorems 1.8–1.10].

Our principal goal in this paper is to offer a radial, and weighted, analogue of (5.1.2) by replacing the gradient with the radial derivative ∂_{r,x_0} centered about a point $x_0 \in \mathbb{R}^n$, given by

$$\partial_{r,x_0} := |x - x_0|^{-1}(x - x_0) \cdot \nabla, \tag{5.1.5}$$

for $x \in \mathbb{R}^n \setminus \{x_0\}, r = |x - x_0|, n \in \mathbb{N}, n \geqslant 2$, denoting $\partial_{r,0} =: \partial_r$. Obviously,

$$|(\nabla f)(x)| \geqslant |(\partial_{r,x_0} f)(x)|, \quad x \in \mathbb{R}^n \backslash \{x_0\}, \quad f \in C_0^\infty(\mathbb{R}^n). \tag{5.1.6}$$

With (5.1.6) in mind, we will give a power weighted analogue of (5.1.1), (5.1.2) when ∇ is replaced by ∂_{r,x_0}. More precisely, we will prove the inequality

$$\int_\Omega d^n x \, |x - x_0|^\alpha |(\partial_{r,x_0} f)(x)|^2 \geqslant \frac{(n-2+\alpha)^2}{4} \int_\Omega d^n x \, |x - x_0|^{\alpha-2} |f(x)|^2, \tag{5.1.7}$$

valid for $f \in C_0^\infty(\Omega \backslash \{x_0\})$, (again, interpreting the right-hand side of (5.1.7) as zero if $\alpha = 2 - n$), where $\alpha \in \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, Ω is open with $x_0 \in \Omega$, and

$$\begin{aligned} &\int_\Omega d^n x \, |x - x_0|^\alpha |(\partial_{r,x_0} f)(x)|^2 \\ &\quad \geqslant \int_\Omega d^n x \, |x - x_0|^{\alpha-2} |f(x)|^2 \bigg\{ \frac{(n-2+\alpha)^2}{4} + \frac{1}{4} \sum_{k=1}^N \prod_{j=1}^k [\ln_j(\gamma/|x - x_0|)]^{-2} \bigg\}, \end{aligned} \tag{5.1.8}$$

valid for $f \in C_0^\infty(\Omega \backslash \{x_0\})$, assuming that $\alpha \in \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, Ω is open and bounded with $x_0 \in \Omega$, and $\gamma > 0$ satisfies $0 < |x - x_0| < \mathrm{diam}(\Omega) < \gamma/e_N$.

While (5.1.7) is well known, see, for instance, [10, p. 19], [11, Theorem 1.2.5] (in the case where $p = 2$, $\varepsilon = 0$), [46,50], and [108], inequality (5.1.8) is the principal result of this paper.

5.2 Refinements of Hardy's inequality

In this section we present our radial and logarithmic refinements of Hardy's inequality.

We start with some preliminary results. We introduce differential operators, $T_{\alpha,0}$, $\alpha \in \mathbb{R}$, on $C_0^\infty(\Omega)$ if $0 \notin \Omega$, respectively, $C_0^\infty(\Omega \backslash \{0\})$ if $0 \in \Omega$, $\Omega \subseteq \mathbb{R}^n$ open, and $T_{\alpha,N}$, $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$, on $C_0^\infty(B_n(0;\rho) \backslash \{0\})$, $n \geqslant 2$, as follows:

$$T_{\alpha,0} := |x|^\alpha \partial_r + [(n - 2 + 2\alpha)/2]|x|^{\alpha-1}, \quad N = 0, \tag{5.2.1}$$

$$T_{\alpha,N} := |x|^\alpha \partial_r + [(n - 2 + 2\alpha)/2]|x|^{\alpha-1} + (1/2)|x|^{\alpha-1} \sum_{k=1}^N \prod_{j=1}^k [\ln_j(\gamma/|x|)]^{-1}, \quad N \in \mathbb{N}. \tag{5.2.2}$$

Then their formal adjoints (with respect to $L^2(\Omega) := L^2(\Omega; d^n x)$), denoted by $T_{0,\alpha}^+$, $\alpha \in \mathbb{R}$, and defined on $C_0^\infty(\Omega)$, respectively, on $C_0^\infty(\Omega \backslash \{0\})$, and $T_{\alpha,N}^+$, $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$, defined on $C_0^\infty(B_n(0;\rho) \backslash \{0\})$, are given by (cf. (5.1.3), (5.1.4))

$$T_{\alpha,0}^+ = -|x|^\alpha \partial_r - (n/2)|x|^{\alpha-1}, \quad N = 0, \tag{5.2.3}$$

$$T_{\alpha,N}^+ = -|x|^\alpha \partial_r - (n/2)|x|^{\alpha-1} + (1/2)|x|^{\alpha-1} \sum_{k=1}^N \prod_{j=1}^k [\ln_j(\gamma/|x|)]^{-1}, \quad N \in \mathbb{N}. \tag{5.2.4}$$

Remark 5.2.1. In the following we will employ a standard convention when repeated use of differential expressions is involved: given differential expressions S_j, $j = 1, 2$, their product $S_1 S_2$ is used in the usual (operator) sense, that is,

$$(S_1 S_2 f)(x) = (S_1(S_2 f))(x), \tag{5.2.5}$$

for f in the underlying function space, and similarly for products of three and more differential expressions. $\diamond$

Next, we note that one obtains inductively, for $\alpha \in \mathbb{R}, N \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\begin{aligned} &\partial_r |x|^{\alpha-1} \prod_{j=1}^{N} [\ln_j(\gamma/|x|)]^{-1} = |x|^{\alpha-1} \prod_{j=1}^{N} [\ln_j(\gamma/|x|)]^{-1} \partial_r \\ &\quad + (\alpha-1)|x|^{\alpha-2} \prod_{j=1}^{N} [\ln_j(\gamma/|x|)]^{-1} + |x|^{\alpha-2} \prod_{j=1}^{N} [\ln_j(\gamma/|x|)]^{-1} \sum_{k=1}^{N} \prod_{j=1}^{k} [\ln_j(\gamma/|x|)]^{-1}, \end{aligned} \tag{5.2.6}$$

where again $\sum_{k=1}^{0}(\,\cdot\,) := 0$, $\prod_{j=1}^{0}(\,\cdot\,) = 1$.

Using (5.2.6), one can prove the following lemma, which will be useful in establishing Theorem 5.2.4.

Lemma 5.2.2. *Let $n \in \mathbb{N}, n \geqslant 2$, $N \in \mathbb{N}_0$, and $\alpha \in \mathbb{R}$. Then*

$$\begin{aligned} &T_{\alpha,N}^{+} |x|^{\alpha-1} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} + |x|^{\alpha-1} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} T_{\alpha,N} \\ &\quad = -|x|^{2\alpha-2} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-2}. \end{aligned} \tag{5.2.7}$$

Proof. Applying (5.2.6) yields

$$\begin{aligned} &T_{\alpha,N}^{+} |x|^{\alpha-1} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} \\ &= -|x|^{\alpha} \partial_r |x|^{\alpha-1} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} - (n/2)|x|^{2\alpha-2} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} \\ &\quad + (1/2)|x|^{2\alpha-2} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} \sum_{k=1}^{N} \prod_{j=1}^{k} [\ln_j(\gamma/|x|)]^{-1} \end{aligned}$$

$$
\begin{aligned}
&= \Big(- |x|^{2\alpha-1} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} \partial_r - (\alpha-1)|x|^{2\alpha-2} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} \\
&\quad - |x|^{2\alpha-2} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} \sum_{k=1}^{N+1} \prod_{j=1}^{k} [\ln_j(\gamma/|x|)]^{-1} \Big) - (n/2)|x|^{2\alpha-2} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} \\
&\quad + (1/2)|x|^{2\alpha-2} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} \sum_{k=1}^{N} \prod_{j=1}^{k} [\ln_j(\gamma/|x|)]^{-1} \\
&= -|x|^{2\alpha-1} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} \partial_r - [(n-2+2\alpha)/2]|x|^{2\alpha-2} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} \\
&\quad - (1/2)|x|^{2\alpha-2} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} \sum_{k=1}^{N} \prod_{j=1}^{k} [\ln_j(\gamma/|x|)]^{-1} - |x|^{2\alpha-2} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-2} \\
&= -|x|^{\alpha-1} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-1} T_{\alpha,N} - |x|^{2\alpha-2} \prod_{j=1}^{N+1} [\ln_j(\gamma/|x|)]^{-2}. \qquad (5.2.8)
\end{aligned}
$$

$\square$

Lemma 5.2.3. *Let $N \in \mathbb{N}_0, n \in \mathbb{N}$, $n \geqslant 2$, and $\alpha \in \mathbb{R}$. Then*

$$
\begin{aligned}
T_{\alpha,N}^{+} T_{\alpha,N} &= -|x|^{2\alpha} \partial_r^2 - (n-1+2\alpha)|x|^{2\alpha-1}\partial_r - [(n-2+2\alpha)/2]^2 |x|^{2\alpha-2} \\
&\quad - (1/4)|x|^{2\alpha-2} \sum_{k=1}^{N} \prod_{j=1}^{k} [\ln_j(\gamma/|x|)]^{-2}.
\end{aligned} \qquad (5.2.9)
$$

Proof. We use induction on $N \in \mathbb{N}_0$. For $N = 0$,

$$
\begin{aligned}
T_{\alpha,0}^{+} T_{\alpha,0} &= \big(- |x|^{\alpha} \partial_r - (n/2)|x|^{\alpha-1} \big) \big(|x|^{\alpha} \partial_r + [(n-2+2\alpha)/2]|x|^{\alpha-1} \big) \\
&= -|x|^{\alpha} \partial_r |x|^{\alpha} \partial_r - [(n-2+2\alpha)/2]|x|^{\alpha} \partial_r |x|^{\alpha-1} \\
&\quad - (n/2)|x|^{2\alpha-1} \partial_r - [n(n-2+2\alpha)/4]|x|^{2\alpha-2} \\
&= -|x|^{2\alpha} \partial_r^2 - (n-1+2\alpha)|x|^{2\alpha-1} \partial_r - [(n-2+2\alpha)/2]^2 |x|^{2\alpha-2}, \qquad (5.2.10)
\end{aligned}
$$

For $N = 1$, a direct computation, employing (5.2.6), yields,

$$
\begin{aligned}
T_{\alpha,1}^{+} T_{\alpha,1} &= \big(T_0^{+} + (1/2)|x|^{\alpha-1} [\ln(\gamma/|x|)]^{-1} \big) \big(T_0 + (1/2)|x|^{\alpha-1} [\ln(\gamma/|x|)]^{-1} \big) \\
&= T_0^{+} T_0 + (1/2) \big(T_0^{+} |x|^{\alpha-1} [\ln(\gamma/|x|)]^{-1} + |x|^{\alpha-1} [\ln(\gamma/|x|)]^{-1} T_0 \big) \\
&\quad + (1/4)|x|^{2\alpha-2} [\ln(\gamma/|x|)]^{-2}
\end{aligned}
$$

$$
\begin{aligned}
&= T_0^+ T_0 - (1/2)|x|^{2\alpha-2}[\ln(\gamma/|x|)]^{-2} + (1/4)|x|^{2\alpha-2}[\ln(\gamma/|x|)]^{-2} \\
&= -|x|^{2\alpha}\partial_r^2 - (n-1+2\alpha)|x|^{2\alpha-1}\partial_r - [(n-2+2\alpha)/2]^2|x|^{2\alpha-2} \\
&\quad - (1/4)|x|^{2\alpha-2}[\ln(\gamma/|x|)]^{-2}. \qquad (5.2.11)
\end{aligned}
$$

Assuming (5.2.9) holds for $N \in \mathbb{N}$, an application of Lemma 5.2.2 then yields for $N+1$,

$$
\begin{aligned}
&T_{\alpha,N+1}^+ T_{\alpha,N+1} \\
&= \Bigg(T_{\alpha,N}^+ + (1/2)|x|^{\alpha-1}\prod_{j=1}^{N+1}[\ln_j(\gamma/|x|)]^{-1}\Bigg)\Bigg(T_{\alpha,N} + (1/2)|x|^{\alpha-1}\prod_{j=1}^{N+1}[\ln_j(\gamma/|x|)]^{-1}\Bigg) \\
&= T_{\alpha,N}^+ T_{\alpha,N} + \frac{1}{2}\Bigg(T_{\alpha,N}^+|x|^{\alpha-1}\prod_{j=1}^{N+1}[\ln_j(\gamma/|x|)]^{-1} + |x|^{\alpha-1}\prod_{j=1}^{N+1}[\ln_j(\gamma/|x|)]^{-1}T_{\alpha,N}\Bigg) \\
&\quad + (1/4)|x|^{2\alpha-2}\prod_{j=1}^{N+1}[\ln_j(\gamma/|x|)]^{-2} \\
&= T_{\alpha,N}^+ T_{\alpha,N} - (1/4)|x|^{2\alpha-2}\prod_{j=1}^{N+1}[\ln_j(\gamma/|x|)]^{-2} \\
&= -|x|^{2\alpha}\partial_r^2 - (n-1+2\alpha)|x|^{2\alpha-1}\partial_r - [(n-2+2\alpha)/2]^2|x|^{2\alpha-2} \\
&\quad - (1/4)|x|^{2\alpha-2}\sum_{k=1}^{N+1}\prod_{j=1}^{k}[\ln_j(\gamma/|x|)]^{-2}. \qquad (5.2.12)
\end{aligned}
$$

□

Given these preliminaries, now we can show the following result.

Theorem 5.2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, with $x_0 \in \Omega$.*

(i) *Then, for all $f \in C_0^\infty(\Omega\backslash\{x_0\})$,*

$$
\begin{aligned}
\int_\Omega d^n x\, |x-x_0|^\alpha |(\nabla f)(x)|^2 &\geqslant \int_\Omega d^n x\, |x-x_0|^\alpha |(\partial_{r,x_0} f)(x)|^2 \\
&\geqslant \frac{(n-2+\alpha)^2}{4}\int_\Omega d^n x\, |x-x_0|^{\alpha-2}|f(x)|^2. \qquad (5.2.13)
\end{aligned}
$$

(ii) *Let* $N \in \mathbb{N}$, *and suppose in addition that* $\Omega \subset \mathbb{R}^n$ *is bounded. Assume* $\gamma > 0$ *is such that* $0 < \operatorname{diam}(\Omega) < \gamma/e_N$, *where* e_N *is as in* (5.1.4), *and let* $\ln_j(\,\cdot\,), j \in \mathbb{N}$, *be as in* (5.1.3), (5.1.4). *Then for all* $f \in C_0^\infty(\Omega\backslash\{x_0\})$ *we have*

$$\int_\Omega d^n x\, |x-x_0|^\alpha |(\nabla f)(x)|^2 \geqslant \int_\Omega d^n x\, |x-x_0|^\alpha |(\partial_{r,x_0} f)(x)|^2$$
$$\geqslant \int_\Omega d^n x\, |x-x_0|^{\alpha-2} |f(x)|^2 \bigg\{\frac{(n-2+\alpha)^2}{4} + \frac{1}{4}\sum_{k=1}^N \prod_{j=1}^k [\ln_j(\gamma/|x-x_0|)]^{-2}\bigg\}. \quad (5.2.14)$$

Proof. It suffices to focus on item (ii) only. As a first step we establish the latter in the special case where $\Omega = B_n(0;\rho)$, $x_0 = 0$, with $\rho, \gamma > 0$ and $\rho < \gamma/e_N$. Thus, we will prove that, for all $f \in C_0^\infty(B_n(0;\rho))$,

$$\int_{B_n(0;\rho)} d^n x\, |(\nabla f)(x)|^2 \geqslant \int_{B_n(0;\rho)} d^n x\, |(\partial_r f)(x)|^2$$
$$\geqslant \int_{B_n(0;\rho)} d^n x\, |x|^{-2} |f(x)|^2 \bigg\{\frac{(n-2+\alpha)^2}{4} + \frac{1}{4}\sum_{k=1}^N \prod_{j=1}^k [\ln_j(\gamma/|x|)]^{-2}\bigg\}. \quad (5.2.15)$$

Define $T_{\alpha,N}$ and $T_{\alpha,N}^+$ as in (5.2.1)–(5.2.4), respectively. For simplicity, we will work with $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$ for $n \in \mathbb{N}$. However, all integrals extend to $f \in C_0^\infty(B_n(0;\rho))$.

