# Solutions to Analysis Bank

**II.1** Let $\mathcal{A}$ be $\sigma$-algebra of $X$. $f : X \to X$ a map.

Need to prove $\{ f^{-1}(S) ; S \in \mathcal{A} \}$ is a $\sigma$-algebra "$\mathcal{B}$".

i) $\phi = f^{-1}(\phi) \in \mathcal{B}$

ii) Let $S \in \mathcal{A}$, then $X \setminus f^{-1}(S) = f^{-1}(X \setminus S)$.

Hence $X \setminus f^{-1}(S) \in \mathcal{B}$.

iii) $f^{-1}\left( \bigcup_{i=1}^{\infty} E_i \right) = \bigcup_{i=1}^{\infty} f^{-1}(E_i) \in \mathcal{B}$.

Hence by definition $\mathcal{B}$ is a $\sigma$-algebra. $\blacksquare$

**II.8** Consider the function :-

$$f : [0,1] \to [0,1]$$

where $f(t) = l(E \cap [0,t])$

Let $\varepsilon > 0$ and $t_0 \in [0,1]$. Without loss of generality assume $t > t_0$. Then

$f(t) - f(t_0) = l(E \cap [t_0, t])$. ~~Let [illegible]~~.

Then $l(E \cap [t_0, t])$ ~~[illegible]~~ $\leq l([t_0, t]) = t - t_0$.

Hence if $|t - t_0| < \varepsilon$

$\Rightarrow$ $|f(t) - f(t_0)| < \varepsilon$.

Therefore $f$ is continuous.

we have $f(0) = 0$ and $f(1) = l(E) > 0$

$\therefore$ By intermediate value theorem $\exists$ $t$ s.t.

$$f(t) = \frac{l(E)}{2}.$$

i.e. $$l(E \cap [0, t]) = \frac{l(E)}{2}.$$

$\therefore$ $A = E \cap [0, t]$.

II.9 we have $l(E) = \inf \left\{ \sum_1^\infty l(I_k) : E \subset \bigcup_1^\infty I_k \right\}$.

Let $\varepsilon > 0$, $\exists$ $\{I_k\}$ s.t. $\sum_1^\infty l(I_k) < l(E) + \varepsilon$.

Given that $l(E \cap I_k) \leq \frac{l(I_k)}{2}$.

Hence $$\sum 2\, l(E \cap I_k) \leq \sum_{k=1}^\infty l(I_k) < l(E) + \varepsilon.$$

we have that

$$\sum l(E \cap I_k) \geq l\left(\bigcup_{k=1}^\infty E \cap I_k\right) = l\left(E \cap \bigcup_{k=1}^\infty I_k\right)$$
$$= l(E).$$

$\therefore$ $$2\, l(E) \leq 2 \sum l(E \cap I_k) < l(E) + \varepsilon$$

$\Rightarrow$ $l(E) < \varepsilon \ \forall \ \varepsilon > 0$ $\Rightarrow$ $l(E) = 0$.

II. 11 We will prove this in general for absolutely continuous function $f$. That is $\forall \varepsilon > 0 \ \exists \delta$ s.t. for pairwise disjoint intervals $(x_k, y_k)$, $1 \le k \le n$ satisfying $\sum_{k=1}^{n} (y_k - x_k) < \delta \Rightarrow \sum_{k=1}^{n} |f(y_k) - f(x_k)| < \varepsilon$.

Since $A$ is measurable set ~~$\exists$~~ ~~$\delta(\varepsilon)$~~ ~~s.t.~~ and $A \subseteq [0,1]$ then $\exists \{I_k\}$ s.t. $\sum_{k=1}^{\infty} l(I_k) < \delta$ and $A \subseteq \bigcup_{k=1}^{\infty} I_k$. ~~I can~~ we can choose $\{I_k\}$ s.t. they are pairwise disjoint and we can find finitely many $I_k$ s.t. $A \subseteq \dot{\bigcup}_{k=1}^{n} I_k$.

Hence $f(A) \subseteq \dot{\bigcup}_{k=1}^{n} f(I_k)$.

$\therefore \ l(f(A)) \le \sum_{k=1}^{n} |f(y_k) - f(x_k)| < \varepsilon$.

$\therefore \ l(f(A)) < \varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow \ l(f(A)) = 0$.

