Net Regular Signed Trees

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Abstract

A signed graph is called net regular if the sum of the signs of every edge incident to each vertex is constant. Graphs that admit a signing making them net regular are called net regularizable. In this paper, net regular signed trees are studied, including general properties, conditions for a tree to be net regularizable, and generating functions.

1 Introduction

The notion of a signed graph was introduced by Harary [3]. A signed graph consists of a graph and a labeling of the edges with ± 1. The signed degree or net degree of a vertex is the sum of the signs of the edges incident to it. Net degree has been well studied ([1, 4, 5, 6, 8, 9, 10, 11, 15, 16]). A signed graph is called net regular if its net degree is constant ([12, 13]). In [13], regular net regular graphs were examined and the question of examining other types of net regular graphs was raised. In this paper we look at net regular trees. Definitions are given in §2, general properties and two equivalent conditions for a tree to be net regularizable appear in §3 (Theorems 3.6 and 3.9), §4 provides an algorithm and computes the initial terms of the generating function for the number of net regularizable trees (Theorem 4.1), and §5 contains some closing remarks.

2 Preliminaries

In this paper, we write $T = (V, E)$ for a *finite tree*. We assume that T is *neither empty nor the singleton* graph to avoid trivialities.

A *signed tree* is a pair (T, σ) with $\sigma : E \to \{\pm 1\}$ a labeling of the edges. For $v \in V$, the *net degree* $d^{\pm}(v)$ of v is defined as the sum of the signs of the edges incident to v,

$$d^{\pm}(v) = \sum_{u \in N(v)} \sigma(uv),$$

where $N(v)$ denotes the neighborhood of v. A signed tree is called *net regular* if the function $d^{\pm}$ is constant. In that case we write $d^{\pm}(T)$ for the common value. A tree is called *net regularizable* if there exists a signing making it net regular.

Since we are assuming $|V| \geq 2$, T has leaves and therefore the only possible value of $d^{\pm}(T)$ for a net regular tree is ± 1. Without loss of generality, we assume that all net regular trees have net degree

$$d^{\pm}(T) = 1.$$

If T is net regular, it follows immediately that the degree $d(v)$ of each vertex $v \in V$ is odd, that is

$$d(v) \equiv 1 \mod 2.$$

By the handshaking lemma, $|V| \equiv 0 \bmod 2$. In fact, by a general result on signed graphs in [1] or by Theorem 3.6 below,

$$|V| \equiv 2 \mod 4.$$

Naturally, these two conditions are not sufficient to imply net regularity.

3 Structure of Net-Regular Trees

In this section we determine a number of equivalent conditions for a tree T to be net regularizable.

Theorem 3.1. *Any tree may be embedded in a net regularizable tree.*

Proof. Start with a tree T and label each edge with -1. At each vertex, add enough leaves, labeled with $+1$, to make the tree net regular. $\square$

Definition 3.2. (1) A tree of the form given in Figure 1 is called a *chair.*

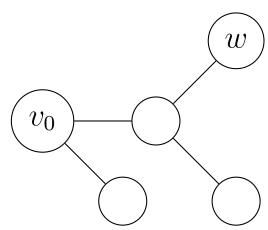

Figure 1: Chair

Namely, a chair is a star graph on four vertices to which an edge is adjoined at one of the leaves.

(2) A subtree $C = (V_C, E_C)$ of $T = (V, E)$ is called an *external chair* of T if (a) C is a chair and (b) if the only edges of $E \backslash E_C$ incident to a vertex of C are incident to v_0 (*i.e.*, C connects to the rest of T only at v_0).

Definition 3.3. For the edge $uv \in E$, deleting uv from T results in two trees. One contains v and the other contains w. Write $T\,(v; u)$ for the tree containing v.

Note that v is a vertex in $T\,(v; u)$ and that T is the disjoint union of the trees $T\,(v; u)$ and $T\,(u; v)$ connected by the edge uv.