By Lemma 5.2.3, one has

$$0 \leqslant \int_{B_n(0;\rho)} d^n x\, |(T_{\alpha,N} f)(x)|^2 = \int_{B_n(0;\rho)} d^n x\, \overline{f(x)} (T_{\alpha,N}^+ T_{\alpha,N} f)(x)$$
$$= -\int_{B_n(0;\rho)} d^n x\, |x|^{2\alpha} \overline{f(x)} (\partial_r^2 f)(x) - (n-1+2\alpha) \int_{B_n(0;\rho)} d^n x\, |x|^{2\alpha-1} \overline{f(x)} (\partial_r f)(x)$$
$$- [(n-2+2\alpha)/2]^2 \int_{B_n(0;\rho)} d^n x\, |x|^{2\alpha-2} |f(x)|^2$$
$$- (1/4) \sum_{k=1}^N \int_{B_n(0;\rho)} d^n x\, |x|^{2\alpha-2} |f(x)|^2 \prod_{j=1}^k [\ln_j(\gamma/|x|)]^{-2}. \quad (5.2.16)$$

Considering the identity,

$$\begin{aligned}\int\limits_{B_n(0;\rho)} d^n x\,|x|^{2\alpha}\overline{f(x)}(\partial_r^2 f)(x) &= -\int\limits_{B_n(0;\rho)} d^n x\,|x|^{2\alpha}|(\partial_r f)(x)|^2 \\ &\quad -(n-1+2\alpha)\int\limits_{B_n(0;\rho)} d^n x\,|x|^{2\alpha-1}\overline{f(x)}(\partial_r f)(x), \quad f\in C_0^\infty(B_n(0;\rho)),\end{aligned} \tag{5.2.17}$$

(5.2.16) becomes

$$\begin{aligned}0 &\leqslant \int\limits_{B_n(0;\rho)} d^n x\,|(T_{\alpha,N}f)(x)|^2 \\ &= \int\limits_{B_n(0;\rho)} d^n x\,|x|^{2\alpha}|(\partial_r f)(x)|^2 - [(n-2+2\alpha)/2]^2 \int\limits_{B_n(0;\rho)} d^n x\,|x|^{2\alpha-2}|f(x)|^2 \\ &\quad -(1/4)\sum_{k=1}^N \int\limits_{B_n(0;\rho)} d^n x\,|x|^{2\alpha-2}|f(x)|^2 \prod_{j=1}^k [\ln_j(\gamma/|x|)]^{-2},\end{aligned} \tag{5.2.18}$$

Rearranging (5.2.18), and replacing $\alpha\in\mathbb{R}$ by $\alpha/2\in\mathbb{R}$, yields

$$\begin{aligned}&\int\limits_{B_n(0;\rho)} d^n x\,|x|^{\alpha}|(\partial_r f)(x)|^2 \\ &\quad\geqslant \int\limits_{B_n(0;\rho)} d^n x\,|x|^{\alpha-2}|f(x)|^2 \bigg\{\frac{(n-2+\alpha)^2}{4} + \frac{1}{4}\sum_{k=1}^N\prod_{j=1}^k [\ln_j(\gamma/|x|)]^{-2}\bigg\}.\end{aligned} \tag{5.2.19}$$

Next, let $\Omega = B_n(x_0;\rho)\subset\mathbb{R}^n$. The proof of (5.2.14) is entirely similar to that of (5.2.15), upon replacing $T_{\alpha,N}$ by

$$\begin{aligned}T_{\alpha,N,x_0} &:= |x-x_0|^{\alpha}\partial_{r,x_0} + [(n-2+2\alpha)/2]|x-x_0|^{\alpha-1} \\ &\qquad + (1/2)|x-x_0|^{\alpha-1}\sum_{k=1}^N\prod_{j=1}^k [\ln_j(\gamma/|x-x_0|)]^{-1},\end{aligned} \tag{5.2.20}$$

and similarly, replacing $T_{\alpha,N}^+$ by

$$\begin{aligned}T_{\alpha,N,x_0}^+ &= -\,|x-x_0|^{\alpha}\partial_{r,x_0} - (n/2)|x-x_0|^{\alpha-1} \\ &\qquad + (1/2)|x-x_0|^{\alpha-1}\sum_{k=1}^N\prod_{j=1}^k [\ln_j(\gamma/|x-x_0|)]^{-1}.\end{aligned} \tag{5.2.21}$$

It then follows that

$$
\begin{aligned}
&T^{+}_{\alpha,N,x_0}T_{\alpha,N,x_0} = -|x-x_0|^{2\alpha}\partial^2_{r,x_0} - (n-1+2\alpha)|x-x_0|^{2\alpha-1}\partial_{r,x_0} \\
&- [(n-2+2\alpha)/2]^2|x-x_0|^{2\alpha-2} - (1/4)|x-x_0|^{2\alpha-2}\sum_{k=1}^{N}\prod_{j=1}^{k}[\ln_j(\gamma/|x-x_0|)]^{-2},
\end{aligned}
\tag{5.2.22}
$$

and continuing as in the proof of (5.2.15) yields (5.2.14) for $\Omega = B_n(x_0;\rho)$.

For an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$ with some fixed $x_0 \in \Omega$, we pick some $\rho > 0$ such that $0 < \operatorname{diam}(\Omega) < \rho < \gamma/e_N$. Since $C_0^\infty(\Omega) \subseteq C_0^\infty(B_n(x_0;\rho))$ (extending functions in $C_0^\infty(\Omega)$ by zero outside Ω), inequality (5.2.14) follows. $\square$

Remark 5.2.5. (i) If in addition, we assume $\alpha \geqslant 2-n$ then the functions $f \in C_0^\infty(\Omega\backslash\{x_0\})$ in Theorem 5.2.4 can be replaced by $f \in C_0^\infty(\Omega)$.

(ii) Upon referring to the spherically symmetric case and oscillation theory for the second order differential expression

$$
-\frac{d^2}{dr^2} - \frac{1}{4r^2} - \frac{1}{4r^2}\sum_{k=1}^{N}\prod_{j=1}^{k}[\ln_j(\gamma/r)]^{-2}, \tag{5.2.23}
$$

with $r > 0$ for $N = 0$ and $0 < r < \gamma/e_N$ for $n \in \mathbb{N}$, discussed in [56], one verifies that the constants $(n-2)^2/4$ and $1/4$ in (5.2.14) are optimal.

(iii) We note that our proof of (5.2.15), most likely, is not the shortest possible one, but brevity was not the point we had in mind. Instead, as was demonstrated in [50] (see also [108]), the value of our strategy of proof, relying on factorizations as in (5.2.9), lies in the wide applicability of this approach to higher-order inequalities, such as the well-known Rellich inequality and beyond. This will be more systematically explored elsewhere [52]. We also note that Theorem 5.2.4 can be further generalized by combining the results found in Chapter Four with the method of polar coordinates shown in Chapter Six. $\diamond$

We conclude with some applications of (5.2.13), (5.2.14) to Schrödinger operators with strongly singular potentials. Let $J \subseteq \mathbb{N}$ be an index set, and let

$\{x_j\}_{j\in J} \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 2$, be a set of points such that

$$\inf_{\substack{j,j'\in J\\ j\neq j'}} |x_j - x_{j'}| \geqslant \varepsilon_0 \tag{5.2.24}$$

for some $\varepsilon_0 > 0$. In addition, let $n \in \mathbb{N}$, let $\xi_j, \eta_j \in \mathbb{R}$, $j \in J$, and let $\delta, \gamma, \xi, \eta \in (0,\infty)$ with

$$|\xi_j| \leqslant \xi < (n-2)^2/4, \quad |\eta_j| \leqslant \eta < 1/4, \quad j \in J, \quad 0 < \varepsilon_0 < 4\gamma/e_N, \quad n \geqslant 3. \tag{5.2.25}$$

Next, we introduce the potential

$$W(x) = \sum_{j\in J} e^{-\delta|x-x_j|}\bigg[\frac{\xi_j}{|x-x_j|^2} + \eta_j \chi_{B_n(x_j;\varepsilon_0/4)}(x) \sum_{\ell=1}^{m}\prod_{k=1}^{\ell}[\ln_j(\gamma/|x-x_j|)]^{-2}\bigg],$$
$$x \in \mathbb{R}^n \setminus \{x_j\}_{j\in J},\ n \geqslant 3, \tag{5.2.26}$$

with χ_M the characteristic function of $M \subset \mathbb{R}^n$.

Then an application of (5.2.14) (actually, (5.2.15) with $\rho = \varepsilon_0/4$) combined with [54, Theorem 3.2] shows that W (and hence, any scalar potential V satisfying $|V| \leqslant |W| + W_0$ a.e. on $\mathbb{R}^n$, with $0 \leqslant W_0 \in L^\infty(\mathbb{R}^n)$) is form bounded with respect to $H_0 = -\Delta$, $\mathrm{dom}(H_0) = H^2(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, $n \geqslant 3$, with form bound strictly less than one (cf. also [31, p. 28–29], and the example in [54, p. 1033–1034]). In this context we recall that $\mathrm{dom}\big(H_0^{1/2}\big) = H^1(\mathbb{R}^n)$, and that $C_0^\infty(\mathbb{R}^n)$ is a form core for H_0.

Finally, when we replace (5.2.26) by

$$W(x) = \sum_{j\in J} e^{-\delta|x-x_j|}\eta_j \chi_{B_n(x_j;\varepsilon_0/4)}(x) \sum_{\ell=1}^{m}\prod_{k=1}^{\ell}[\ln_j(\gamma/|x-x_j|)]^{-2},$$
$$x \in \mathbb{R}^2 \setminus \{x_j\}_{j\in J}, \tag{5.2.27}$$

with $\delta, \gamma, \eta \in (0,\infty)$ and $|\eta_j| \leqslant \eta < 1/4$, $j \in J$, $0 < \varepsilon_0 < 4\gamma/e_N$, these form boundedness considerations with respect to $H_0 = -\Delta$, $\mathrm{dom}(H_0) = H^2(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$, with form bound strictly less than one, extend to the case where $n = 2$.

CHAPTER SIX

On Multidimensional Power Weighted Birman–Hardy–Rellich-type Inequalities with Radial Refinements

6.1 Introduction

We begin by recalling the Birman–Hardy–Rellich integral inequalities

$$\int_0^\infty dx\,\big|f^{(m)}(x)\big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}}\int_0^\infty dx\,x^{-2m}|f(x)|^2, \quad m\in\mathbb{N}, \tag{6.1.1}$$

valid for $f \in C_0^\infty((0,\infty))$, proved by M. Š. Birman in [19]. Since the establishment of (6.1.1), a great amount of research has been dedicated to the pursuit of improving these inequalities, such as: extending to $p \in [1,\infty)$, logarithmically weaker singular potentials, optimal function spaces, vector-valued functions, weight functions, multidimensional domains, and radial derivatives/Laplacian. This note focuses on the last five items, discussing new results in these areas.

We introduce the radial derivative ∂_r, given by

$$\partial_r := |x|^{-1}x\cdot\nabla, \qquad x\in\mathbb{R}^n\backslash\{0\},\ r=|x|,\ n\in\mathbb{N},\ n\geqslant 2, \tag{6.1.2}$$

as well as the radial Laplacian Δ_r,

$$\Delta_r := r^{1-n}\partial_r r^{n-1}\partial_r = \partial_r^2 + (n-1)r^{-1}\partial_r. \tag{6.1.3}$$

Recall that the Laplacian Δ, in n-dimensional polar coordinates, can be expressed as

$$\Delta = \Delta_r + r^{-2}\Delta_{S^{n-1}}, \tag{6.1.4}$$

where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on the $(n-1)$-sphere $S^{n-1}\subset\mathbb{R}^n$. We will be considering the radial differential expressions $\partial_r(-\Delta_r)^{m-1}$, and respectively $(-\Delta_r)^m$, for $m\in\mathbb{N}$, interpreting $\partial_r(-\Delta_r)^0 := \partial_r$.

Given $m, n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, we define the constants

$$\begin{aligned} A'(m,n,\alpha) &:= \prod_{j=1}^{\lfloor m/2 \rfloor} \left(\frac{n+2m-4j-\alpha}{2}\right)^2, \\ A''(m,n,\alpha) &:= \prod_{k=0}^{\lfloor (m-1)/2 \rfloor} \left(\frac{n-2m+4k+\alpha}{2}\right)^2, \\ A(m,n,\alpha) &:= A'(m,n,\alpha)A''(m,n,\alpha), \end{aligned} \tag{6.1.5}$$

denoting $\prod_{j=1}^{0}(\,\cdot\,) := 1$.

Remark 6.1.1. Given $m, n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, we make the following observations:

(i) If m is odd,

$$A(m,n,\alpha) = \prod_{j=1}^{(m-1)/2} \left(\frac{n-2+4j-\alpha}{2}\right)^2 \prod_{k=1}^{(m+1)/2} \left(\frac{n+2-4k+\alpha}{2}\right)^2. \tag{6.1.6}$$

(ii) If m is even,

$$A(m,n,\alpha) = \prod_{j=1}^{m/2} \left(\frac{n-4+4j-\alpha}{2}\right)^2 \prod_{k=1}^{m/2} \left(\frac{n-4k+\alpha}{2}\right)^2. \tag{6.1.7}$$

◇

The multidimensional analogue of (6.1.1) for the special Hardy case $m = 1$, with the gradient ∇ replaced by radial derivative ∂_r, was established in [51],

$$\begin{aligned} \int_\Omega d^n x\, |x|^\alpha \big|(\partial_r f)(x)\big|^2 \geqslant \frac{(n-2+\alpha)^2}{4} \int_\Omega d^n x\, |x|^{\alpha-2} |f(x)|^2, \\ f \in C_0^\infty(\mathbb{R}^n \backslash \{0\}), \quad n \in \mathbb{N}, \end{aligned} \tag{6.1.8}$$

as a consequence of a more general result that included iterated logarithmic refinement terms. In this note, we continue the notion of radial refinements in (6.1.8) to include all higher ordered derivatives in this multidimensional context, replacing the standard differential expressions $\nabla(-\Delta)^m$, $(-\Delta)^m$, $m \in \mathbb{N}$, (see for instance [12–15, 114]) with their radial counterparts $\partial_r(-\Delta_r)^m$, $(-\Delta_r)^m$, $m \in \mathbb{N}$.

In section 6.2 we give a brief review of integration in polar coordinates, recalling how the Lebesgue-induced Borel measure $\widetilde{m}$ on $(0,\infty) \times S^{n-1}$ can be expressed as a product $\widetilde{m} = \sigma \times \omega$, where σ denotes the measure $r^{n-1}dr$ on $(0, \infty)$ and $\omega = \omega_{n-1}$ denotes the surface measure on the unit sphere S^{n-1}. This construction yields the integral identity,

$$\int_{\mathbb{R}^n} d^n x\, f(x) = \int_{S^{n-1}} d\omega(\theta) \int_0^\infty r^{n-1} dr\, f(r, \theta), \quad f \in L^1(\mathbb{R}^n), \tag{6.1.9}$$

with an analogous result for $f \in L^1(B_n(0; \rho))$, where $B_n(0; \rho) \subset \mathbb{R}^n$ denotes the open ball in $\mathbb{R}^n$ of radius $\rho \in (0, \infty)$ centered at the origin. In section 6.3, we establish the radial power weighted Birman-type inequalities

$$\int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big|(\partial_r(-\Delta_r)^{m-1} f)(x)\big|^2 \geqslant A(2m-1, n, \alpha) \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-4m+2} |f(x)|^2, \tag{6.1.10}$$

and analogously for the even ordered derivatives

$$\int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big|((-\Delta_r)^m f)(x)\big|^2 \geqslant A(2m, n, \alpha) \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-4m} |f(x)|^2, \tag{6.1.11}$$

for all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, where $m, n \in \mathbb{N}$, and $\alpha \in \mathbb{R}$.