As $f = x^2$ and $f = \sqrt{x}$ are absolutely continuous, the result follows ∎

II. 18 Let $B = ([0,1])^d$, and $f : B \to \mathbb{R}$ is continuous.

Let $F(x, y) = \chi_{\Gamma}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$

Then by Tonelli's theorem

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} F(x,y)\, dy\, dx = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}} F(x,y)\, dy \right) dx$$

For $x$ fixed, $F_x(y) = \begin{cases} 1 & \text{if } y = f(x) \\ 0 & \text{otherwise} \end{cases}$

Hence $\int_{\mathbb{R}} F_x(y)\, dy = 0$

$\therefore \int_{\mathbb{R}^d} \int_{\mathbb{R}} F(x,y)\, dy\, dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}} F_x(y)\, dy\, dx = 0$

$LHS = l(\Gamma) = RHS = 0.$ ∎

II. 19) Let $\mu$ be finitely additive and $\mu(X) < \infty$.

Suppose $\mu$ is countably additive.

Let $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n \supseteq \dots$ with $\cap A_n = \phi$.

To prove $\lim_{n \to \infty} \mu(A_n) = 0$

Let $B_1 = A_1 \setminus A_2$ ... and $B_n = A_n \setminus A_{n+1}$.

Then $\{B_i\}_1^\infty$ are disjoint sets.

Then $A_n = \bigcup_{k=n}^{\infty} B_k$

$$\Rightarrow \quad \mu(A_n) = \sum_{i=n}^{\infty} \mu(B_i).$$

$\because \mu(X) < \infty \Rightarrow s = \sum_{i=1}^{\infty} \mu(B_i) = \mu(\bigcup_{i=1}^{\infty} B_i) < \infty$.

Hence suppose $s_n = \sum_{i=1}^{n} \mu(B_i)$ and $s_n < s_{n+1}$ and $s_n$ is bounded. Every bounded increasing sequence has a limit.

$\therefore \lim_{n\to\infty} s_n = s < \infty$. $\Rightarrow \lim_{n\to\infty} (s - s_n) = 0$.

Since $\mu(A_n) = s - s_n$, $\Rightarrow \lim_{n\to\infty} \mu(A_n) = 0$.

On the contrary suppose if ~~let~~ $\bigcap_{n=1}^{\infty} A_n = \phi$ and $A_1 \supseteq A_2 \supseteq \ldots$ Then $\lim_{n\to\infty} \mu(A_n) = 0$.

Let $\{B_i\}$ be disjoint sets. and ~~$B_k = A_1 \setminus \bigcup_{i=1}^{k} B_i$~~ $A_k = A_1 \setminus \bigcup_{i=1}^{k} B_i$

$A_1 = \bigcup_{i=1}^{\infty} B_i$,

Then ~~$\mu(A_k) = \mu(A)$~~

clearly $A_1 \supseteq A_2 \supseteq \ldots$.

Then $\mu(A_k) = \mu(A_1) - \mu(B_1 \cup \ldots \cup B_k)$

$$= \mu(A_1) - \sum_{i=1}^{k} \mu(B_i)$$

$$\lim_{k\to\infty} \mu(\cancel{B_k}\, A_k) = 0 \Rightarrow \mu(\cancel{B_1}\, A_1) = \sum_{i=1}^{\infty} \mu(B_i)$$

$$\therefore \quad \mu\left(\bigcup_{1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$$

$\blacksquare$

II.22 Consider $1_{-A} * 1_B(x)$.

$$\int_{\mathbb{R}^n} 1_{-A} * 1_B(x)\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{-A}(x-y)\, 1_B(y)\,dy\,dx$$

(By Tonelli) $$= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} 1_{-A}(x-y)\, 1_B(y)\,dx\right)dy$$

$$1_{-A}(x-y) = \begin{cases} 1 & \text{if } x-y \in -A \text{ i.e. } x \in y-A \\ 0 & \text{if } x-y \notin -A \text{ i.e. } x \notin y-A \end{cases}$$

$$= 1_{y-A}(x).$$

$$\Rightarrow \text{RHS} = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} 1_{y-A}(x)\, 1_B(y)\,dx\right)dy$$