Lemma 3.4. *Suppose T is net regular with edge $w_{-1}w_0 \in E$ and w_0 is not a leaf. Pick a vertex w in $T\,(w_0; w_{-1})$ (necessarily a leaf in T) so that* $\operatorname{dist}(w_0, w)$ *is maximal. If* $\operatorname{dist}(w_0, w) \geq 2$*, then w sits in an external chair of $T\,(w_0; w_{-1})$ (and of T) as in Figure 1 with labeling as in Figure 2.*

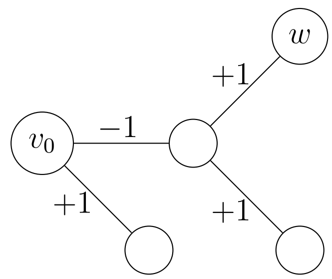

Figure 2: Chair Labels

Proof. Let $w_{-1}, w_0, w_1, \ldots, w_N = w$ be the vertices in the path from w_{-1} to w. Assume $N \geq 2$ and let $v_{-1} = w_{N-3}$, $v_0 = v_{N-2}$, and $v_1 = w_{N-1}$. By maximality, w is a leaf and net regularity forces $\sigma(v_1 w) = +1$.

Regardless of whether $\sigma(v_0 v_1) = \pm 1$, net regularity requires that, in addition to v_0 and w, v_1 be incident to at least one other vertex, w'. Again by maximality, w' must also be a leaf and net regularity forces $\sigma(v_1 w') = +1$. If v_1 were incident to any additional vertices, they too would be leaves and the edges labeled by $+1$ which makes net regularity impossible. It follows that v_1 is incident to only v_0, w, and w'. Finally, net regularity implies that $\sigma(v_0 v_1) = -1$ so that we get a subgraph as in Figure 3.

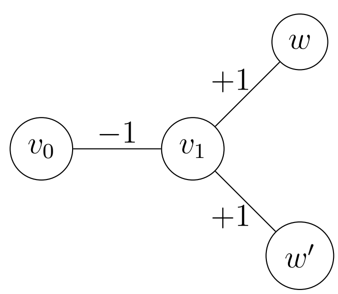

Figure 3:

It remains only to see that v_0 is also incident to a leaf. Besides $v_{-1}v_0$ and $v_0 v_1$, by net regularity, v_0 must be incident to at least one more edge. Write all those edges as $v_0 u_1, \ldots, v_0 u_M$. If at least one of these is a leaf, then we are done. By way of proof by contradiction, suppose this is not the case. Then, by maximality and by the same considerations as above, each edge $v_0 u_i$ leads to a subgraph as in Figure 4. As this makes net regularity impossible at v_0,

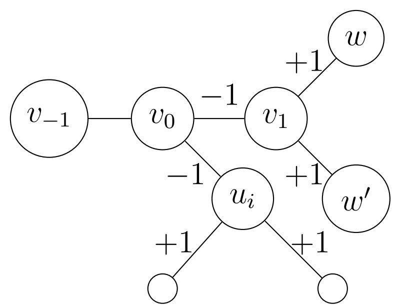

Figure 4:

we have our contradiction. $\square$

Definition 3.5. (1) We say T is *constructible from T' by iteratively attaching chairs* if either $T = T'$ or there is a sequence of trees $T_0 = T', T_1, \ldots, T_N = T$ so that T_{i+1} is constructed from T_i and a chair C by identifying a vertex of T_{i+1} with the vertex v_0 of C (see Figure 1).

(2) We say T *reduces to T' after an iterative removal of chairs* if T is constructible from T' by iteratively attaching chairs.

Recall that a tree T is assumed to be finite and neither empty nor the singleton graph in this paper. We write P_n for the path graph with n vertices.

Theorem 3.6. *Let T be a tree. There exists a signing making T net regular if and only if T is constructible from P_2 by iteratively attaching chairs. In that case, the sign of the original P_2 edge is $+1$ and the sign of each successive chair is given in Figure 2.*

Proof. It is obvious that a tree iteratively constructible from P_2 by attaching chairs (with the signing as indicated above) is net regular. For the opposite direction, argue by induction on the number of vertices $|T|$. The base case of $|T| = 2$ (*i.e.*, $T = P_2$) is trivial.

Now if T is net regular with $|T| > 2$, pick a leaf w_{-1}. By net regularity, w_{-1} is incident to exactly one edge, $w_{-1}w_0$, with sign $+1$. Pick a vertex w in $T(w_0; w_{-1})$ so that $\operatorname{dist}(w_0, w)$ is maximal. If $\operatorname{dist}(w_0, w) = 1$, then T must consist of a central point, w_0, with a bunch of spokes. Precisely, T consists of the vertices $w_{-1}, w_0, w = w_1, \ldots, w_N$ with edges $w_0w_{-1}, w_0w_1, \ldots, w_0w_N$. It is clear that such a graph is not net regular. Therefore, $\operatorname{dist}(w_0, w) \geq 2$ and Lemma 3.4 applies.