In section 6.4 we reconsider the weighted Birman-type inequalities on bounded domains $\Omega \subset \mathbb{R}^n$ with the standard power weight $|x|^\alpha, \alpha \in \mathbb{R}$, replaced by functions given as the shortest, and farthest, distance to the boundary $\partial\Omega = \overline{\Omega} \backslash \Omega^\circ$, in the special case $\Omega = B_n(0; \rho)$. These distance functions δ and φ on Ω, are given by

$$\delta(x) := \inf_{y \in \partial\Omega} |x - y|, \qquad \varphi(x) := \sup_{y \in \partial\Omega} |x - y|, \qquad x \in \Omega. \tag{6.1.12}$$

Following similar methods to section 6.3, we establish Birman-type inequalities,

$$\int_a^b dr\, \delta(r)^\alpha \big|F^{(m)}(r)\big|^2 \geqslant A(m, 1, \alpha) \int_a^b dr\, \delta(r)^{\alpha-2m} |F(r)|^2. \tag{6.1.13}$$

for all $F \in H_0^m((a, b); \delta(r)^\alpha dr)$, where $m \in \mathbb{N}, a, b, \alpha \in \mathbb{R}$, $a < b$ with $\alpha < 1$, and also

$$\int_a^b dr\, \varphi(r)^\alpha \big|F^{(m)}(r)\big|^2 \geqslant A(m, 1, \alpha) \int_a^b dr\, \varphi(r)^{\alpha-2} |F(r)|^2. \tag{6.1.14}$$

for all $F \in H_0^m((a,b);\varphi(r)^\alpha dr)$, where $m \in \mathbb{N}, a, b, \alpha \in \mathbb{R}$, $a < b$ with $\alpha > 2m-1$, where, given a weight function $0 \leqslant w \in L^1_{loc}((a,b);dx)$, $H_0^m((a,b);w(r)dr)$ denotes the standard, w-weighted Sobolev space on (a,b) obtained upon completion of $C_0^\infty((a,b))$ in the norm of $H^m((a,b);w(r)dr)$. The multidimensional analogue of (6.1.13), (6.1.14), is then established:

$$\begin{aligned} &\int_{B_n(0;\rho)} d^n x\, \delta(x)^\alpha \big|(\partial_r(-\Delta_r)^{m-1}f)(x)\big|^2 \\ &\quad \geqslant A(2m-1,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \delta(x)^{\alpha-4m+2}|f(x)|^2, \end{aligned} \tag{6.1.15}$$

and

$$\begin{aligned} &\int_{B_n(0;\rho)} d^n x\, \delta(x)^\alpha \big|((-\Delta_r)^m f)(x)\big|^2 \\ &\quad \geqslant A(2m,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \delta(x)^{\alpha-4m}|f(x)|^2, \end{aligned} \tag{6.1.16}$$

for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$, where $m,n \in \mathbb{N}$, $\rho \in (0,\infty)$, $\alpha \in \mathbb{R}$, and $\alpha < 2-n$. A similar result to (6.1.15), (6.1.16), with certain modified constants $\widetilde{A}(m,n,\alpha)$, $m,n \in \mathbb{N}, \alpha \in \mathbb{R}$, which we introduce in (6.4.11), is then established for the farthest distance to the boundary:

$$\begin{aligned} &\int_{B_n(0;\rho)} d^n x\, \varphi(x)^\alpha \big|(\partial_r(-\Delta_r)^{m-1}f)(x)\big|^2 \\ &\quad \geqslant \widetilde{A}(2m-1,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha-4m+2}|f(x)|^2, \end{aligned} \tag{6.1.17}$$

for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$, where $m,n \in \mathbb{N}$, $\rho \in (0,\infty)$, $\alpha \in \mathbb{R}$, and $\alpha > 4m-2-n$; as well as

$$\begin{aligned} &\int_{B_n(0;\rho)} d^n x\, \varphi(x)^\alpha \big|((-\Delta_r)^m f)(x)\big|^2 \\ &\quad \geqslant \widetilde{A}(2m,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha-4m}|f(x)|^2, \end{aligned} \tag{6.1.18}$$

for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$, where $\alpha > 4m-n$.

Furthermore, we show that all inequalities given are strict for all $f \not\equiv 0$ (resp. $F \not\equiv 0$). Finally, in section 6.5 we indicate that all results naturally extend to $\mathcal{H}$-valued functions, where $\mathcal{H}$ is a complex, separable Hilbert space.

We begin by briefly reviewing n-dimensional integration using polar coordinates.

6.2 *Integration with Polar Coordinates*

In this section, we briefly review a canonical result of n-dimensional integration in polar coordinates that allows one to express the Lebesgue measure in $\mathbb{R}^n$ as a product of the measure $r^{n-1}dr$ on $(0,\infty)$ and the surface measure on the unit sphere. See, for instance [48, Section 2.7] for more details.

Denoting the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ by S^{n-1}, the polar coordinates of $x \in \mathbb{R}^n \backslash \{0\}$ are given by

$$r = |x| \in (0,\infty), \qquad \theta = x/|x| \in S^{n-1}. \tag{6.2.1}$$

Recall $\mathbb{R}^n \backslash \{0\} \cong (0,\infty) \times S^{n-1}$ via the bijection

$$\Phi : \mathbb{R}^n \backslash \{0\} \to (0,\infty) \times S^{n-1}, \quad x \mapsto (r,\theta), \tag{6.2.2}$$

with inverse $\Phi^{-1}(r,\theta) = r\theta$, allowing us to identify functions $f(x)$ on $\mathbb{R}^n \backslash \{0\}$ as functions $f(r,\theta)$ on $(0,\infty) \times S^{n-1}$.

We denote by $\widetilde{m}$ the Borel measure on $(0,\infty) \times S^{n-1}$ given by

$$\widetilde{m}(E) := m(\Phi^{-1}(E)), \quad E \subseteq (0,\infty) \times S^{n-1}, \tag{6.2.3}$$

where m denotes standard Lebesgue measure on $\mathbb{R}^n$. Also, we define the measure σ on $(0,\infty)$ via

$$\sigma(E) := \int_E r^{n-1} dr, \quad E \subset (0,\infty). \tag{6.2.4}$$

Finally, let $\omega = \omega_{n-1}$ denote the unique Borel measure on S^{n-1} (see the proof of [48, Theorem 2.49] for an explicit construction) satisfying

$$\widetilde{m} = \sigma \times \omega. \tag{6.2.5}$$

Given this notation, and recalling Fubini-Tonelli, for any $f \in L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} d^n x\, f(x) = \int_0^\infty r^{n-1} dr \int_{S^{n-1}} d\omega(\theta)\, f(r,\theta) = \int_{S^{n-1}} d\omega(\theta) \int_0^\infty r^{n-1} dr\, f(r,\theta). \tag{6.2.6}$$

Analogously, if $B_n(0;\rho) \subset \mathbb{R}^n$ denotes $\{x \in \mathbb{R}^n : |x| < \rho\}$ the open ball in $\mathbb{R}^n$ of radius $\rho \in (0,\infty)$ centered at the origin, then using a similar construction as above, one has for any $f \in L^1(B_n(0;\rho))$,

$$\int_{B_n(0;\rho)} d^n x\, f(x) = \int_{S^{n-1}} d\omega(\theta) \int_0^\rho r^{n-1} dr\, f(r,\theta). \tag{6.2.7}$$

Furthermore, if $f \in L^1(\mathbb{R}^n)$ (resp. $f \in L^1(B_n(0;\rho))$) satisfies $f(x) = g(|x|)$ for some function g on $(0,\infty)$ (resp. $(0,\rho)$), then

$$\int_{\mathbb{R}^n} d^n x\, f(x) = \omega(S^{n-1}) \int_0^\infty r^{n-1} dr\, g(r), \tag{6.2.8}$$

and respectively,

$$\int_{B_n(0;\rho)} d^n x\, f(x) = \omega(S^{n-1}) \int_0^\rho r^{n-1} dr\, g(r). \tag{6.2.9}$$

The identities (6.2.6) and (6.2.7) will be particularly useful throughout sections 6.3 and 6.4, respectively.

6.3 Radial Power Weighted Birman–Hardy–Rellich-type Inequalities

Using the elementary notion of polar coordinates from section 6.2, we establish the multidimensional Birman inequalities with power weights and radial refinements.

6.3.1 Hardy and Rellich Inequalities

We first give a trivial proof of the multidimensional Hardy inequality with the gradient ∇ replaced by the radial derivative ∂_r using polar coordinates from section 6.2. A more general result, that includes logarithmic refinement terms, was proven in [51] using factorizations of differential operators.

Proposition 6.3.1. *Fix $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then for all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$,*

$$\int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big|(\partial_r f)(x)\big|^2 \geqslant \frac{(n-2+\alpha)^2}{4} \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-2} |f(x)|^2. \tag{6.3.1}$$

Moreover, inequality (6.3.1) *is strict for $f \not\equiv 0$.*

Proof. Let $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$. Using (6.2.6), and applying the one-dimensional Hardy inequality to

$$F(\,\cdot\,) := f(\,\cdot\,, \theta) \in C_0^\infty((0,\infty)), \tag{6.3.2}$$

for each fixed $\theta \in S^{n-1}$, yields

$$\begin{aligned}
\int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big|(\partial_r f)(x)\big|^2 &= \int_{S^{n-1}} d\omega(\theta) \int_0^\infty dr\, r^{n-1+\alpha} \big|(\partial_r f)(r,\theta)\big|^2 \\
&\geqslant \int_{S^{n-1}} d\omega(\theta) \, \frac{(n-2+\alpha)^2}{4} \int_0^\infty dr\, r^{n-3+\alpha} |f(r,\theta)|^2 \\
&= \frac{(n-2+\alpha)^2}{4} \int_{S^{n-1}} d\omega(\theta) \int_0^\infty dr\, r^{n-3+\alpha} |f(r,\theta)|^2 \\
&= \frac{(n-2+\alpha)^2}{4} \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-2} |f(x)|^2.
\end{aligned} \tag{6.3.3}$$

Furthermore, if $f \not\equiv 0$ then $F \not\equiv 0$ and hence the inequality in (6.3.3) is strict. □

We now establish the power weighted Rellich inequality with the Laplacian Δ replaced by its radial counterpart Δ_r.

Lemma 6.3.2. *Fix $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then for all $F \in C_0^\infty((0,\infty))$,*

$$\begin{aligned}
&\int_0^\infty dr\, r^{n-1+\alpha} \big|F''(r) + (n-1) r^{-1} F'(r)\big|^2 \\
&\quad \geqslant \frac{(n-\alpha)^2 (n-4+\alpha)^2}{16} \int_0^\infty dr\, r^{n-5+\alpha} |F(r)|^2.
\end{aligned} \tag{6.3.4}$$

Moreover, inequality (6.3.4) *is strict for $F \not\equiv 0$.*

Proof. Let $F \in C_0^\infty((0,\infty))$, and without loss of generality assume F is $\mathbb{R}$-valued.

Integration by parts, and two applications of the Hardy inequality, yield

$$\int_0^\infty dr\, r^{n-1+\alpha} \big[F''(r) + (n-1) r^{-1} F'(r)\big]^2$$

$$
\begin{aligned}
&= \int_0^\infty dr\, r^{n-1+\alpha}\big[F''(r)\big]^2 + 2(n-1)\int_0^\infty dr\, r^{n-2+\alpha}F'(r)F''(r) \\
&\quad + (n-1)^2\int_0^\infty dr\, r^{n-3+\alpha}\big[F'(r)\big]^2 \\
&= \int_0^\infty dr\, r^{n-1+\alpha}\big[F''(r)\big]^2 + [(n-1)^2 - (n-1)(n-2+\alpha)]\int_0^\infty dr\, r^{n-3+\alpha}\big[F'(r)\big]^2 \\
&\geqslant [\tfrac{1}{4}(n-2+\alpha)^2 + (n-1)^2 - (n-1)(n-2+\alpha)]\int_0^\infty dr\, r^{n-3+\alpha}\big[F'(r)\big]^2 \\
&= \frac{(n-\alpha)^2}{4}\int_0^\infty dr\, r^{n-3+\alpha}\big[F'(r)\big]^2 \\
&\geqslant \frac{(n-\alpha)^2(n-4+\alpha)^2}{16}\int_0^\infty dr\, r^{n-5+\alpha}[F(r)]^2. \qquad (6.3.5)
\end{aligned}
$$

Furthermore, if $F \not\equiv 0$ the last inequality in (6.3.5) is strict. □

Proposition 6.3.3. *Fix $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then for all $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$,*

$$
\int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big|(-\Delta_r f)(x)\big|^2 \geqslant \frac{(n-\alpha)^2(n-4+\alpha)^2}{16}\int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-4}|f(x)|^2. \qquad (6.3.6)
$$

Moreover, inequality (6.3.6) *is strict for $f \not\equiv 0$.*

Proof. Let $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$. Using (6.2.6), and applying Lemma 6.3.2 to

$$
F(\,\cdot\,) := f(\,\cdot\,,\theta) \in C_0^\infty((0,\infty)), \qquad (6.3.7)
$$

for each fixed $\theta \in S^{n-1}$, yields

$$
\begin{aligned}
&\int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big|(-\Delta_r f)(x)\big|^2 \\
&\quad = \int_{S^{n-1}} d\omega(\theta)\int_0^\infty dr\, r^{n-1+\alpha}\big|(\partial_r^2 f)(r,\theta) + (n-1)r^{-1}(\partial_r f)(r,\theta)\big|^2 \\
&\quad \geqslant \int_{S^{n-1}} d\omega(\theta)\,\frac{(n-\alpha)^2(n-4+\alpha)^2}{16}\int_0^\infty dr\, r^{n-5+\alpha}|f(r,\theta)|^2 \\
&\quad = \frac{(n-\alpha)^2(n-4+\alpha)^2}{16}\int_{S^{n-1}} d\omega(\theta)\int_0^\infty dr\, r^{n-5+\alpha}|f(r,\theta)|^2 \\
&\quad = \frac{(n-\alpha)^2(n-4+\alpha)^2}{16}\int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-4}|f(x)|^2. \qquad (6.3.8)
\end{aligned}
$$

Furthermore, if $f \not\equiv 0$ then $F \not\equiv 0$ so (6.3.8) is strict by Lemma 6.3.2. □

6.3.2 Birman Inequality

We conclude this section by proving the multidimensional power weighted Birman inequalities with $\nabla(-\Delta)^m$ and $(-\Delta)^m$ replaced by their radial analogues $\partial_r(-\Delta_r)^m$, and $(-\Delta_r)^m$, respectively.

Theorem 6.3.4. *Fix $m, n \in \mathbb{N}$, and $\alpha \in \mathbb{R}$. For all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$,*

$$(i) \int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big|(\partial_r(-\Delta_r)^{m-1} f)(x)\big|^2 \geqslant A(2m-1, n, \alpha) \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-4m+2} |f(x)|^2; \tag{6.3.9}$$

$$(ii) \int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big|((-\Delta_r)^m f)(x)\big|^2 \geqslant A(2m, n, \alpha) \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-4m} |f(x)|^2. \tag{6.3.10}$$

Moreover, inequalities (6.3.9)*and* (6.3.10) *are strict for $f \not\equiv 0$.*

Proof. For part (i), we use induction over $m \in \mathbb{N}$. The case $m = 1$ holds by Proposition 6.3.1. Assuming (6.3.9) holds for $m \in \mathbb{N}$, inductive hypothesis and Proposition 6.3.3 yields for $m + 1$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |x|^\alpha \big|(\partial_r(-\Delta_r)^m f)(x)\big|^2 \\
&\quad = \int_{\mathbb{R}^n} |x|^\alpha \big|[\partial_r(-\Delta_r)^{m-1}(-\Delta_r) f](x)\big|^2 \\
&\quad \geqslant A(2m-1, n, \alpha) \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-4m+2} \big|(-\Delta_r f)(x)\big|^2 \\
&\quad \geqslant A(2m-1, n, \alpha) A(2, n, \alpha - 4m + 2) \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-4m-2} |f(x)|^2.
\end{aligned} \tag{6.3.11}$$

Recalling Remark 6.1.1 (i),

$$\begin{aligned}
&A(2m-1, n, \alpha) A(2, n, \alpha - 4m + 2) \\
&\quad = \prod_{j=1}^{m-1} \left(\frac{n - 2 + 4j - \alpha}{2}\right)^2 \prod_{k=1}^{m} \left(\frac{n + 2 - 4k + \alpha}{2}\right)^2 \\
&\qquad \times \left(\frac{n - (\alpha - 4m + 2)}{2}\right)^2 \left(\frac{n - 4 + (\alpha - 4m + 2)}{2}\right)^2 \\
&\quad = \prod_{j=1}^{m-1} \left(\frac{n - 2 + 4j - \alpha}{2}\right)^2 \prod_{k=1}^{m} \left(\frac{n + 2 - 4k + \alpha}{2}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{n-2+4m-\alpha}{2}\right)^2 \left(\frac{n-4m-2+\alpha}{2}\right)^2 \\
&= \prod_{j=1}^{m} \left(\frac{n-2+4j-\alpha}{2}\right)^2 \prod_{k=1}^{m+1} \left(\frac{n+2-4k+\alpha}{2}\right)^2 \\
&= A(2m+1, n, \alpha). \qquad (6.3.12)
\end{aligned}
$$

Furthermore, if $f \not\equiv 0$ the last inequality in (6.3.11) is strict by Proposition 6.3.3.