$$= \int_{\mathbb{R}^n} \ell(y-A)\, \cancel{\ell(B)}\, .\, 1_B(y)\,dy$$

Since Lebesgue measure is translation invariant & $\ell(-A) = \ell(A)$

$$\Rightarrow \text{RHS} = \ell(A)\,\ell(B) > 0.$$

Now $$\text{LHS} = \int\left(\int 1_{-A}(x-y)\, 1_B(y)\,dy\right)dx$$

$$= \int\int 1_{(x+A)}(y)\, 1_B(y)\,dy\,dx$$

Since $1_{-A}(x-y) = 1_{(x+A)}(y)$

$\therefore$ $LHS = \int \int 1_{(x+A)}(y)\, 1_B(y)\, dy\, dx$

$= \int \int 1_{(x+A)\cap B}\, dy\, dx$

$= \int \ell((x+A)\cap B)\, dx \;=\; \ell(A)\ell(B) > 0.$

$\therefore$ $\exists$ $x_0$ s.t. $\ell((x_0 + A)\cap B) > 0$.

II.24 Suppose $f_n : X \to \mathbb{R}$ measurable. Let

$S = \{x \mid \lim_{n\to\infty} f_n(x)$ exists $\}$. i.e.

If $x \in S \Rightarrow \lim_{n\to\infty} f_n(x) = \underbrace{\limsup_{n\to\infty} f_n(x)}_{= g} = \underbrace{\liminf_{n\to\infty} f_n(x)}_{= h}$.

We have $g$ and $h$ are measurable, hence $g-h$ are measurable.

$\therefore$ $S = (g-h)^{-1}(\{0\})$ is a measurable set as

$\{0\}$ is a measurable set

III. 1

(a) $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sin\left(\frac{\pi x}{n}\right) = \sin(0) = 0.$

Let $K$ be compact then $\exists\, n_0$ s.t. $\forall\, n \geq n_0$ s.t.

$K \subseteq [-n, n]$.

Then $\sup_{x\in K} |f_n(x)| = \sup_{x\in K} \left|\sin\left(\frac{\pi x}{n}\right)\right| \leq \sup_{x\in K} \left|\frac{\pi x}{n}\right| \leq \frac{n_0 \pi}{n}$

$\therefore$ for $\varepsilon > 0$ let $n_1$ be s.t. $\frac{n_0}{n_1}\pi < \varepsilon$.

$\therefore$ $f_n \to 0$ on compact sets.

If $f_n \to 0$ on $\mathbb{R}$ uniformly $\Rightarrow$ $\forall \varepsilon > 0$ $\exists\, n_0$ s.t.

$\sup_{x\in\mathbb{R}} |f_n(x)| < \varepsilon$ $\forall\, n \geq n_0$

But choose $x = \frac{n}{2}$.

Then $|f_n(x)| = \sin\left(\frac{\pi n}{n 2}\right) = 1 > \varepsilon$ if $\varepsilon < 1$.

Hence $f_n \not\to 0$ uniformly on $\mathbb{R}$.

(b) $\int_{-\infty}^{\infty} f(x)\,dx = 0$

$\lim_{n\to\infty} \int_{-n}^{n} \sin\left(\frac{\pi x}{n}\right) = \lim_{n\to\infty} \left.\frac{\cos\frac{\pi x}{n}}{\pi/n}\right|_{-n}^{n} = \lim_{n\to\infty} \frac{n}{\pi}\left[\cos\pi - \cos(-\pi)\right]$

$= \lim_{n\to\infty} \frac{n}{\pi}\left[-1+1\right] = 0.$

Assumptions of LDCT are not satisfied. Since $\nexists\, g \in L^1$ s.t. $|g(x)| \geq |f_n(x)|$

If $\exists$ such a $g$ then $\int |f_n(x)| \le \int |g| < \infty$ $\forall n \in \mathbb{N}$.