Let C be an external chair of T with vertex v_0 of C labeled as in Figure 1. Notice that net regularity forces C to have the labeling from Figure 2. Define T' to be the tree obtained from T by removing all edges of C and all vertices of C except for v_0. By the labeling of C, it follows that T' is still net regular (with the signing given by restriction from T to T'). By construction, $2 \leq |T'| < |T|$. The induction hypothesis implies that T' is iteratively constructible from P_2 by attaching chairs which, in turn, shows that T is as well. $\square$

As an immediate corollary of Theorem 3.6, we recover the fact that

$$|T| \equiv 2 \mod 4$$

when T is net regularizable.

Corollary 3.7. *If T is constructible from P_2 by iteratively attaching k chairs, then $|T| = 2 + 4k$ and there are precisely k edges labeled with a -1.*

Lemma 3.8. *Suppose T is net regular with edge $w_{-1}w_0 \in E$.*

(1) If $\sigma(w_{-1}w_0) = +1$, then $T(w_0; w_{-1})$ reduces to the graph P_1, as in Figure 5, after an iterative removal of chairs.

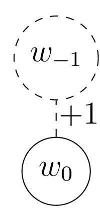

Figure 5:

(2) If $\sigma(w_{-1}w_0) = -1$, then $T(w_0; w_{-1})$ reduces to the graph P_3, as in Figure 6, after an iterative removal of chairs.

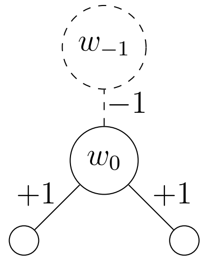

Figure 6:

Proof. By repeated application of Lemma 3.4, after an iterative removal of chairs, $T(w_0; w_{-1})$ reduces to a graph T' satisfying $\operatorname{dist}(w_0, w) = 1$ for any other vertex w of T' (if there are any). The result now follows by net regularity. $\square$

Theorem 3.9. *Let T be a tree. There exists a signing making T net regular if and only if:*

(1) For each edge uv, $|T(v;u)|$ is congruent to 1 or $3 \bmod 4$.

(2) For each vertex v, the number of incident edges vu with $|T(v;u)| \equiv 1$ is one greater than the number of incident edges vu with $|T(v;u)| \equiv 3$.

In this case, $\sigma(uv) = +1$ when $|T(v;u)| \equiv 1$ and $\sigma(uv) = -1$ when $|T(v;u)| \equiv 3$.

Proof. Clearly conditions (1) and (2) are sufficient. To see they are necessary, suppose T is net regular. Let uv be an edge with $\sigma(uv) = +1$. By Lemma 3.8, $T(v;u)$ reduces to P_1 after an iterative removal of chairs so that $|T(v;u)| \equiv 1$. Similarly, if $\sigma(uv) = -1$, then $T(v;u)$ reduces to P_3 after an iterative removal of chairs so that $|T(v;u)| \equiv 3$. $\square$

For the following, recall that we normalize all net regular trees to have net degree $d^{\pm}(T) = 1$.

Corollary 3.10. *A net regularizable tree has a unique choice of signing that makes it net regular.*

Corollary 3.11. *If T is net regular and uv is an edge, then T is constructible from that particular edge uv by iteratively attaching chairs if and only if $\sigma(uv) = +1$ if and only if $|T(v;u)| \equiv 1 \bmod 4$.*

4 Generating Functions

Write $t_{\mathrm{NR}}(k)$ for the *number of net regularizable trees* with $2+4k$ vertices and $t(n)$ for the *number of (unlabeled) trees* with n vertices. Write $\mathrm{NR}(x) = \sum_0^\infty t_{\mathrm{NR}}(n)\, x^n$ for the *generating function for the number of net regularizable trees.* It is natural to suspect that $t_{\mathrm{NR}}(k)$ may not have a closed formula. Instead, in this section, we calculate the first few terms of $\mathrm{NR}(x)$. Before we begin, we observe that

$$t_{\mathrm{NR}}(k) \geq t(k)$$

by Theorem 3.1 and Corollary 3.7. This appears to be extremely far from sharp.