Part (ii) follows similarly, using Proposition 6.3.3 for the case $m = 1$ and recalling Remark 6.1.1 (ii) to show $A(2m, n, \alpha)A(2, n, \alpha - 4m) = A(2m+2, n, \alpha)$. $\square$

Remark 6.3.5. The constants $A(m, n, \alpha)$ in Theorem 6.3.4 are the well-known optimal constants for such inequalities. See for instance [12, 13, 15, 29, 53, 59, 114].

Furthermore, the function space $C_0^\infty(\mathbb{R}^n \backslash \{0\})$ in Theorem 6.3.4 can be replaced by $C_0^\infty(\mathbb{R}^n)$ in part (i) if we further assume $\alpha > 4m - 2 - n$, and respectively, in part (ii) if $\alpha > 4m - n$. $\diamond$

6.4 Weighted Sobolev Spaces and Distance to the Boundary

Having established the radial Birman inequalities with standard power weights $|x|^\alpha$, $\alpha \in \mathbb{R}$, we now wish to consider the more general power-type weight functions given by the shortest, and farthest, distance to the boundary $\partial\Omega = \overline{\Omega} \backslash \Omega^\circ$ of a bounded open set $\Omega \subset \mathbb{R}^n$. In particular, we consider the special case $\Omega = B_n(0; \rho)$, $\rho \in (0, \infty)$. Our method of proof involves first extending the one-dimensional power weighted case to optimal Sobolev-type function spaces of left/right endpoint vanishing functions on the interval (a, b), $a, b \in \mathbb{R}, a < b$, then establishing the distance-weighted Birman inequalities by dividing the interval (a, b) in halves, and finally, extending to multi-dimensions as in section 6.3.

Thus, we begin by introducing the following Sobolev-type spaces:

Given $m \in \mathbb{N}$, $a, b \in \mathbb{R}$, $a < b$, let $0 \leqslant w \in L^1_{loc}((a, b); dr)$ be an arbitrary weight function. We introduce the function space $H_L^m([a,b]; w(r)dr)$, given by

$$\begin{aligned} H_L^m([a,b]; w(r)dr) &:= \{F : [a,b] \to \mathbb{C} \mid F^{(m)} \in L^2((a,b); w(r)dr); \\ &\qquad F^{(j)} \in AC([a,b]); F^{(j)}(a) = 0, j = 0, 1, \dots, m-1\}, \end{aligned} \tag{6.4.1}$$

and analogously, the space $H_R^m([a,b]; w(r)dr)$ of right vanishing functions,

$$\begin{aligned} H_R^m([a,b]; w(r)dr) &:= \{F : [a,b] \to \mathbb{C} \mid F^{(m)} \in L^2((a,b); w(r)dr); \\ &\qquad F^{(j)} \in AC([a,b]); F^{(j)}(b) = 0, j = 0, 1, \dots, m-1\}. \end{aligned} \tag{6.4.2}$$

We denote by $H_0^m((a,b); w(r)dr)$ the standard weighted Sobolev space on (a,b) obtained upon completion of $C_0^\infty((a,b))$ in the norm of $H^m((a,b); w(r)dr)$;

$$\begin{aligned} H^m((a,b); w(r)dr) = \big\{F : [a,b] \to \mathbb{C} \,\big|\, & F^{(j)} \in AC([a,b]),\ j = 0, 1, \dots, m-1; \\ & F^{(k)} \in L^2((a,b); w(r)dr),\ k = 0, 1, \dots, m\big\}, \end{aligned} \tag{6.4.3}$$

and

$$\begin{aligned} H_0^m((a,b); w(r)dr) = \big\{F \in H^m((a,b); w(r)dr) \,\big|\, & F^{(j)}(a) = F^{(j)}(b) = 0, \\ & j = 0, 1, \dots, m-1\big\}. \end{aligned} \tag{6.4.4}$$

Observe that for $m \in \mathbb{N}$, $m \geqslant 2$, and $k = 1, \dots, m-1$,

$$F \in H_{L/R}^m([a,b]; w(r)dr) \text{ implies } F^{(k)} \in H_{L/R}^{m-k}([a,b]; w(r)dr), \tag{6.4.5}$$

and analogously,

$$F \in H_0^m((a,b); w(r)dr) \text{ implies } F^{(k)} \in H_0^{m-k}((a,b); w(r)dr). \tag{6.4.6}$$

This fact will be useful throughout section 6.4.

The weight functions w considered in (6.4.1) and (6.4.2) will be the standard power-type functions $(r-a)^\alpha$, $(b-r)^\alpha$, for appropriate $\alpha \in \mathbb{R}$, yielding the spaces $H_{L/R}^m([a,b]; (r-a)^\alpha dr)$ and $H_{L/R}^m([a,b]; (b-r)^\alpha dr)$, respectively.

For the Sobolev space (6.4.4), we will consider the shortest, and farthest, distance functions to the boundary $\{a, b\}$ of the interval (a, b). We define these functions in the more general multidimensional setting below:

Given $n \in \mathbb{N}$, suppose $\Omega \subset \mathbb{R}^n$ is bounded and open with boundary $\partial\Omega = \overline{\Omega} \backslash \Omega^\circ$. We introduce the shortest distance function δ on Ω,

$$\delta(x) := \inf_{y \in \partial\Omega} |x - y|, \quad x \in \Omega, \tag{6.4.7}$$

and the farthest distance function φ on Ω, given by

$$\varphi(x) := \sup_{y \in \partial\Omega} |x - y|, \quad x \in \Omega. \tag{6.4.8}$$

For the special case $n = 1$, $\Omega = (a, b) \subset \mathbb{R}$, (6.4.7) and (6.4.8) become

$$\delta(r) = \min\{r - a, b - r\}, \qquad \varphi(r) = \max\{r - a, b - r\}, \qquad r \in (a, b). \tag{6.4.9}$$

Furthermore, if $n \in \mathbb{N}$, $\Omega = B_n(0; \rho)$ or $B_n(0; \rho)\backslash\{0\}$, then both δ and φ are radial. That is,

$$\delta(x) = \delta(|x|), \qquad \varphi(x) = \varphi(|x|), \qquad x \in B_n(0; \rho)\backslash\{0\}. \tag{6.4.10}$$

Finally, we introduce the modified constants $\widetilde{A}(m, n, \alpha)$, $m, n \in \mathbb{N}, \alpha \in \mathbb{R}$, given by

$$\widetilde{A}(m, n, \alpha) := \begin{cases} \left(\dfrac{n-2+\alpha}{2}\right)^2 \displaystyle\prod_{j=1}^{(m-1)/2} \left(\dfrac{3n-4j-2+\alpha}{2}\right)^2 \prod_{k=1}^{(m-1)/2} \left(\dfrac{n-4k-2+\alpha}{2}\right)^2, & \text{if } m \text{ is odd}, \\ \displaystyle\prod_{j=1}^{m/2} \left(\dfrac{3n-4j+\alpha}{2}\right)^2 \prod_{k=1}^{m/2} \left(\dfrac{n-4k+\alpha}{2}\right)^2, & \text{if } m \text{ is even}, \end{cases} \tag{6.4.11}$$

again with $\prod_{k=1}^{0}(\,\cdot\,) := 1$.

Remark 6.4.1. In the special case $n = 1$, observe that

$$\widetilde{A}(m,1,\alpha) = A(m,1,\alpha), \tag{6.4.12}$$

for all $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$. $\diamond$

Given these preliminaries, we now establish the distance-weighted Birman-type inequalities in one-dimension on the optimal function spaces, as well as their multidimensional analogues with radial refinements.

6.4.1 Hardy and Rellich Inequalities

We first establish the multidimensional radial Hardy inequality with the shortest/farthest distance weights, beginning with a trivial proof of the one-dimensional case on the optimal spaces of left/right vanishing functions of Sobolev type. An alternate proof of Lemma 6.4.2 can be derived mutatis mutandis from the unweighted result in [53, Theorem 7.1], and an even more general result can be found in [29, Example 3.2.3], for instance.

Lemma 6.4.2. *Fix $a, b, c, \alpha \in \mathbb{R}$, $a < c < b$, and assume $\alpha < 1$. The following hold:*

(i) For all $F \in H^1_L([a,b];(r-a)^\alpha dr)$,

$$\int_a^c dr\,(r-a)^\alpha \big|F'(r)\big|^2 \geqslant \frac{(1-\alpha)^2}{4}\int_a^c dr\,(r-a)^{\alpha-2}|F(r)|^2. \tag{6.4.13}$$

(ii) For all $F \in H^1_R([a,b];(b-r)^\alpha dr)$,

$$\int_c^b dr\,(b-r)^\alpha \big|F'(r)\big|^2 \geqslant \frac{(1-\alpha)^2}{4}\int_c^b dr\,(b-r)^{\alpha-2}|F(r)|^2. \tag{6.4.14}$$

Moreover, inequalities (6.4.13) *and* (6.4.14) *are strict for $F \not\equiv 0$.*

Proof. For part (i), without loss of generality take $a = 0$ and assume $0 \neq F \in H^1_L([0,b];r^\alpha dr)$ is $\mathbb{R}$-valued.

Integration by parts, and the Cauchy-Schwarz inequality (see Remark 6.4.3 below), yield

$$\int_0^c dr\,r^{\alpha-2}[F(r)]^2 = \frac{1}{\alpha-1}\int_0^c dr\,(r^{\alpha-1})'[F'(r)]^2$$

$$
\begin{aligned}
&= \frac{1}{\alpha-1}\left\{ r^{\alpha-1}F(r)\Big|_0^c - 2\int_0^c dr\, r^{\alpha-1}F(r)F'(r)\right\} \\
&= \frac{-c^{\alpha-1}F(c)}{1-\alpha} + \frac{2}{1-\alpha}\int_0^c dr\, r^{\alpha-1}F(r)F'(r) \\
&\leqslant \frac{2}{1-\alpha}\int_0^c dr\, r^{\alpha-1}F(r)F'(r) \\
&= \frac{2}{1-\alpha}\int_0^c dr\, \big(r^{\alpha/2-1}F(r)\big)\big(r^{\alpha/2}F'(r)\big) \\
&\leqslant \frac{2}{1-\alpha}\bigg(\int_0^c dr\, r^{\alpha-2}[F(r)]^2\bigg)^{1/2}\bigg(\int_0^c dr\, r^{\alpha}\big[F'(r)\big]^2\bigg)^{1/2}.
\end{aligned} \tag{6.4.15}
$$

Rearranging (6.4.15) yields the desired result;

$$
\int_0^c dr\, r^{\alpha}\big|F'(r)\big|^2 \geqslant \frac{(1-\alpha)^2}{4}\int_0^c dr\, r^{\alpha-2}|F(r)|^2. \tag{6.4.16}
$$

Strictness follows by Cauchy-Schwarz.

Part (ii) then holds via reflection across the interval midpoint $r = (a+b)/2$. $\square$

Remark 6.4.3. The condition $c \in (a,b)$ in Lemma 6.4.2 can be further extended to $c \in (a,b]$ for part (i) and $c \in [a,b)$ for part (ii). Indeed, for the space $H^1_L([a,b];(r-a)^{\alpha}dr)$ and respectively $H^1_R([a,b];(b-r)^{\alpha}dr)$, one can replace the boundary conditions at $r=a$ (resp. at $r=b$) by $F/(r-a) \in L^2((a,b);(r-a)^{\alpha}dr)$ (resp. by $F/(b-r) \in L^2((a,b);(b-r)^{\alpha}dr)$; hence, the Cauchy-Schwarz inequality in (6.4.15) may still be applied as both integrals are finite. See, for instance, [53] for similar discussions in the unweighted case $\alpha = 0$. $\diamond$

A similar result for $\alpha \in \mathbb{R}$, $\alpha > 1$, holds analogously.

Lemma 6.4.4. *Fix $a,b,c,\alpha \in \mathbb{R}$, $a < c < b$, and assume $\alpha > 1$. The following hold:*

(i) *For all $F \in H^1_L([a,b];(b-r)^{\alpha}dr)$,*

$$
\int_a^c dr\, (b-r)^{\alpha}\big|F'(r)\big|^2 \geqslant \frac{(1-\alpha)^2}{4}\int_a^c dr\, (b-r)^{\alpha-2}|F(r)|^2. \tag{6.4.17}
$$

(ii) *For all $F \in H^1_R([a,b];(r-a)^{\alpha}dr)$,*

$$
\int_c^b dr\, (r-a)^{\alpha}\big|F'(r)\big|^2 \geqslant \frac{(1-\alpha)^2}{4}\int_c^b dr\, (r-a)^{\alpha-2}|F(r)|^2. \tag{6.4.18}
$$

Moreover, inequalities (6.4.17) *and* (6.4.18) *are strict for $F \not\equiv 0$.*

Using Lemmas 6.4.2 and 6.4.4, we arrive at the following:

Proposition 6.4.5. *Fix $a, b, \alpha \in \mathbb{R}$, $a < b$. The following hold:*

(i) If $\alpha < 1$, then for all $F \in H_0^1((a,b);\delta(r)^\alpha dr)$,

$$\int_a^b dr\, \delta(r)^\alpha \big|F'(r)\big|^2 \geqslant \frac{(1-\alpha)^2}{4} \int_a^b dr\, \delta(r)^{\alpha-2} |F(r)|^2. \tag{6.4.19}$$

(ii) If $\alpha > 1$, then for all $F \in H_0^1((a,b);\varphi(r)^\alpha dr)$,

$$\int_a^b dr\, \varphi(r)^\alpha \big|F'(r)\big|^2 \geqslant \frac{(1-\alpha)^2}{4} \int_a^b dr\, \varphi(r)^{\alpha-2} |F(r)|^2. \tag{6.4.20}$$

Moreover, inequalities (6.4.19) *and* (6.4.20) *are strict for $F \not\equiv 0$.*

Proof. For part (i), split the interval $(a,b) = (a,(a+b)/2] \cup [(a+b)/2, b)$, applying Lemma 6.4.2 (i) to $(a,(a+b)/2)$ and Lemma 6.4.2 (ii) to $((a+b)/2, b)$. Strictness follows from either Lemma 6.4.2 part (i) or (ii).

Part (ii) follows analogously by applying Lemma 6.4.4 (i), (ii). □

We now provide the following multidimensional and radial analogue of Proposition 6.4.5.

Theorem 6.4.6. *Fix $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $\rho \in (0,\infty)$. The following hold:*

(i) If $\alpha < 2-n$, then for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$,

$$\int_{B_n(0;\rho)} d^n x\, \delta(x)^\alpha \big|(\partial_r f)(x)\big|^2 \geqslant \frac{(n-2+\alpha)^2}{4} \int_{B_n(0;\rho)} d^n x\, \delta(x)^{\alpha-2} |f(x)|^2. \tag{6.4.21}$$

(ii) If $\alpha > 2-n$, then for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$,

$$\int_{B_n(0;\rho)} d^n x\, \varphi(x)^\alpha \big|(\partial_r f)(x)\big|^2 \geqslant \frac{(n-2+\alpha)^2}{4} \int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha-2} |f(x)|^2. \tag{6.4.22}$$

Moreover, inequalities (6.4.21) *and* (6.4.22) *are strict for $f \not\equiv 0$.*

Proof. The proofs of (i) and (ii) are analogous to that of Proposition 6.3.1, using (6.2.7) and Proposition 6.4.5, and recalling (6.4.10). □

Next, we prove the distance-weighted radial Rellich inequality, beginning the following preliminary result.

Lemma 6.4.7. *Fix $n \in \mathbb{N}, a, b, c, \alpha \in \mathbb{R}$, $a < c < b$, and assume $\alpha < 2 - n$. The following hold:*

(i) For all $F \in H_L^2([a,b]; (r-a)^{n-1+\alpha}dr)$,

$$\begin{aligned}&\int_a^c dr\,(r-a)^{n-1+\alpha}\big|F''(r) + (n-1)r^{-1}F'(r)\big|^2 \\ &\quad \geqslant \frac{(n-\alpha)^2(n-4+\alpha)^2}{16}\int_a^c dr\,(r-a)^{n-5+\alpha}|F(r)|^2.\end{aligned} \tag{6.4.23}$$

(ii) For all $F \in H_R^2([a,b]; (b-r)^{n-1+\alpha}dr)$,

$$\begin{aligned}&\int_c^b dr\,(b-r)^{n-1+\alpha}\big|F''(r) + (n-1)r^{-1}F'(r)\big|^2 \\ &\quad \geqslant \frac{(n-\alpha)^2(n-4+\alpha)^2}{16}\int_c^b dr\,(b-r)^{n-5+\alpha}|F(r)|^2.\end{aligned} \tag{6.4.24}$$

Moreover, inequalities (6.4.23) and (6.4.24) are strict for $F \not\equiv 0$.