We have $\frac{1}{2}\chi_{[-\frac{5n}{6}, -\frac{n}{6}]} + \frac{1}{2}\chi_{[\frac{n}{6}, \frac{5n}{6}]} \le |f_n(x)| \le |g(x)|$

$$\Rightarrow \int \frac{1}{2}\chi_{[-\frac{5n}{6}, -\frac{n}{6}]} + \frac{1}{2}\chi_{[\frac{n}{6}, \frac{5n}{6}]} \le \int |g(x)| < \infty \quad \forall n$$

$$\Rightarrow \frac{2n}{3} \le \int |g(x)| < \infty \quad \forall n.$$

which is a contradiction. $\blacksquare$

III.3

(a) $f_n(x) = \chi_{[1,n]}(x)\left(1 - \frac{x}{n}\right)^n$

$$\int_1^n \left(1 - \frac{x}{n}\right)^n dx = \frac{\left(1 - \frac{x}{n}\right)^{n+1}}{\frac{n+1}{-n}}\Bigg|_1^n$$

$$= \frac{-n}{n+1}\left[-\left(1 - \frac{1}{n}\right)^{n+1}\right] = \frac{n}{n+1}\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n}\right)^n$$

$$\therefore \lim_{n\to\infty} \int_1^n \left(1 - \frac{x}{n}\right)^n dx = 1 \cdot e^{-1} = e^{-1}.$$

(b) $f_n(x) = \chi_{[1,2n]}\left(1 - \frac{x}{n}\right)^n$. Then

$$\int_1^{2n} \left(1 - \frac{x}{n}\right)^n = \left(1 - \frac{x}{n}\right)^{n+1}\left(\frac{-n}{n+1}\right)\Bigg|_1^{2n} = \frac{-n}{n+1}\left[(-1)^{n+1} - \left(1 - \frac{1}{n}\right)^{n+1}\right]$$

If $n = 2k+1$ then $\lim_{k\to\infty} \int \phi_{2k+1}(x) =$ (continued on next page)

$$= \lim_{k\to\infty} \frac{-(2k+1)}{(2k+2)}\left[1 - \left(1-\frac{1}{2k+1}\right)^{2k+1}\left(1-\frac{1}{2k+1}\right)\right]$$

$$= -1\left[1-e^{-1}\right] = e^{-1} - 1.$$

If $n = 2k$ then

$$\lim_{k\to\infty} \int f_{2k}(x) = \lim_{k\to\infty} \frac{-2k}{2k+1}\left[-1-\left(1-\frac{1}{2k}\right)^{2k}\left(1-\frac{1}{2k}\right)\right]$$

$$= -1\left[-1-e^{-1}\right] = e^{-1}+1.$$

$\therefore$ lim does not exist. $\blacksquare$

III. 6 Suppose $\{a_n\}$ and $\{b_n\}$ are bounded. Then $\exists$ subsequence $\{a_{n_k}\}$ and $\{b_{n_k}\}$ s.t. $a_{n_k} \to a$ and $b_{n_k} \to b$ and $|a_{n_k}| \le m$ and $|b_{n_k}| \le M$ $\forall$ $k$.

Clearly $|f_n(x)| \le m + M$.

$\therefore$ By LDCT.

$$1 = \int_0^1 1\,dx = \lim_{k\to\infty} \int_0^1 a_{n_k} \sin(2\pi n_k x) + b_{n_k}\cos(2\pi n_k x)$$

$$= \lim_{k\to\infty}\left[-a\frac{\cos 2\pi n_k x}{2\pi n_k}\Big|_0^1 + b\frac{\sin 2\pi n_k x}{2\pi n_k}\Big|_0^1\right]$$

$= 0 \ne \int_0^1 1\,dx = 1$. Hence we get a contradiction $\blacksquare$

(III.7)

Let $f_n(x) = \cos^{2n}(\pi f(x)) \leq 1$ on $X$.

$$\therefore \lim_{n\to\infty} \int_X \cos^{2n}(\pi f(x))\, dx = \int_X \lim_{n\to\infty} \cos^{2n}(\pi f(x))\, d\mu(x)$$

(By LDCT).

Now $$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } f(x) \notin \mathbb{Z} \\ 1 & \text{if } f(x) \in \mathbb{Z} \end{cases}$$

$$= \chi_{\{x : f(x) \in \mathbb{Z}\}}.$$

$$\therefore \text{LHS} = \mu\{x : f(x) \in \mathbb{Z}\}.$$

(III.13) $f_n(x) = \dfrac{n}{x(\ln(x))^n}$ ; $x \geq e$

(a) $\displaystyle\int_e^\infty \frac{n}{x(\ln x)^n}\, dx$. Put $\ln x = y \Rightarrow \frac{1}{x} dx = dy$

$$\text{LHS} = \int_1^\infty \frac{n}{y^n}\, dy = \frac{n}{1-n} y^{-n+1}\Big|_1^\infty = \frac{n}{n-1} < \infty.$$

for $n \geq 2$.