Theorem 4.1. *The generating function for the number of net regularizable trees is*

$$\begin{aligned}\mathrm{NR}(x) = 1 &+ x + 2x^2 + 6x^3 + 22x^4 + 95x^5 + 465x^6 + 2{,}470x^7 \\ &+ 13{,}965x^8 + 82{,}333x^9 + 501{,}469x^{10} + 3{,}131{,}490x^{11} + 19{,}955{,}360x^{12} \\ &+ 129{,}294{,}514x^{13} + 849{,}505{,}193x^{14} + 5{,}648{,}076{,}997x^{15} + \cdots\end{aligned}$$

The proof of this result will occupy the next few pages.

4.1 Chair Trees

Define a *chair tree* to be a tree constructible from a fixed initial marked chair C_0 by iteratively attaching chairs to any vertex except v_0 (Figure 1). Write $\mathrm{CT}(x) = \sum_1^\infty a_n x^n$ for the corresponding generating function. Precisely, a_n is the number of chair trees constructed from n chairs. By trivial inspection and a few minutes of drawing, $\mathrm{CT}(x) = x + 3x^2 + 15x^3 + \cdots$.

Define a *multiset of chair trees* to be a union of chair trees all of whose distinguished vertices v_0 are identified. Write $\mathrm{MCT}(x)$ to be the generating function for the multiset of chair trees. The empty multiset of chair trees is allowed so that $\mathrm{MCT}(0) = 1$. As is well known,

$$\mathrm{MCT}(x) = e^{\sum_{k=1}^\infty \frac{1}{k}\mathrm{CT}\left(x^k\right)}. \tag{1}$$

By hand it can be easily seen that $\mathrm{MCT}(x) = 1 + x + 4x^2 + 19x^3 + \cdots$.

Finally, define a P_3 *chair tree* to be tree built by attaching (at v_0) a multiset of chair trees at each vertex of P_3. Write $\mathrm{PCT}(x)$ for the generating function for the P_3 chair trees. Again, the empty P_3 multiset of chair trees is allowed so that $PCT\left(0\right)=1$. By hand, it can be shown that $PCT\left(x\right)=1+2x+9x^2+\cdots$.

By examining the diagram of the fixed initial marked chair in Figure 7 and adding a multiset of chair trees at v_4, a P_3 chair tree at v_1, v_2, v_3, and

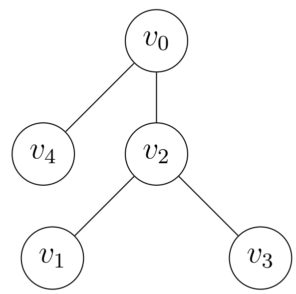

Figure 7:

also counting the initial marked tree, we see that

$$\mathrm{CT}\left(x\right)=x\ \mathrm{MCT}\left(x\right)\ \mathrm{PCT}\left(x\right). \tag{2}$$

Notice that the generating function for $\mathrm{PCT}\left(x\right)$ involves many symmetries that may not at first be apparent. For example, if a chair is added at v_2, then there are actually *three* equivalent children of v_2 to which another chair may be added.

In preparation for determining various of these generating functions, let

$$p=\left(p_1,p_2,\ldots\right)=(\overbrace{q_1,\ldots,q_1}^{a_1},\overbrace{q_2,\ldots,q_2}^{a_2},\ldots)$$

be a partition of $n\in\mathbb{N}$ so that $p_1\geq p_2\geq\cdots>0$ with $\sum_i p_i=n$ and $q_1>q_2>\cdots>0$ with $\sum_i a_i q_i=n$. Then if $F\left(x\right)$ is the generating function for a combinatorial class of objects $\mathcal{F}$, recall that the generating function for selecting *exactly* n *unordered* objects from $\mathcal{F}$ is

$$F_n\left(x\right)\equiv\frac{1}{n!}\sum_{p\in\mathrm{Part}(n)}\frac{n!}{\prod_j q_j^{a_j}a_j!}\prod_i F\left(x^{p_i}\right). \tag{3}$$

For example, the partitions of 2 are $(1,1)$ and (2) so that

$$F_2\left(x\right)=\frac{1}{2}\left(F\left(x\right)^2+F\left(x^2\right)\right)$$

and the partitions of 3 are $(1,1,1)$, $(2,1)$, and (3) so that

$$F_3(x) = \frac{1}{6}\left(F(x)^3 + 3F\left(x^2\right)F(x) + 2F\left(x^3\right)\right).$$

Turn now to $\mathrm{PCT}(x)$. By separating out the chairs attached to the central vertex v of P_3 as in Figure 8 (where a attached chairs are pictured)