Proof. For part (i), assume $a = 0$ and $F \in H_L^2([0,b]; r^{n-1+\alpha}dr)$ is $\mathbb{R}$-valued, without loss of generality.

Similarly to the proof of Lemma 6.3.2,

$$\begin{aligned}&\int_0^c dr\,r^{n-1+\alpha}\big[F''(r) + (n-1)r^{-1}F'(r)\big]^2 \\ &\quad = \int_0^c dr\,r^{n-1+\alpha}\big[F''(r)\big]^2 + 2(n-1)\int_0^c dr\,r^{n-2+\alpha}F'(r)F''(r) \\ &\qquad + (n-1)^2\int_0^c dr\,r^{n-3+\alpha}\big[F'(r)\big]^2.\end{aligned} \tag{6.4.25}$$

Integrating by parts, we observe that

$$\begin{aligned}\int_0^c dr\,r^{n-2+\alpha}F'(r)F''(r) &= r^{n-2+\alpha}\big[F'(r)\big]^2\Big|_0^c - (n-2+\alpha)\int_0^c dr\,r^{n-3+\alpha}\big[F'(r)\big]^2 \\ &\quad - \int_0^c dr\,r^{n-2+\alpha}F''(r)F'(r).\end{aligned} \tag{6.4.26}$$

Since $F'(0) = 0$, rearranging (6.4.26) yields

$$2(n-1)\int_0^c dr\,r^{n-2+\alpha}F'(r)F''(r)$$

$$= (n-1)c^{n-2+\alpha}\big[F'(c)\big]^2 - (n-1)(n-2+\alpha)\int_0^c dr\, r^{n-3+\alpha}\big[F'(r)\big]^2$$
$$\geqslant -(n-1)(n-2+\alpha)\int_0^c dr\, r^{n-3+\alpha}\big[F'(r)\big]^2. \tag{6.4.27}$$

Combining (6.4.27) to (6.4.26), we apply Lemma 6.4.2 (i) twice, recalling (6.4.5), to yield

$$\begin{aligned}
&\int_0^c dr\, r^{n-1+\alpha}\big[F''(r) + (n-1)r^{-1}F'(r)\big]^2 \\
&\quad\geqslant \int_0^c dr\, r^{n-1+\alpha}\big[F''(r)\big]^2 + [(n-1)^2 - (n-1)(n-2+\alpha)]\int_0^c dr\, r^{n-3+\alpha}\big[F'(r)\big]^2 \\
&\quad\geqslant [\tfrac{1}{4}(n-2+\alpha)^2 + (n-1)^2 - (n-1)(n-2+\alpha)]\int_0^c dr\, r^{n-3+\alpha}\big[F'(r)\big]^2 \\
&\quad= \frac{(n-\alpha)^2}{4}\int_0^c dr\, r^{n-3+\alpha}\big[F'(r)\big]^2 \\
&\quad\geqslant \frac{(n-\alpha)^2(n-4+\alpha)^2}{16}\int_0^c dr\, r^{n-5+\alpha}[F(r)]^2.
\end{aligned} \tag{6.4.28}$$

Furthermore, if $F \not\equiv 0$ the first inequality in (6.4.28) is strict by Lemma 6.4.2 (i), and the fact (6.4.5).

Part (ii) holds via reflection across the interval midpoint $r = (a+b)/2$. □

We now prove a similar result for $n \in \mathbb{N}, \alpha \in \mathbb{R}$, $\alpha > 4-n$, that yields a different constant if $n \in \mathbb{N}, n \geqslant 2$.

Lemma 6.4.8. *Fix $n \in \mathbb{N}, a, b, c, \alpha \in \mathbb{R}$, $a < c < b$, and assume $\alpha > 4-n$. The following hold:*

(i) *For all $F \in H_L^2([a,b]; (b-r)^{n-1+\alpha}dr)$,*

$$\begin{aligned}
&\int_a^c dr\, (b-r)^{n-1+\alpha}\big|F''(r) + (n-1)r^{-1}F'(r)\big|^2 \\
&\quad\geqslant \frac{(3n-4+\alpha)^2(n-4+\alpha)^2}{16}\int_a^c dr\, (b-r)^{n-5+\alpha}|F(r)|^2.
\end{aligned} \tag{6.4.29}$$

(ii) *For all $F \in H_R^2([a,b]; (r-a)^{n-1+\alpha}dr)$,*

$$\begin{aligned}
&\int_c^b dr\, (r-a)^{n-1+\alpha}\big|F''(r) + (n-1)r^{-1}F'(r)\big|^2 \\
&\quad\geqslant \frac{(3n-4+\alpha)^2(n-4+\alpha)^2}{16}\int_c^b dr\, (r-a)^{n-5+\alpha}|F(r)|^2.
\end{aligned} \tag{6.4.30}$$

Moreover, inequalities (6.4.29) *and* (6.4.30) *are strict for* $F \not\equiv 0$.

Proof. For part (i), assume $a = 0$ and $F \in H^2_L([0,b]; (b-r)^{n-1+\alpha}dr)$ is $\mathbb{R}$-valued, without loss of generality.

Similarly to the proof of Lemma 6.4.7,

$$\begin{aligned}
&\int_0^c dr\,(b-r)^{n-1+\alpha}\big[F''(r) + (n-1)r^{-1}F'(r)\big]^2 \\
&\quad = \int_0^c dr\,(b-r)^{n-1+\alpha}\big[F''(r)\big]^2 + 2(n-1)\int_0^c dr\,(b-r)^{n-2+\alpha}F'(r)F''(r) \\
&\qquad + (n-1)^2\int_0^c dr\,(b-r)^{n-3+\alpha}\big[F'(r)\big]^2. \qquad (6.4.31)
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_0^c dr\,(b-r)^{n-2+\alpha}F'(r)F''(r) = (b-r)^{n-2+\alpha}\big[F'(r)\big]^2\Big|_0^c \qquad (6.4.32)\\
&\quad + (n-2+\alpha)\int_0^c dr\,(b-r)^{n-3+\alpha}\big[F'(r)\big]^2 - \int_0^c dr\,(b-r)^{n-2+\alpha}F''(r)F'(r).
\end{aligned}$$

Since $F'(0) = 0$, rearranging (6.4.32) yields

$$\begin{aligned}
&2(n-1)\int_0^c dr\,(b-r)^{n-2+\alpha}F'(r)F''(r) \\
&\quad = (n-1)(b-c)^{n-2+\alpha}\big[F'(c)\big]^2 + (n-1)(n-2+\alpha)\int_0^c dr\,(b-r)^{n-3+\alpha}\big[F'(r)\big]^2 \\
&\quad \geqslant (n-1)(n-2+\alpha)\int_0^c dr\,(b-r)^{n-3+\alpha}\big[F'(r)\big]^2. \qquad (6.4.33)
\end{aligned}$$

Combining (6.4.33) to (6.4.31), we apply Lemma 6.4.4 (i) twice, recalling (6.4.5), to yield

$$\begin{aligned}
&\int_0^c dr\,(b-r)^{n-1+\alpha}\big[F''(r) + (n-1)r^{-1}F'(r)\big]^2 \\
&\quad \geqslant \int_0^c dr\,(b-r)^{n-1+\alpha}\big[F''(r)\big]^2 \\
&\qquad + [(n-1)^2 + (n-1)(n-2+\alpha)]\int_0^c dr\,(b-r)^{n-3+\alpha}\big[F'(r)\big]^2 \\
&\quad \geqslant [\tfrac{1}{4}(n-2+\alpha)^2 + (n-1)^2 + (n-1)(n-2+\alpha)]\int_0^c dr\,(b-r)^{n-3+\alpha}\big[F'(r)\big]^2 \\
&\quad = \frac{(3n-4+\alpha)^2}{4}\int_0^c dr\,(b-r)^{n-3+\alpha}\big[F'(r)\big]^2
\end{aligned}$$

$$\geqslant \frac{(3n-4+\alpha)^2(n-4+\alpha)^2}{16} \int_0^c dr\,(b-r)^{n-5+\alpha}[F(r)]^2. \tag{6.4.34}$$

Furthermore, if $F \not\equiv 0$ the first inequality in (6.4.34) is strict by Lemma 6.4.4 (i) and recalling (6.4.5).

Part (ii) holds via reflection across the midpoint $r = (a+b)/2$. $\square$

Proposition 6.4.9. *Fix $n \in \mathbb{N}$ and $a, b, \alpha \in \mathbb{R}$, $a < b$. The following hold:*

(i) *If $\alpha < 2 - n$, then for all $F \in H_0^2([a,b]; \delta(r)^\alpha dr)$,*

$$\begin{aligned} &\int_a^b dr\,\delta(r)^{n-1+\alpha}\big|F''(r) + (n-1)r^{-1}F'(r)\big|^2 \\ &\qquad \geqslant \frac{(n-\alpha)^2(n-4+\alpha)^2}{16} \int_0^\infty dr\,\delta(r)^{n-5+\alpha}|F(r)|^2. \end{aligned} \tag{6.4.35}$$

(ii) *If $\alpha > 4 - n$, then for all $F \in H_0^2([a,b]; \varphi(r)^\alpha dr)$,*

$$\begin{aligned} &\int_a^b dr\,\varphi(r)^{n-1+\alpha}\big|F''(r) + (n-1)r^{-1}F'(r)\big|^2 \\ &\qquad \geqslant \frac{(3n-4+\alpha)^2(n-4+\alpha)^2}{16} \int_0^\infty dr\,\varphi(r)^{n-5+\alpha}|F(r)|^2. \end{aligned} \tag{6.4.36}$$

Moreover, inequalities (6.4.35) *and* (6.4.36) *are strict for $F \not\equiv 0$.*

Proof. Again, part (i) follows by splitting the interval $(a,b) = (a,(a+b)/2] \cup [(a+b)/2, b)$ and applying Lemma 6.4.7 (i) and (ii) to $(a,(a+b)/2)$ and $((a+b)/2, b)$, respectively. Part (ii) follows analogously using Lemma 6.4.8 (i), (ii). $\square$

Corollary 6.4.10. *Fix $a, b, \alpha \in \mathbb{R}$, $a < b$. The following hold:*

(i) *If $\alpha < 1$, then for all $F \in H_0^2([a,b]; \delta(r)^\alpha dr)$,*

$$\int_a^b dr\,\delta(r)^\alpha \big|F''(r)\big|^2 \geqslant \frac{(1-\alpha)^2(3-\alpha)^2}{16} \int_0^\infty dr\,\delta(r)^{\alpha-4}|F(r)|^2. \tag{6.4.37}$$

(ii) *If $\alpha > 3$, then for all $F \in H_0^2([a,b]; \varphi(r)^\alpha dr)$,*

$$\int_a^b dr\,\varphi(r)^\alpha \big|F''(r)\big|^2 \geqslant \frac{(1-\alpha)^2(3-\alpha)^2}{16} \int_0^\infty dr\,\varphi(r)^{\alpha-4}|F(r)|^2. \tag{6.4.38}$$

Moreover, inequalities (6.4.37) *and* (6.4.38) *are strict for $F \not\equiv 0$.*

We now give the multidimensional and radial extension of Proposition 6.4.9.

Theorem 6.4.11. *Fix $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $\rho \in (0, \infty)$. The following hold:*

(i) If $\alpha < 2 - n$, then for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$,

$$\begin{aligned} &\int_{B_n(0;\rho)} d^n x\, \delta(x)^\alpha \big|(-\Delta_r f)(x)\big|^2 \\ &\quad \geqslant \frac{(n-\alpha)^2(n-4+\alpha)^2}{16} \int_{B_n(0;\rho)} d^n x\, \delta(x)^{\alpha-4} |f(x)|^2. \end{aligned} \tag{6.4.39}$$

(ii) If $\alpha > 4 - n$, then for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$,

$$\begin{aligned} &\int_{B_n(0;\rho)} d^n x\, \varphi(x)^\alpha \big|(-\Delta_r f)(x)\big|^2 \\ &\quad \geqslant \frac{(3n-4+\alpha)^2(n-4+\alpha)^2}{16} \int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha-4} |f(x)|^2. \end{aligned} \tag{6.4.40}$$

Moreover, inequalities (6.4.39) *and* (6.4.40) *are strict for $f \not\equiv 0$.*

Proof. The proof is similar to that of Proposition 6.3.3, using (6.2.7) and applying Proposition 6.4.9 (i), (ii). □

6.4.2 Birman Inequality

We now establish the distance-weighted Birman-type inequalities with $\nabla(-\Delta)^m$ and $(-\Delta)^m$ replaced by $\partial_r(-\Delta_r)^m$, and $(-\Delta_r)^m$, respectively.

Theorem 6.4.12. *Fix $m \in \mathbb{N}$ and $a, b, \alpha \in \mathbb{R}$, $a < b$. The following hold:*

(i) If $\alpha < 1$, then for all $F \in H_0^m((a,b); \delta(r)^\alpha dr)$,

$$\int_a^b dr\, \delta(r)^\alpha \big|F^{(m)}(r)\big|^2 \geqslant A(m,1,\alpha) \int_a^b dr\, \delta(r)^{\alpha-2m} |F(r)|^2. \tag{6.4.41}$$

(ii) If $\alpha > 2m - 1$, then for all $F \in H_0^m((a,b); \varphi(r)^\alpha dr)$,

$$\int_a^b dr\, \varphi(r)^\alpha \big|F^{(m)}(r)\big|^2 \geqslant A(m,1,\alpha) \int_a^b dr\, \varphi(r)^{\alpha-2m} |F(r)|^2. \tag{6.4.42}$$

Moreover, inequalities (6.4.41) *and* (6.4.42) *are strict for $F \not\equiv 0$.*

Proof. The proof follows by iterating Proposition 6.4.5, while recalling (6.4.6). □

Theorem 6.4.13. *Fix $m, n \in \mathbb{N}$, $\rho \in (0,\infty)$, $\alpha \in \mathbb{R}$, and assume $\alpha < 2-n$. For all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$,*

$$(i) \int_{B_n(0;\rho)} d^n x\, \delta(x)^\alpha \big|(\partial_r(-\Delta_r)^{m-1} f)(x)\big|^2 \geqslant A(2m-1,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \delta(x)^{\alpha-4m+2} |f(x)|^2; \tag{6.4.43}$$

$$(ii) \int_{B_n(0;\rho)} d^n x\, \delta(x)^\alpha \big|((-\Delta_r)^{m} f)(x)\big|^2 \geqslant A(2m,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \delta(x)^{\alpha-4m} |f(x)|^2. \tag{6.4.44}$$

Moreover, inequalities (6.4.43) *and* (6.4.44) *are strict for $f \not\equiv 0$.*

Proof. The proof is similar to that of Theorem 6.3.4, using Theorem 6.4.6 (i) (resp. Theorem 6.4.11 (i)) for the case $m = 1$, Theorem 6.4.11 (i) for the induction step applied to $\alpha_0 = \alpha - 4m + 2 < 2 - n$ (resp. $\alpha_0 = \alpha - 4m < 2 - n$), and finally, recalling Remark 6.1.1 (i) (resp (ii)) to show $A(2m-1,n,\alpha)A(2,n,\alpha-4m+2) = A(2m+1,n,\alpha)$ (resp $A(2m,n,\alpha)A(2,n,\alpha-4m) = A(2m+2,n,\alpha)$). □

Theorem 6.4.14. *Fix $m, n \in \mathbb{N}$, $\rho \in (0,\infty)$, $\alpha \in \mathbb{R}$. The following hold:*

(i) If $\alpha > 4m-2-n$ then for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$,

$$\begin{aligned} &\int_{B_n(0;\rho)} d^n x\, \varphi(x)^\alpha \big|(\partial_r(-\Delta_r)^{m-1} f)(x)\big|^2 \\ &\quad \geqslant \widetilde{A}(2m-1,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha-4m+2} |f(x)|^2. \end{aligned} \tag{6.4.45}$$

(ii) If $\alpha > 4m-n$ then for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$,

$$\begin{aligned} &\int_{B_n(0;\rho)} d^n x\, \varphi(x)^\alpha \big|((-\Delta_r)^{m} f)(x)\big|^2 \\ &\quad \geqslant \widetilde{A}(2m,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha-4m} |f(x)|^2. \end{aligned} \tag{6.4.46}$$

Moreover, inequalities (6.4.45) *and* (6.4.46) *are strict for $f \not\equiv 0$.*

Proof. For part (i), we use induction over $m \in \mathbb{N}$. The case $m = 1$ holds by Theorem 6.4.6 (ii).

Assume (6.4.45) holds for $m \in \mathbb{N}$; then for $m+1$ let $\alpha > 4m+2-n$. By inductive hypothesis, and Theorem 6.4.11 (ii) applied to $\alpha_0 = \alpha - 4m + 2 > 4 - n$, one has

$$\begin{aligned}
&\int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha} \big|(\partial_r(-\Delta_r)^m f)(x)\big|^2 \\
&\quad = \int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha} \big|[\partial_r(-\Delta_r)^{m-1}(-\Delta_r) f](x)\big|^2 \\
&\quad \geqslant \widetilde{A}(2m-1,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha-4m+2} \big|(-\Delta_r f)(x)\big|^2 \\
&\quad \geqslant \widetilde{A}(2m-1,n,\alpha)\widetilde{A}(2,n,\alpha-4m+2) \int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha-4m-2} |f(x)|^2.
\end{aligned} \tag{6.4.47}$$

Recalling (6.4.11), one computes

$$\begin{aligned}
&\widetilde{A}(2m-1,n,\alpha)\widetilde{A}(2,n,\alpha-4m+2) \\
&\quad = \left(\frac{n-2+\alpha}{2}\right)^2 \prod_{j=1}^{m-1} \left(\frac{3n-4j-2+\alpha}{2}\right)^2 \prod_{k=1}^{m-1} \left(\frac{n-4k-2+\alpha}{2}\right)^2 \\
&\qquad \times \left(\frac{3n-4+(\alpha-4m+2)}{2}\right)^2 \left(\frac{n-4+(\alpha-4m+2)}{2}\right)^2 \\
&\quad = \left(\frac{n-2+\alpha}{2}\right)^2 \prod_{j=1}^{m-1} \left(\frac{3n-4j-2+\alpha}{2}\right)^2 \prod_{k=1}^{m-1} \left(\frac{n-4k-2+\alpha}{2}\right)^2 \\
&\qquad \times \left(\frac{3n-4m-2+\alpha}{2}\right)^2 \left(\frac{n-4m-2+\alpha}{2}\right)^2 \\
&\quad = \left(\frac{n-2+\alpha}{2}\right)^2 \prod_{j=1}^{m} \left(\frac{3n-4j-2+\alpha}{2}\right)^2 \prod_{k=1}^{m} \left(\frac{n-4k-2+\alpha}{2}\right)^2 \\
&\quad = \widetilde{A}(2m+1,n,\alpha).
\end{aligned} \tag{6.4.48}$$

Furthermore, if $f \not\equiv 0$ the last inequality in (6.4.47) is strict by Theorem 6.4.11 (ii).

Part (ii) is proven analogously, using Theorem 6.4.11 (ii) for both the case $m=1$ and the induction step applied to $\alpha_0 = \alpha - 4m > 4 - n$, and finally, recalling (6.4.11) to show $\widetilde{A}(2m,n,\alpha)\widetilde{A}(2,\alpha-4m) = \widetilde{A}(2m+2,n,\alpha)$. $\square$

Remark 6.4.15. The constants $A(m,1,\alpha)$ in Theorem 6.4.12 (i), and $A(m,n,\alpha)$ in Theorem 6.4.13, are sharp, by considering the half intervals $(a,(a+b)/2)$ and $((a+$

$b)/2,b)$. See for instance [53, Theorem 7.1] for the special case $\alpha = 0$. Optimality in Theorem 6.4.12 (*ii*) and Theorem 6.4.14 is not currently known. $\diamond$

6.5 The Vector-Valued Case

In this section, we establish that the weighted radial Birman-type inequalities generalize mutatis mutandis to the vector-valued case in which f is $\mathcal{H}$-valued, with $\mathcal{H}$ a separable, complex Hilbert space (recall section 2.8 for basic properties of Bochner integrability and associated vector-valued L^p- and Sobolev spaces).

Indeed, the method of polar integration in (6.2.6) (resp. (6.2.7)) extends to

$$\int_{\mathbb{R}^n} d^n x\, f(x) = \int_{S^{n-1}} d\omega(\theta) \int_0^\infty r^{n-1} dr\, f(r,\theta), \quad f \in L^1(\mathbb{R}^n;\mathcal{H}), \tag{6.5.1}$$

and analogously for $f \in L^1(B_n(0;\rho);\mathcal{H})$, $\rho \in (0,\infty)$. In addition, all considered function spaces naturally generalize to their respective $\mathcal{H}$-valued analogues.

Thus, the main results can be generalized as follows:

Theorem 6.5.1. *Fix $m,n \in \mathbb{N}$, $\alpha \in \mathbb{R}$. For all $f \in C_0^\infty(\mathbb{R}^n\backslash\{0\};\mathcal{H})$,*

$$(i) \int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big\|(\partial_r(-\Delta_r)^{m-1} f)(x)\big\|_{\mathcal{H}}^2 \tag{6.5.2}$$
$$\geqslant A(2m-1,n,\alpha) \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-4m+2} \|f(x)\|_{\mathcal{H}}^2;$$

$$(ii) \int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big\|((-\Delta_r)^m f)(x)\big\|_{\mathcal{H}}^2 \geqslant A(2m,n,\alpha) \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-4m} \|f(x)\|_{\mathcal{H}}^2. \tag{6.5.3}$$

Moreover, inequalities (6.5.2)*and* (6.5.3) *are strict for $f \not\equiv 0$.*

Theorem 6.5.2. *Fix $m \in \mathbb{N}$ and $a,b,\alpha \in \mathbb{R}$, $a < b$. The following hold:*

(*i*) *If $\alpha < 1$, then for all $F \in H_0^m((a,b);\delta(r)^\alpha dr;\mathcal{H})$,*

$$\int_a^b dr\, \delta(r)^\alpha \big\|F^{(m)}(r)\big\|_{\mathcal{H}}^2 \geqslant A(m,1,\alpha) \int_a^b dr\, \delta(r)^{\alpha-2m} \|F(r)\|_{\mathcal{H}}^2. \tag{6.5.4}$$

(*ii*) *If $\alpha > 2m-1$, then for all $F \in H_0^m((a,b);\varphi(r)^\alpha dr;\mathcal{H})$,*

$$\int_a^b dr\, \varphi(r)^\alpha \big\|F^{(m)}(r)\big\|_{\mathcal{H}}^2 \geqslant A(m,1,\alpha) \int_a^b dr\, \varphi(r)^{\alpha-2m} \|F(r)\|_{\mathcal{H}}^2. \tag{6.5.5}$$

Moreover, inequalities (6.5.4) *and* (6.5.5) *are strict for $F \not\equiv 0$.*

Theorem 6.5.3. *Fix $m,n \in \mathbb{N}$, $\rho \in (0,\infty)$, $\alpha \in \mathbb{R}$, and assume $\alpha < 2-n$. For all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\};\mathcal{H})$,*

$$(i) \int_{B_n(0;\rho)} d^n x\, \delta(x)^\alpha \big\|(\partial_r(-\Delta_r)^{m-1}f)(x)\big\|_{\mathcal{H}}^2 \tag{6.5.6}$$
$$\geqslant A(2m-1,n,\alpha)\int_{B_n(0;\rho)} d^n x\, \delta(x)^{\alpha-4m+2}\|f(x)\|_{\mathcal{H}}^2;$$

$$(ii) \int_{B_n(0;\rho)} d^n x\, \delta(x)^\alpha \big\|((-\Delta_r)^m f)(x)\big\|_{\mathcal{H}}^2 \tag{6.5.7}$$
$$\geqslant A(2m,n,\alpha)\int_{B_n(0;\rho)} d^n x\, \delta(x)^{\alpha-4m}\|f(x)\|_{\mathcal{H}}^2.$$

Moreover, inequalities (6.5.6) *and* (6.5.7) *are strict for $f \not\equiv 0$.*

Theorem 6.5.4. *Fix $m,n \in \mathbb{N}$, $\rho \in (0,\infty)$, $\alpha \in \mathbb{R}$. The following hold:*

(i) If $\alpha > 4m-2-n$ then for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\};\mathcal{H})$,

$$\begin{aligned}\int_{B_n(0;\rho)} d^n x\, \varphi(x)^\alpha \big\|(\partial_r(-\Delta_r)^{m-1}f)(x)\big\|_{\mathcal{H}}^2 \\ \geqslant \widetilde{A}(2m-1,n,\alpha)\int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha-4m+2}\|f(x)\|_{\mathcal{H}}^2.\end{aligned} \tag{6.5.8}$$

(ii) If $\alpha > 4m-n$ then for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\};\mathcal{H})$,

$$\begin{aligned}\int_{B_n(0;\rho)} d^n x\, \varphi(x)^\alpha \big\|((-\Delta_r)^m f)(x)\big\|_{\mathcal{H}}^2 \\ \geqslant \widetilde{A}(2m,n,\alpha)\int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha-4m}\|f(x)\|_{\mathcal{H}}^2.\end{aligned} \tag{6.5.9}$$

Moreover, inequalities (6.5.8) *and* (6.5.9) *are strict for $f \not\equiv 0$.*

One can follow the special scalar case treated in the proof of Theorem 6.3.4 line by line.

Remark 6.5.5. As in the scalar case, the constants $A(m,n,\alpha)$ (resp. $A(m,1,\alpha)$) in Theorems 6.5.1, 6.5.2 (i), and 6.5.3 are sharp, and in Theorem 6.5.1 the function space $C_0^\infty(\mathbb{R}^n\backslash\{0\};\mathcal{H})$ can be replaced by $C_0^\infty(\mathbb{R}^n;\mathcal{H})$ if we further assume that $\alpha > 4m-2-n$ for part (i) and respectively $\alpha > 4m-n$ for part (ii). $\diamond$

CHAPTER SEVEN

Conclusion

In Chapter Two, we investigated the classical Birman–Hardy–Rellich sequence of inequalities for $f \in C_0^m((0,\infty))$, $m \in \mathbb{N}$,

$$\int_0^\infty dx\, \big|f^{(m)}(x)\big|^2 \geqslant \frac{[(2m-1)!!]^2}{2^{2m}} \int_0^\infty dx\, \frac{|f(x)|^2}{x^{2m}}, \tag{7.0.1}$$

established in 1961 by M. Š. Birman [19, p. 48]. Introducing the Hilbert space $H_L^m([0,\infty))$, $m \in \mathbb{N}$, given by

$$\begin{split} H_L^m([0,\infty)) := \big\{f : [0,\infty) \to \mathbb{C} \,\big|\, f^{(j)} \in AC_{loc}([0,\infty));\ f^{(m)} \in L^2((0,\infty)); \\ f^{(j)}(0) = 0,\ j = 0,1,\dots,m-1\big\}, \end{split} \tag{7.0.2}$$

and employing the integral inequality in Theorem 2.2.1, we established several important properties of $H_L^m([0,\infty))$. In Theorem 2.4.4 we gave our own proof of (7.0.1) on the larger space $H_L^m([0,\infty))$, using the subtle, yet important, implication

$$f \in H_L^m([0,\infty)) \text{ implies } f' \in H_L^{m-1}([0,\infty)). \tag{7.0.3}$$

We showed that equality in each of these inequalities is only achieved when $f \equiv 0$, and that the constants $[(2m-1)!!]^2/2^{2m}$ are all sharp. We then introduced a sequence of generalized continuous Cesàro operators, $T_m, m \in \mathbb{N}$, via

$$\begin{split} &(T_m f)(x) := \frac{1}{x^m} \int_0^x dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m f(t_m), \qquad x \in (0,\infty), \\ &f \in \operatorname{dom}(T_m) = L^2((0,\infty)). \end{split} \tag{7.0.4}$$

associated to the Birman–Hardy–Rellich-type inequalities, and computed their spectra. Turning our attention to the finite interval case $[0,b]$, we proved the Birman inequalities also hold on the standard Sobolev space $H_0^m((0,b))$. Finally, we asserted that all previous results extend to $\mathcal{H}$-valued functions, with $\mathcal{H}$ a separable, complex Hilbert space.

In Chapter Three, we generalized the classical power weighted Hardy inequality for $p \in [1,\infty), \alpha \in \mathbb{R}$,

$$\int_0^\infty dx\, x^\alpha |f'(x)|^p \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^\infty dx\, x^{\alpha-p}|f(x)|^p, \quad f \in C_0^\infty((0,\infty)) \quad (7.0.5)$$

as a consequence of a more abstract inequality within the context of $\mathcal{B}$-valued functions, where $\mathcal{B}$ is an arbitrary separable Banach space. In particular, we proved

$$\begin{aligned} &\int_a^b dx\, w_1(x)^p[-w_1'(x)]^{1-p} w_2(x)^p \|F(x)\|_{\mathcal{B}}^p \\ &\quad \geqslant p^{-p} \int_a^b dx\, [-w_1'(x)] \bigg(\int_a^x dx'\, w_2(x')\|F(x')\|_{\mathcal{B}}\bigg)^p, \end{aligned} \qquad F \in C_0((a,b);\mathcal{B}). \quad (7.0.6)$$

where $-\infty \leqslant a < b \leqslant \infty$, $p \in [1,\infty)$, $0 \leqslant w_1 \in AC_{loc}((a,b))$, $0 \leqslant [-w_1']$ a.e. on (a,b), $0 \leqslant w_2 \in L^1_{loc}((a,b);dx)$, and $[-w_1']^{1-p}w_2^p \in L^1_{loc}((a,b);dx)$, and its companion result with $\int_a^x dx' \ldots$ replaced by $\int_x^b dx' \ldots$., containing (7.0.5) in Example 3.2.3, and via iteration, the entire sequence of power weighted Birman–Hardy–Rellich inequalities for $p \in [1,\infty), \alpha \in \mathbb{R}$. We then extended Example 3.2.3 to the operator-valued context, proving that if $p \in [1,\infty)$, $b \in (0,\infty) \cup \{\infty\}$, and $\alpha < p-1$, then for weakly measurable $F : (0,b) \to \mathcal{B}(\mathcal{H})$ with $\|F(\,\cdot\,)\|_{\mathcal{B}_p(\mathcal{H})} \in L^p((0,b);x^\alpha dx)$,

$$\operatorname{tr}_{\mathcal{H}}\bigg(\int_0^b dx\ x^\alpha\, |F(x)|^p\bigg) \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \operatorname{tr}_{\mathcal{H}}\bigg(\int_0^b dx\, x^{\alpha-p}\bigg|\int_0^x dx'\, F(x')\bigg|^p\bigg), \quad (7.0.7)$$

together with its companion result for $\alpha > p-1$. For the special case $p \in [1,2]$ and $\alpha < p-1$, we proved that for weakly measurable $F : (0,\infty) \to \mathcal{B}(\mathcal{H})$ with $F(\,\cdot\,) \geqslant 0$ a.e. on $(0,\infty)$, and $\int_0^\infty dx\, x^\alpha F(x)^p \in \mathcal{B}(\mathcal{H})$,

$$\int_0^\infty dx\ x^\alpha\, F(x)^p \geqslant \left(\frac{|\alpha - p + 1|}{p}\right)^p \int_0^\infty dx\, x^{\alpha-p}\bigg(\int_0^x dx'\, F(x')\bigg)^p, \qquad (7.0.8)$$

together with its companion result for $\alpha > p-1$. Iterating the process, we established

$$\int_0^\infty dx\, x^\alpha |f^{(m)}(x)|^p \geqslant \frac{\prod_{j=1}^k |\alpha - jp + 1|^p}{p^{kp}} \int_0^\infty dx\, x^{\alpha-kp}|f^{(m-k)}(x)|^p, \qquad (7.0.9)$$

for $p \in [1,2],\ 1 \leqslant k \leqslant m,\ m \in \mathbb{N},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0,\infty);\mathcal{B}(\mathcal{H}))$.