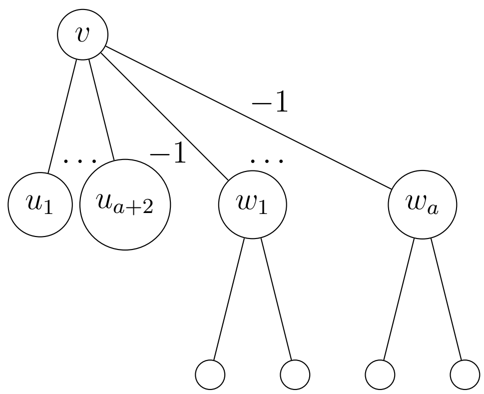

Figure 8:

it follows that

$$\mathrm{PCT}(x) = \sum_{a\geq 0} x^a \,\mathrm{PCT}_a(x)\,\mathrm{MCT}_{a+2}(x). \tag{4}$$

Combined with Equations 1 and 2, Equation 4 gives an easily automated iterative method for calculating $\mathrm{CT}(x)$, $\mathrm{MCT}(x)$, and $\mathrm{PCT}(x)$. For example, if $\mathrm{CT}(x) = \sum_{k=0}^{\infty} a_k x^k$ and $\mathrm{PCT}(x) = \sum_{k=0}^{\infty} b_k x^k$, one finds that Equation 4 implies

$$b_0 = 1, \quad b_1 = 1 + a_1, \quad b_2 = 2 + 3a_1 + a_1^2 + a_2,$$
$$b_3 = \frac{1}{3}(12 + 22a_1 + 12a_1^2 + 2a_1^3 + 6a_2 + 6a_1a_2 + 3a_3), \ldots$$

Combining this with Equations 1 and 2, we can inductively solve for $\mathrm{CT}(x)$ and obtain

$$\begin{aligned}\mathrm{CT}(x) = x &+ 3x^2 + 15x^3 + 79x^4 + 463x^5 + 2{,}842x^6 + 18{,}261x^7 + 120{,}834x^8 \\ &+ 819{,}229x^9 + 5{,}658{,}536x^{10} + 39{,}685{,}005x^{11} + 281{,}826{,}519x^{12} \\ &+ 2{,}022{,}583{,}829x^{13} + 14{,}645{,}875{,}257x^{14} + 106{,}873{,}747{,}884x^{15} + \cdots.\end{aligned}$$

In turn, this allows to calculate that

$$\begin{aligned}\mathrm{MCT}(x) &= 1+x+4x^2+19x^3+104x^4+612x^5+3{,}821x^6+24{,}746x^7+165{,}060x^8\\ &+ 1{,}125{,}442x^9 + 7{,}810{,}707x^{10} + 54{,}988{,}526x^{11} + 391{,}760{,}249x^{12}\\ &+ 28{,}19{,}145{,}479x^{13} + 20{,}461{,}211{,}968x^{14} + 149{,}608{,}592{,}569x^{15} + \cdots\end{aligned}$$

and

$$\begin{aligned}\mathrm{PCT}(x) &= 1 + 2x + 9x^2 + 43x^3 + 242x^4 + 1437x^5 + 9058x^6 + 59{,}062x^7\\ &+ 396{,}207x^8 + 2{,}713{,}848x^9 + 18{,}906{,}784x^{10} + 133{,}534{,}659x^{11}\\ &+ 953{,}964{,}640x^{12} + 6{,}881{,}027{,}743x^{13} + 50{,}044{,}650{,}836x^{14} + \cdots .\end{aligned}$$

4.2 Dissymmetry Theorem

With an eye towards using the Dissymmetry Theorem ([2], §4.3.3) that came out of the work of Otter [7], let $\mathrm{NR}^{mv}(x)$ be the generating function (indexed by the number of chairs) for net regular trees with a *marked vertex*, $\mathrm{NR}^{me}(x)$ be the generating function (indexed by the number of chairs) for net regular trees with a *marked edge*, and $\mathrm{NR}^{mde}(x)$ be the generating function (indexed by the number of chairs) for net regular trees with a *marked directed edge*.