In Chapter Four, we generalize the one-dimensional Birman inequalities to include power weights x^{α}, $\alpha \in \mathbb{R}$, and recursively defined logarithms $\ln_j(,\cdot\,)$, $j \in \mathbb{N}$,

$$\ln_1(\,\cdot\,) := \ln(\,\cdot\,), \quad \ln_{j+1}(\,\cdot\,) := \ln(\ln_j(\,\cdot\,)), \quad j \in \mathbb{N}, \tag{7.0.10}$$

with

$$e_0 := 0, \quad e_{j+1} := e^{e_j}, \quad j \in \mathbb{N}_0, \tag{7.0.11}$$

as well as their normalized counterparts $X_j(\,\cdot\,)$, $j \in \mathbb{N}$,

$$X_1(\,\cdot\,) := (1 - \ln(\,\cdot\,))^{-1}, \quad X_{j+1}(\,\cdot\,) := X_1(X_j(\,\cdot\,)), \quad j \in \mathbb{N}. \tag{7.0.12}$$

We introduced the following elementary variable transformation, attributed to P. Hartman [72] and E. Müeller-Pfeiffer [96]: Given $m, N, \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 1, 2, \ldots, 2m-1$, assume $f \in C^{\infty}((e_N, \infty))$. Set

$$\begin{aligned} &x = e^t, \qquad dx = e^t dt, \qquad t \in (e_{N-1}, \infty), \\ &f(x) \equiv f(e^t) = e^{(m - \frac{1+\alpha}{2})t} w(t), \quad w \in C_0^{\infty}((e_{N-1}, \infty)). \end{aligned} \tag{7.0.13}$$

Using this change of variables, we proved the following Birman–Hardy–Rellich-type inequality on the exterior interval (ρ, ∞) for any $\rho \in \mathbb{N}$,

$$\begin{aligned} \int_{\rho}^{\infty} dx\, x^{\alpha} \big|f^{(m)}(x)\big|^2 &\geqslant A(m,\alpha) \int_{\rho}^{\infty} dx\, x^{\alpha-2m} |f(x)|^2 \\ &+ B(m,\alpha) \sum_{k=1}^{N} \int_{\rho}^{\infty} dx\, x^{\alpha-2m} \prod_{i=1}^{k} [\ln_i(x/\gamma)]^{-2} |f(x)|^2 \\ &+ \sum_{j=2}^{m} |c_{2j}(m,\alpha)| A(j,0) \int_{\rho}^{\infty} dx\, x^{\alpha-2m} [\ln(x/\gamma)]^{-2j} |f(x)|^2 \\ &+ \sum_{j=2}^{m} |c_{2j}(m,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_{\rho}^{\infty} dx\, x^{\alpha-2m} [\ln(x/\gamma)]^{-2j} \prod_{i=1}^{k} [\ln_{i+1}(x/\gamma)]^{-2} |f(x)|^2, \end{aligned} \tag{7.0.14}$$

for $f \in C_0^{\infty}((\rho, \infty))$, where $m, N \in \mathbb{N}, \alpha \in \mathbb{R}$, and $\rho, \gamma \in (0, \infty)$ with $\rho \geqslant e_N \gamma$. We also established (7.0.14) for the normalized logarithms $X_j(\,\cdot\,)$, $j \in \mathbb{N}$, as well. By modifying the transformation (7.0.13), we then obtained analogous results on the interior interval $(0,\rho)$, for any $\rho \in (0,\infty)$;

$$\begin{aligned}
\int_0^\rho dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(m,\alpha) \int_0^\rho dx\, x^{\alpha-2m}|f(x)|^2 \\
+ B(m,\alpha) \sum_{k=1}^{N} \int_0^\rho dx\, x^{\alpha-2m} \prod_{i=1}^{k} [\ln_i(\gamma/x)]^{-2}|f(x)|^2 \\
+ \sum_{j=2}^{m} |c_{2j}(m,\alpha)| A(j,0) \int_0^\rho dx\, x^{\alpha-2m} [\ln(\gamma/x)]^{-2j}|f(x)|^2 \\
+ \sum_{j=2}^{m} |c_{2j}(m,\alpha)| B(j,0) \sum_{k=1}^{N-1} \int_0^\rho dx\, x^{\alpha-2m} [\ln(\gamma/x)]^{-2j} \prod_{i=1}^{k} [\ln_{i+1}(\gamma/x)]^{-2}|f(x)|^2,
\end{aligned} \tag{7.0.15}$$

for $f \in C_0^\infty((0,\rho))$, where $m, N \in \mathbb{N}, \alpha \in \mathbb{R}$, and $\rho, \gamma \in (0,\infty)$ with $\gamma \geqslant e_N \rho$, again, showing (7.0.15) remains true for the normalized logarithms $X_j(\,\cdot\,)$, $j \in \mathbb{N}$, as well. The inequalities (7.0.14) and (7.0.15) for $X_j(\,\cdot\,)$, $j \in \mathbb{N}$, were then further improved by replacing the N-th sum with an infinite series. Finally, we extended all previous Birman-type inequalities to the more general vector-valued case, replacing complex-valued $f(x)$ by $f(x) \in \mathcal{H}$, with $\mathcal{H}$ a complex, separable Hilbert space. For $m \geqslant 2$ these inequalities are new because the constants $A(m,\alpha)$ and $B(m,\alpha)$ are optimal, the weight parameter $\alpha \in \mathbb{R}$ is unrestricted, the conditions on the logarithmic parameters γ and τ are sharp, the two integral terms with $c_{2j}(m,\alpha)$ are new, and the inequalities are proven for both iterated logarithms $\ln_j(\,\cdot\,)$, $X_j(\,\cdot\,)$, and on both the external interval (ρ,∞) and internal interval $(0,\rho)$ for any $\rho \in (0,\infty)$.

In Chapter Five, we then turned our attention to the multidimensional setting, recalling the classical Hardy inequality for $f \in C_0^\infty(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ open, $n \in \mathbb{N}$, $n \geqslant 2$,

$$\int_\Omega d^n x\, |(\nabla f)(x)|^2 \geqslant \frac{(n-2)^2}{4} \int_\Omega d^n x\, |x|^{-2}|f(x)|^2. \tag{7.0.16}$$

and its logarithmic refinement, derived in [50],

$$\begin{aligned}&\int_\Omega d^nx\,|(\nabla f)(x)|^2\\&\quad\geqslant\int_\Omega d^nx|x-x_0|^{-2}|f(x)|^2\bigg\{\frac{(n-2)^2}{4}+\frac{1}{4}\sum_{k=1}^{N}\prod_{j=1}^{k}[\ln_j(\gamma/|x-x_0|)]^{-2}\bigg\},\end{aligned}\tag{7.0.17}$$

for $f\in C_0^\infty(\Omega)$, assuming $\Omega\subset\mathbb{R}^n$, $n\in\mathbb{N}$, $n\geqslant 2$, Ω is open and bounded with $x_0\in\Omega$, $N\in\mathbb{N}$, and the logarithmic terms $\ln_j(\,\cdot\,)$, $j\in\mathbb{N}$, are recursively given by

$$\ln_1(\,\cdot\,):=\ln(\,\cdot\,),\quad \ln_{j+1}(\,\cdot\,):=\ln(\ln_j(\,\cdot\,)),\quad j\in\mathbb{N},\tag{7.0.18}$$

for $\gamma>0$, $x\in\mathbb{R}^n\backslash\{x_0\}$, with $0<|x-x_0|<\mathrm{diam}(\Omega)<\gamma/e_N$, where

$$e_1:=1,\quad e_{j+1}:=e^{e_j},\quad j\in\mathbb{N}.\tag{7.0.19}$$

Using factorizations of certain differential operators $T_{\alpha,N}$, $\alpha\in\mathbb{R}, N\in\mathbb{R}$, we further improved (7.0.17) to a power weighted analogue with the gradient ∇ replaced by the radial derivative ∂_{r,x_0} centered about a point $x_0\in\mathbb{R}^n$, via

$$\partial_{r,x_0}:=|x-x_0|^{-1}(x-x_0)\cdot\nabla,\tag{7.0.20}$$

for $x\in\mathbb{R}^n\setminus\{x_0\}, r=|x-x_0|, n\in\mathbb{N}, n\geqslant 2$. In particular, we proved

$$\begin{aligned}&\int_\Omega d^nx\,|x-x_0|^\alpha|(\partial_{r,x_0}f)(x)|^2\\&\quad\geqslant\int_\Omega d^nx\,|x-x_0|^{\alpha-2}|f(x)|^2\bigg\{\frac{(n-2)^2}{4}+\frac{1}{4}\sum_{k=1}^{N}\prod_{j=1}^{k}[\ln_j(\gamma/|x-x_0|)]^{-2}\bigg\},\end{aligned}\tag{7.0.21}$$

valid for $f\in C_0^\infty(\Omega)$, with $\Omega\subset\mathbb{R}^n$, $n\in\mathbb{N}$, $n\geqslant 2$, Ω is open and bounded with $x_0\in\Omega$, $N\in\mathbb{N}$.

In Chapter Six, we continued the pursuit of radial refinements from Chapter Five to the case of the multidimensional weighted Birman inequalities. Recalling the radial derivative ∂_r,

$$\partial_r:=|x|^{-1}x\cdot\nabla,\qquad x\in\mathbb{R}^n\backslash\{0\},\ r=|x|,\ n\in\mathbb{N},\ n\geqslant 2,\tag{7.0.22}$$

as well as the radial Laplacian Δ_r,

$$\Delta_r := r^{1-n}\partial_r r^{n-1}\partial_r = \partial_r^2 + (n-1)r^{-1}\partial_r, \tag{7.0.23}$$

we considered the radial differential expressions $\partial_r(-\Delta_r)^m$ and $(-\Delta_r)^m$, for $m \in \mathbb{N}$, in replace of the usual differential expressions $\nabla(-\Delta)^m$ and $(-\Delta)^m$, $m \in \mathbb{N}$, respectively. Giving a brief review of integration in polar coordinates in $\mathbb{R}^n \cong (0,\infty) \times S^{n-1}$, we used the standard identity

$$\int_{\mathbb{R}^n} d^n x\, f(x) = \int_{S^{n-1}} d\omega(\theta) \int_0^\infty r^{n-1} dr\, f(r,\theta), \quad f \in L^1(\mathbb{R}^n). \tag{7.0.24}$$

to establish the multidimensional power weighted Birman inequalities with radial refinements,

$$\int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big|(\partial_r(-\Delta_r)^{m-1} f)(x)\big|^2 \geqslant A(2m-1,n,\alpha) \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-4m+2} |f(x)|^2, \tag{7.0.25}$$

and analogously for the even-ordered derivatives,

$$\int_{\mathbb{R}^n} d^n x\, |x|^\alpha \big|((-\Delta_r)^m f)(x)\big|^2 \geqslant A(2m,n,\alpha) \int_{\mathbb{R}^n} d^n x\, |x|^{\alpha-4m} |f(x)|^2, \tag{7.0.26}$$

for all $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$, where $m,n \in \mathbb{N}$, and $\alpha \in \mathbb{R}$. We then reconsider the weighted Birman-type inequalities on bounded domains $\Omega \subset \mathbb{R}^n$ with the standard power weight $|x|^\alpha, \alpha \in \mathbb{R}$, replaced by the shortest, and farthest, distance to the boundary $\partial\Omega = \overline{\Omega}\backslash\Omega^\circ$, in the special case $\Omega = B_n(0;\rho)$. These distance functions δ and φ on Ω, are given by

$$\delta(x) := \inf_{y \in \partial\Omega} |x-y|, \qquad \varphi(x) := \sup_{y \in \partial\Omega} |x-y|, \qquad x \in \Omega. \tag{7.0.27}$$

Following similar methods to section 6.3, we established the Birman-type inequalities,

$$\int_a^b dr\, \delta(r)^\alpha \big|F^{(m)}(r)\big|^2 \geqslant A(m,1,\alpha) \int_a^b dr\, \delta(r)^{\alpha-2m} |F(r)|^2. \tag{7.0.28}$$

for all $F \in H_0^m((a,b);\delta(r)^\alpha dr)$, where $m \in \mathbb{N}, a, b, \alpha \in \mathbb{R}$, $a < b$ with $\alpha < 1$, and also

$$\int_a^b dr\, \varphi(r)^\alpha \big|F^{(m)}(r)\big|^2 \geqslant A(m,1,\alpha) \int_a^b dr\, \varphi(r)^{\alpha-2}|F(r)|^2. \tag{7.0.29}$$

for all $F \in H_0^m((a,b);\varphi(r)^\alpha dr)$, where $m \in \mathbb{N}, a, b, \alpha \in \mathbb{R}$, $a < b$ with $\alpha > 2m-1$, where $H_0^m((a,b);w(r)dr)$ denotes the standard, w-weighted Sobolev space on (a,b). The multidimensional analogue of (7.0.28), (7.0.29), was then established:

$$\int_{B_n(0;\rho)} d^n x\, \delta(x)^\alpha \big|(\partial_r(-\Delta_r)^{m-1} f)(x)\big|^2 \geqslant A(2m-1,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \delta(x)^{\alpha-4m+2} |f(x)|^2, \tag{7.0.30}$$

and analogously,

$$\int_{B_n(0;\rho)} d^n x\, \delta(x)^\alpha \big|((-\Delta_r)^m f)(x)\big|^2 \geqslant A(2m,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \delta(x)^{\alpha-4m} |f(x)|^2, \tag{7.0.31}$$

for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$, where $m, n \in \mathbb{N}$, $\rho \in (0,\infty)$, $\alpha \in \mathbb{R}$, and $\alpha < 2-n$. A similar result to (7.0.30), (7.0.31), with certain modified constants $\widetilde{A}(m,n,\alpha)$, $m, n \in \mathbb{N}, \alpha \in \mathbb{R}$, introduced in (6.4.11), were then established for the farthest distance to the boundary:

$$\int_{B_n(0;\rho)} d^n x\, \varphi(x)^\alpha \big|(\partial_r(-\Delta_r)^{m-1} f)(x)\big|^2 \geqslant \widetilde{A}(2m-1,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha-4m+2} |f(x)|^2, \tag{7.0.32}$$

for all $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$, where $m, n \in \mathbb{N}$, $\rho \in (0,\infty)$, $\alpha \in \mathbb{R}$, and $\alpha > 4m-2-n$; as well as

$$\int_{B_n(0;\rho)} d^n x\, \varphi(x)^\alpha \big|((-\Delta_r)^m f)(x)\big|^2 \geqslant \widetilde{A}(2m,n,\alpha) \int_{B_n(0;\rho)} d^n x\, \varphi(x)^{\alpha-4m} |f(x)|^2, \tag{7.0.33}$$

for $f \in C_0^\infty(B_n(0;\rho)\backslash\{0\})$, where $\alpha > 4m-n$. We conclude the chapter by extending all results to $\mathcal{H}$-valued functions, where $\mathcal{H}$ is a complex, separable Hilbert space.

BIBLIOGRAPHY

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces* , 2nd ed., 2005.

[2] Adimurthi, N. Chaudhuri, and M. Ramaswami, *An Improved Hardy-Sobolev Inequality and Its Application*, Proceedings of AMS, **130(2)**, 489–505 (2001).

[3] Adimurthi and M. J. Esteban Ceremade *An improved Hardy-Sobolev inequality in $W^{1,p}$ and its application to Schrödinger operators*, Nonlin. Diff. Eq. Appl. **12**, 243–263 (2005).

[4] Adimurthi, M. Grossi, and S. Santra, *Optimal Hardy–Rellich inequalities, maximum principle and related eigenvalue problem*, J. Funct. Anal. **240**, 36—83 (2006).

[5] Adimurthi and S. Santra, *Generalized Hardy–Rellich Inequalities in Critical Dimension and its Applications*, Comm. in Cont. Math. **11** (3), 367—394 (2009).

[6] A. A. Albanese, J. Bonet, and W. J. Ricker, *On the continuous Cesàro operator in certain function spaces*, Positivity **19**, 659–679 (2015).

[7] H. Amann, *Linear and Quasilinear Parabolic Problems*, Monographs in Mathematics, Vol. 89, Birkhäuser, Basel, 1995.

[8] H. Ando and T. Horiuchi, *Missing terms in the weighted Hardy–Sobolev inequalities and its application*, Kyoto J. of Math. **52** (4), 759—796 (2012).

[9] W. Arendt, C. K. Batty, M. Hieber, F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Transforms*, Monographs in Mathematics, Vol. 96, Birkhäuser, Basel, 2001.

[10] A. Balinsky and W. D. Evans, *Spectral Analysis of Relativistic Operators*, Imperial College Press, London, 2011.

[11] A. Balinsky, W. D. Evans, and R. T. Lewis, *The Analysis and Geometry of Hardy's Inequality*, Universitext, Springer, 2015.

[12] G. Barbatis, *Improved Rellich inequalities for the Polyharmonic Operator*, Indiana University Mathematics Journal, **55(4)**, 1401–1422 (2006).

[13] G. Barbatis, *Best constants for higher-order Rellich inequalities in $L^p(\Omega)$*, Math Z., **255**, 877–896 (2007).

[14] G. Barbatis and A. Tertikas, *On a class of Rellich inequalities*, J. Comp. and Applied Math. **194**, 156–172 (2006).

[15] G. Barbatis, S. Fillipas, and A. Tertikas, *Series Expansions for Lp Hardy inequalities*, Indiana University Math. J., **52**, 171–190 (2003).

[16] G. Barbatis, S. Fillipas, and A. Tertikas, *Refined Geometric L^p Hardy Inequalities*, Comm. in Cont. Math. **5** (6), 869–881 (2003).

[17] G. Barbatis, S. Fillipas, and A. Tertikas, *A Unified Approach to Improved L^p Hardy Inequalities with Best Constants*, Trans. American Math. Soc. **356** (6), 2169–2196 (2003).

[18] H. Baumgärtel and M. Wollenberg, *Mathematical Scattering Theory*, Operator Theory: Advances and Applications, Vol. 9, Birkhäuser, Boston, 1983.

[19] M. S. Birman, *The spectrum of singular boundary problems*, Mat. Sb. (N.S.) **55** (97), 125–174 1961 (Russian). Engl. transl. in Amer. Math. Soc. Transl., Ser. 2, **53**, 23–80 (1966).

[20] M. S. Birman and M. Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, Reidel, Dordrecht, 1987.

[21] D. W. Boyd, *The spectrum of the Cesàro operator*, Acta Sci.Math. **29**, 31–34 (1968).

[22] J. S. Bradley, *Hardy inequalities with mixed norms*, Canad. Math. Bull. **21**, 405–408 (1978).

[23] A. Brown, P. R. Halmos, and A. L. Shields, *Cesàro operators*, Acta Sci. Math. **26**, 125–137 (1965).

[24] V. I. Burenkov, *Sobolev Spaces on Domains*, Teubner, Stuttgart, 1998.

[25] P. Caldiroli and R. Musina, *Rellich inequalities with weights*, Calc. Var. **45**, 147–164 (2012).

[26] P. Cembranos and J. Mendoza, *Banach Spaces of Vector-Valued Functions*, Lecture Notes in Mathematics, Vol. 1676, Springer, Berlin, 1997.

[27] R. S. Chisholm and W. N. Everitt, *On bounded integral operators in the space of integrable-square functions*, Proc. Roy. Soc. Edinb. (A), **69**, 199–204 (1970/71).

[28] R. S. Chisholm, W. N. Everitt, and L. L. Littlejohn, *An integral operator inequality with applications*, J. of Inequal. & Applications **3**, 245–266 (1999).

[29] C. Y. Chuah, F. Gesztesy, L. Littlejohn, T. Mei, I. Michael, and M. M. H. Pang, *On Weighted Hardy-Type Inequalities*, Math. Ineq. & App. (to appear).

[30] J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Graduate Studies in Math., Vol. 96, Springer, 1990.

[31] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry*, Texts and Monographs in Physics, Springer, Berlin, 1987.

[32] E. B. Davies, *Spectral Theory and Differential Operators*, Cambridge Studies in Advanced Mathematics, Vol. 42, Cambridge University Press, Cambridge, UK, 1995.

[33] E. B. Davies, *Linear Operators and their Spectra*, Cambridge Studies in Advanced Math., Vol. 106, Cambridge University Press, 2007.

[34] E. B. Davies and A. M. Hinz, *Explicit constants for Rellich inequalities in* $L_p(\Omega)$, Math. Z. **227** (3), 511–523 (1998).

[35] A. Detalla, T. Horiuchi, and H. Ando, *Missing Terms in Hardy–Sobolev Inequalities and its Application*, Far East J. Math. Sci. **14** (3), 333–359 (2004).

[36] A. Detalla, T. Horiuchi, and H. Ando, *Missing Terms in Hardy–Sobolev Inequalities*, Proc. Japan Acad. **80** (A), 160–165 (2004).

[37] A. Detalla, T. Horiuchi, and H. Ando, *Sharp remainder terms of Hardy–Sobolev inequalities*, Math. J., Ibaraki Univ. **37**, 39–52 (2005).

[38] A. Detalla, T. Horiuchi, and H. Ando, *Sharp Remainder Terms of the Rellich Inequality and its Application*, Bull. Malays. Math. Sci. Soc. (2) **35** (2A), 519–528 (2012).

[39] Yu. A. Dubinskii, *Hardy Inequalities with Exceptional Parameter Values and Applications*, Doklady Math. **80** (1), 558–562 (2009).

[40] Yu. A. Dubinskii, *A Hardy-Type Inequality and Its Applications*, Proc. Steklov Inst. Math. **269** (1), 106–126 (2010).

[41] Yu. A. Dubinskii, *Bilateral scales of Hardy inequalities and their applications to some problems in mathematical physics.*, J. Math. Sci. **201** (6), 751–795 (2014).

[42] J. Diestel and J. J. Uhl, *Vector Measures*, Mathematical Surveys, Vol. 15, Amer. Math. Soc., Providence, RI, 1977.

[43] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1989.

[44] D. E. Edmunds and W. D. Evans, *Hardy Operators, Function Spaces, and Embeddings*, Springer, Berlin, 2004.

[45] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Math., Vol. 19, 1997.

[46] W. G. Faris, *Weak Lebesgue spaces and quantum mechanical binding*, Duke Math. J. **43**, 365–373 (1976).

[47] S. Fillipas and A. Tertikas, *Optimizing Improved Hardy Inequalities*, J. Funct. Anal. **192** (1), 186–233 (2002).

[48] G. Folland, *Real Analysis: Modern Techniques and their Applications*, 2nd ed., 1999.

[49] F. Gesztesy, *On non-degenerate ground states for Schrödinger operators*, Rep. Math. Phys. **20**, 93–109 (1984).

[50] F. Gesztesy and L. L. Littlejohn, *Factorizations and Hardy–Rellich-type inequalities*, in *Partial Differential Equations, Mathematical Physics, and Stochastic Analysis. A Volume in Honor of Helge Holden's 60th Birthday*, EMS Congress Reports, F. Gesztesy, H. Hanche-Olsen, E. Jakobsen, Y. Lyubarskii, N. Risebro, and K. Seip (eds.), 207–226 (2018).

[51] F. Gesztesy, L. L. Littlejohn, I. Michael, and M. M. H. Pang, *Radial and Logarithmic Refinements of Hardy's Inequality*, Algebra i Analiz, **30(3)**, 55–65 (2018) (Russian), St. Petersburg Math. J., St. **30**, 429–436 (2019) (English).

[52] F. Gesztesy, L. L. Littlejohn, I. Michael, and M. M. H. Pang, in preparation.

[53] F. Gesztesy, L. L. Littlejohn, I. Michael, and R. Wellman, *On Birman's Sequence of Hardy–Rellich-Type Inequalities*, J. Diff. Eq. **264(4)**, 2761–2801 (2018).

[54] F. Gesztesy, M. Mitrea, I. Nenciu, and G. Teschl, *Decoupling of deficiency indices and applications to Schrödinger-type operators with possibly strongly singular potentials*, Adv. Math. **301**, 1022–1061 (2016).

[55] F. Gesztesy and L. Pittner, *A generalization of the virial theorem for strongly singular potentials*, Rep. Math. Phys. **18**, 149–162 (1980).

[56] F. Gesztesy and M. Ünal, *Perturbative oscillation criteria and Hardy-type inequalities*, Math. Nachr. **189**, 121–144 (1998).

[57] F. Gesztesy, R. Weikard, and M. Zinchenko, *Initial value problems and Weyl–Titchmarsh theory for Schrödinger operators with operator-valued potentials*, Operators and Matrices **7**, 241–283 (2013).

[58] N. Ghoussoub and A. Moradifam, *Bessel pairs and optimal Hardy and Hardy–Rellich inequalities*, Math. Ann. **349** (1), 1–57 (2011).

[59] N. Ghoussoub, A. Moradifam, *Functional Inequalities. New Perspectives and New Applications* , AMS, 2013.

[60] I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, Israel Program for Scientific Translations, Jerusalem, 1965, Daniel Davey & Co., Inc., New York, 1966.

[61] A. Gogatishvili, A. Kufner, L.-E. Persson, and A. Wedestig, *An equivalence theorem for integral conditions related to Hardy's inequality*, Real Anal. Exchange **29**, 867–880 (2003/04).

[62] M. González and F. León-Saavedra, *Cyclic behavior of the Cesàro operator on* $L_2(0, \infty)$, Proc. Amer. Math. Soc. **137**, 2049–2055 (2009).

[63] M. Haase, *The Functional Calculus for Sectorial Operators*, Operatory Theory: Advances and Appllications, Vol. 169, Birkhäuser, 2006.

[64] F. Hansen, *Non-commutative Hardy inequalities*, Bull. Lond. Math. Soc. **41**, 1009–1016 (2009).

[65] F. Hansen, K. Krulić, J. Pečarić, and L.-E. Persson, *Generalized noncommutative Hardy and Hardy–Hilbert type inequalities*, Intl. J. Math. **21**, 1283–1295 (2010).

[66] G. H. Hardy, *Notes on some points in the integral calculus LXIV*, Messenger Math. **57**, 12–16 (1928).

[67] G. H. Hardy, *Notes on some points in the integral calculus, XLI. On the convergence of certain integrals and series*, Messenger Math. **45**, 163–166 (1915).

[68] G. H. Hardy, *Notes on some points in the integral calculus, LI. On Hilbert's double-series theorem, and some connected theorems concerning the convergence of infinite series and integrals*, Messenger Math. **48**, 107–112 (1919).

[69] G. H. Hardy, *Note on a theorem of Hilbert*, Math. Z. **6**, 314–317 (1920).

[70] G. H. Hardy, *Notes on some points in the integral calculus, LX. An inequality between integrals*, Messenger Math. **54**, 150–156 (1925).

[71] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, reprinted, 1988.

[72] P. Hartman, *On the linear logarithmic-exponential differential equation of the second-order*, Amer. J. Math. **70**, 764–779 (1948).

[73] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, 1964.

[74] P. Hartman, *Ordinary Differential Equations*, 2nd ed., SIAM, 2004.

[75] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Göttingen Nachr., 157–227 (1906).

[76] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Colloquium Publications, Vol. 31, rev. ed., Amer. Math. Soc., Providence, RI, 1985.

[77] N. Ioku and M. Ishiwata, *A Scale Invariant Form of a Critical Hardy Inequality*, Int. Math. Res. Not. **2015** (18), 8830–8846 (2015).

[78] F. Jones, *Lebesgue Integration on Euclidean Space*, rev. ed., Jones and Bartlett, 2001.

[79] H. Kalf, *On the characterization of the Friedrichs extension of ordinary or elliptic differential operators with a strongly singular potential*, J. Funct. Anal. **10**, 230–250 (1972).

[80] H. Kalf, U.-W. Schmincke, J. Walter, and R. Wüst, *On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials*, in *Spectral Theory and Differential Equations*, W. N. Everitt (ed.), Lecture Notes in Math., Vol. 448, Springer, Berlin, 1975, pp. 182–226.

[81] H. Kalf and J. Walter, *Strongly singular potentials and essential self-adjointness of singular elliptic operators in* $C_0^\infty(\mathbb{R}^n\backslash\{0\})$, J. Funct. Anal. **10**, 114–130 (1972).

[82] M. Kian, *On a Hardy operator inequality*, Positivity **22**, 773–781 (2018).

[83] T. Kilpelainen, *Weighted Sobolev Spaces and Capacity*, Ann. Acad. Sci. Fenn. Ser. A I Math. **19(1)**, 95–113 (1994).

[84] A. Kufner, *Weighted Sobolev Spaces*, A Wiley-Interscience Publication, John Wiley & Sons, 1985.

[85] A. Kufner, L. Maligranda and L.-E. Persson, *The Hardy Inequality: About its History and Some Related Results*, Vydavatelský Servis, Pllsen, 2007.

[86] A. Kufner and L.-E. Persson, *Weighted Inequalities of Hardy Type*, World Scientific, Singapore, 2003.

[87] A. Kufner, L.-E. Persson, and N. Samko, *Weighted Inequalities of Hardy Type*, 2nd ed., World Scientific, Singapore, 2017.

[88] S. T. Kuroda, *An Introduction to Scattering Theory*, Aarhus University Lecture Notes Series, No. 51, 1978.

[89] M. Lacruz, F. León-Saavedra, S. Petrovic, O. Zabeti, *Extended eigenvalues for Cesàro operators*, J. Math. Anal. Appl. **429**, 623–657 (2015).

[90] G. Leibowitz, *The Cesàro operators and their generalizations: Examples in infinite-dimensional linear analysis*, Amer. Math. Monthly **80**, 654–661 (1973).

[91] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Graduate Studies in Math., Vol. 14, Amer. Math. Soc., Providence, RI., 2001.

[92] A. Meskhi, *Solution of some weight problems for the Riemann–Liouville and Weyl operators*, Georgian Math. J. **5**, 565–574 (1998).

[93] J. Mikusiński, *The Bochner Integral*, Academic Press, New York, 1978.

[94] A. Moradifam, *Optimal Weighted Hardy–Rellich Inequalities on $H^2 \cap H_0^1$*, J. London Math. Soc. **85** (2), 22–40 (2012).

[95] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math. **44**, 31–38 (1972).

[96] E. Müeller-Pfeiffer, *Spectral Theory of Ordinary Differential Operators*, Ellis Horwood Limited, West Sussex, 1981.

[97] M. Nassyrova, *Weighted Inequalities Involving Hardy-type and Limiting Geometric Mean Operators*, doctoral dissertation, Dept. of Math., Luleå University of Technology, March 2002.

[98] M. Nasyrova and V. Stepanov, *On weighted Hardy inequalities on semiaxis for functions vanishing at the endpoints*, J. Inequal. Appl. **1**, 223–238 (1997).

[99] M. Nasyrova and V. Stepanov, *On maximal overdetermined Hardy's inequality of second order on a finite interval*, Math. Bohem. **124**, 293–302 (1999).

[100] J. Newman and M. Solomyak, *Two-sided estimates on singular values for a class of integral operators on the semi-axis*, Integral Eq. Operator Th. **20**, 335–349 (1994).

[101] B. Opic and A. Kufner, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series, Vol. 219, Longman Scientific & Technical, Harlow, 1990.

[102] M. P. Owen, *The Hardy–Rellich inequality for polyharmonic Operators*, Proc. Roy. Soc. Edinburgh, Sect. A **129**, 825–839 (1999).

[103] B. J. Pettis *On integration in vector spaces*, Trans. Am. Math. Soc. **44**, 277–304, (1938).

[104] D. V. Prokhorov, *On the boundedness and compactness of a class of integral operators*, J. London Math. Soc. (2) **61**, 617–628 (2000).

[105] M. Reed, B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self Adjointness* Academic Press Inc., 1975.

[106] F. Rellich, *Halbbeschränkte Differentialoperatoren Höherer Ordnung*, Proc. Int. Congress Math. **3** 243–250 (1954). Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956 (German).

[107] F. Rellich, *Perturbation Theory of Eigenvalue Problems*, Gordon and Breach, 1969.

[108] M. Ruzhansky and N. Yessirkegenov, *Factorizations and Hardy–Rellich inequalities on stratified groups*,

[109] U.-W. Schmincke, *Essential self-adjointness of a Schrödinger operator with strongly singular potential*, Math. Z. **124**, 47-50 (1972).

[110] I. Schur, *Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen*, J. Math. **140**, 1–28 (1911).

[111] M. A. Shubin, *Pseudodifferential Operators and Spectral Theory, Second Ed.* Springer-Verlag Berlin Heidelberg New York, 2001.

[112] F. Takahashi, *A simple proof of Hardy's inequality in a limiting case*, Arch. Math. **104** (1), 77–82 (2015).

[113] G. Talenti, *Osservazioni sopra una classe di disuguaglianze*, Rend. Sem. Mat. Fis. Milano **39**, 171–185 (1969).

[114] A. Tertikas and N. B. Zographopoulos, *Best constants in the Hardy–Rellich inequalities and related improvements*, Adv. Math. **209**, 407–459 (2007).

[115] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Chelsea, New York, 1986.

[116] G. Tomaselli, *A class of inequalities*, Boll. Un. Mat. Ital. (4)**2**, 622–631 (1969).

[117] S. B. Yakubovich, *Index Transforms*, World Scientific, Singapore, 1996.

[118] D. Yafaev, *Sharp constants in the Hardy–Rellich inequalities*, J. Funct. Anal. **168**, 121–144 (1999).

[119] K. Yosida, *Functional Analysis*, 6th ed., Springer, Berlin, 1980.