By looking at the edges incident to a marked vertex v_0 and using Lemma 3.8, it follows that every net regular tree with marked vertex v_0 is constructed iteratively by adding chairs anywhere but v_0 to any diagram of the form found in Figure 9 (the picture below shows a edges labeled with -1): It follows that

$$\mathrm{NR}^{mv}(x) = \sum_{a=0}^{\infty} x^a \,\mathrm{MCT}_{a+1}(x)\,\mathrm{PCT}_a(x).$$

Therefore it is easy to calculate that

$$\begin{aligned}\mathrm{NR}^{mv}(x) &= 1 + 2x + 8x^2 + 39x^3 + 212x^4 + 1{,}251x^5 + 7{,}793x^6 + 50{,}474x^7\\ &+336{,}556x^8 + 2{,}294{,}871x^9 + 15{,}927{,}450x^{10} + 112{,}144{,}478x^{11} + 799{,}058{,}373x^{12}\\ &+5{,}750{,}838{,}752x^{13} + 41{,}744{,}478{,}744x^{14} + 305{,}264{,}349{,}331x^{15} + \cdots .\end{aligned}$$

The study of an edge marked net regular tree breaks into two cases. Write e for the marked edge. Trees with $\sigma(e) = +1$ clearly contribute $\mathrm{MCT}_2(x)$ to $\mathrm{NR}^{me}(x)$. Trees with $\sigma(e) = -1$ are more involved. By looking at the edges incident to one of the vertices of e and using Lemma 3.8, it follows that every

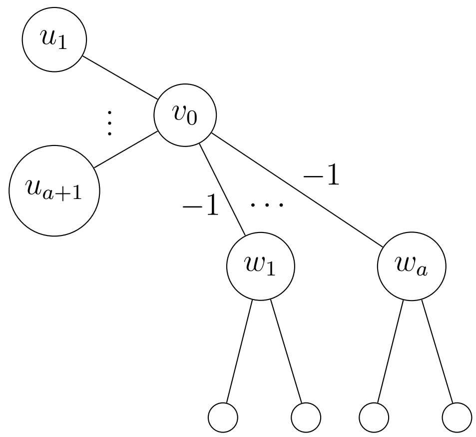

Figure 9:

net regular tree with marked vertex v_0 is constructed iteratively by adding chairs anywhere but v_0 to any diagram of the form found in Figure 10 (we draw only the left side here as the right side is similar): The contribution to $\mathrm{NR}^{me}(x)$ is therefore

$$x\left(\sum_{a=0}^{\infty} x^a\, \mathrm{MCT}_{a+2}(x)\, \mathrm{PCT}_a(x)\right)_2$$

so that

$$\mathrm{NR}^{me}(x) = \mathrm{MCT}_2(x) + x\left(\sum_{a=0}^{\infty} x^a\, \mathrm{MCT}_{a+2}(x)\, \mathrm{PCT}_a(x)\right)_2.$$

Then it is straightforward to then calculate that

$$\begin{aligned}\mathrm{NR}^{me}(x) = {} & 1+2x+7x^2+35x^3+194x^4+1{,}165x^5+7{,}347x^6+48{,}047x^7+322{,}695x^8 \\ & + 2{,}212{,}780x^9 + 15{,}426{,}593x^{10} + 109{,}014{,}425x^{11} + 779{,}106{,}834x^{12} \\ & + 5{,}621{,}553{,}296x^{13} + 40{,}894{,}998{,}297x^{14} + 299{,}616{,}331{,}396x^{15} + \cdots .\end{aligned}$$

For the last piece, it is similarly clear that

$$\mathrm{NR}^{mde}(x) = \mathrm{MCT}(x)^2 + x\left(\sum_{a=0}^{\infty} x^a\, \mathrm{MCT}_{a+2}(x)\, \mathrm{PCT}_a(x)\right)^2.$$

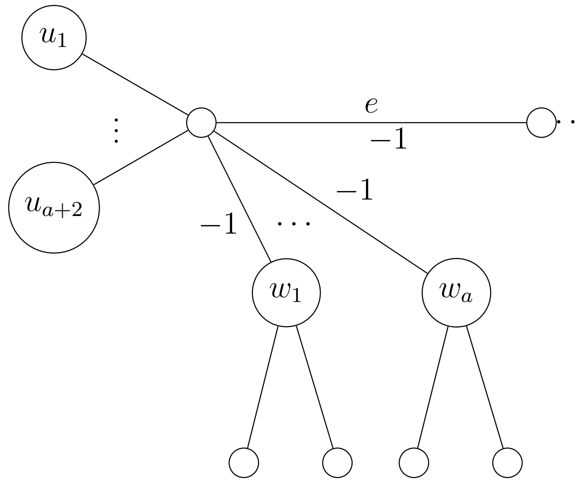

Figure 10:

From this it follows that

$$\mathrm{NR}^{mde}(x) = 1 + 3x + 13x^2 + 68x^3 + 384x^4 + 2{,}321x^5 + 14{,}675x^6 + 96{,}051x^7 + 645{,}286x^8 + 4{,}425{,}318x^9 + 30{,}852{,}574x^{10} + 218{,}027{,}413x^{11} + 1{,}558{,}209{,}847x^{12} + 11{,}243{,}097{,}534x^{13} + 81{,}789{,}971{,}848x^{14} + 599{,}232{,}603{,}730x^{15} + \cdots .$$

Finally, the Dissymmetry Theorem tells us that

$$\mathrm{NR}(x) = \mathrm{NR}^{mv}(x) + \mathrm{NR}^{me}(x) - \mathrm{NR}^{mde}(x).$$

Theorem 4.1 follows.

5 Closing Remarks

It would be interesting to give a closed form relation that implicitly determines $\mathrm{NR}(x)$. Although the results of the previous section allows a fairly rapid calculation of $\mathrm{NR}(x)$ up to degree 15, the techniques do not immediately provide a closed form relation.

It would also be interesting to determine the generating function for net regular trees of a specified diameter d. Toward that end, we mention a few results. First of all, for $k \geq 2$ (for $k = 0$ the only diameter is 1 and for

$k = 1$ the only diameter is 3), it can be shown that net regular trees of order $n = 2 + 4k$ have all diameters $d \in \{4, \dots, n/2\}$.

Given such a d, we will define certain fundamental diagrams from which all diameter d net regular trees may be constructed by the addition of chairs. Begin with an arbitrary choice of $S \subseteq \{1, 2, \dots, d-3\}$ subject to the requirements that $|S| \equiv d - 3 \bmod 2$ and $d + |S| \leq n/2$ (the later condition is automatically satisfied when $d \leq 2 + k$). For such a choice of $|S|$, there is a unique associated word in the letters $\{A, B, *\}$ of length $d-1$ beginning and ending with A in positions 0 and $d-2$ so that (1) a $*$ is written in positions corresponding to S^c and (2) A's and B's are uniquely filled in to the positions corresponding to S so that (a) the number of stars between successive A's or successive B's is even and (b) the number of stars between successive A's and B's is odd. We say two subsets S are *equivalent* if their corresponding words agree up to a reversal of order.

For each representative of an equivalence class of subsets S, begin with the path P_d and add edges as follows: (1) add no edges to the first and last vertex, (2) to the second and penultimate vertex, append a leaf, (3) take the associated word and remove the first and last letters and identify, in order, the resulting word with the remaining vertices, (4) for each $*$, add a leaf, (5) for each A, three leaves, and (6) for each B, add a single incident edge which, in turn, is connected to two leaves. The diagrams constructed in this way are called the *fundamental diagrams*. For example, with $n = 22$, $d = 7$, and $S = \{2, 4\} \subseteq \{1, 2, 3, 4\}$, the corresponding word is $A * B * AA$ and the corresponding fundamental diagram is found in Figure 11.

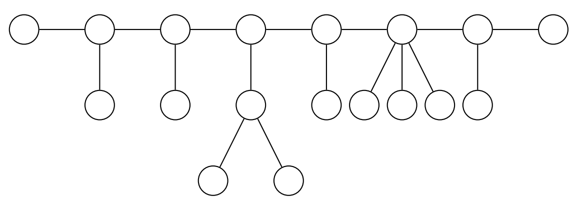

Figure 11:

It can be shown that a fundamental diagram has a unique choice of sign making it net regular. Then it is possible to show that all net regular trees of order n and diameter d arise by adding $(n - d - |S|)/2$ chairs anywhere to a fundamental diagram within the "triangle" that does not increase its diameter (as S varies over all legal equivalence classes of subsets). The number of fundamental diagrams can be counted with the help of Burnside's Counting Lemma. When $d \leq k+2$, it turns out that this number is

$$2^{\lfloor (d-5)/2 \rfloor} + 2^{d-5}$$

(OEIS A005418, [14]) which gives a lower bound on the number net regular trees of order n and diameter d. When $d \geq k+3$, there is a formula given in terms of partial sums of a row of Pascal's triangle and so does not seem to admit a nice closed formula—although growth estimates are still possible.

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