$\therefore$ The integral exists for $n \geq 2$.

(b) $\ln(x) > 1$ for $x > e$

$\Rightarrow (\ln(x))^n \longrightarrow \infty$ for $x > e$

$\Rightarrow \dfrac{1}{(\ln(x))^n} \longrightarrow 0$ for $x > e$.

By L'Hospital's rule for $n$.

$$\lim_{n\to\infty} \frac{n}{x(\ln x)^n} = \lim_{n\to\infty} \frac{1}{x \ln(\ln x) \ln(x)^n} = 0.$$

(c)
$$\lim_{n\to\infty} \int_e^\infty \frac{n}{x(\ln x)^n} dx = \lim_{n\to\infty} \frac{n}{n-1} = 1$$

$$\int_e^\infty \lim_{n\to\infty} \frac{n}{x(\ln x)^n} dx = 0.$$

Hence LDCT is not satisfied. ∎

III.14 $\quad \phi(x) = \int_{\mathbb{R}} \cos(xy) g(y) dy.$

Let $x_n \to x$, then $|\cos(x_n y) g(y)| \leq |g(y)| \in L^1$.

Hence by LDCT

$$\lim_{n\to\infty} \phi(x_n) = \int_{\mathbb{R}} \cos(xy) g(y) dy = \phi(x).$$

$\therefore$ $\phi$ is continuous ∎

III.16 $\quad f_n(x) = \frac{x^n}{n!} e^{-x}$

(a) For $x > 0$, we have $x^{n-1} \leq (n-1)! e^x$ for $n \geq 1$.

$$\therefore \quad f_n(x) = x \frac{x^{n-1}}{n!} e^{-x} \leq \frac{x (n-1)! e^x e^{-x}}{n!} = \frac{x}{n}$$

Hence $\lim_{n\to\infty} f_n(x) = 0 \quad \forall x > 0.$

(b) $I_n = \int_0^\infty f_n(x) = \int_0^\infty \frac{x^n}{n!} e^{-x}\, dx$

$I_n = e^{-x} \frac{x^{n+1}}{(n+1)!} \Big|_0^\infty + \int_0^\infty e^{-x} \frac{x^{n+1}}{(n+1)!}\, dx$ [By IBP].

$\therefore I_n = 0 + \int_0^\infty e^{-x} \frac{x^{n+1}}{(n+1)!}\, dx = I_{n+1}$

$\therefore$ Let $n=0 \Rightarrow \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1.$

Hence $I_n = 1 \quad \forall n \geq 0.$

(c) Let $f_k(x) = \frac{x^n}{n!} \left(1 - \frac{x}{k}\right)^k \chi_{[0,k]}.$ $f_k(x) \leq \frac{x^n}{k!} e^{-x}$

We have Binomial expansion

For fixed $k$, $0 \leq x \leq k$, $1 - \frac{x}{k} < 1.$

Let $\phi(y) = \left(1 - \frac{x}{y}\right)^y = e^{y \ln(1 - \frac{x}{y})}.$

$\phi'(y) = e^{y\ln(1-\frac{x}{y})} \left( y \left(\frac{1}{1-\frac{x}{y}}\right)\left(\frac{x}{y^2}\right) + \ln\left(1 - \frac{x}{y}\right)\right).$

$\therefore$ For $0 < x < y$, $\phi'(y) > 0.$

Hence increasing.

$f_k(x) = \frac{x^n}{n!}\left(1 - \frac{x}{k}\right)^k \chi(0,k) \leq \frac{x^n}{n!}\left(1 - \frac{x}{k}\right)^k \chi(0,k+1)$

$\leq \frac{x^n}{n!}\left(1 - \frac{x}{k+1}\right)^{k+1} \chi(0,k+1) = f_{k+1}(x).$

$\therefore$ By MCT $\lim_{k\to\infty} \int_0^k \frac{x^n}{n!}\left(1 - \frac{x}{k}\right)^k dx = \int_0^\infty \frac{x^n}{n!} e^{-x} dx = 1 \quad \forall n \geq 0$

■

III. 18 $\displaystyle\lim_{n\to\infty}\int_0^\infty \left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right)dx$

$f_n(x) = \left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right).$

Then $\displaystyle\lim_{n\to\infty} f_n(x) = \frac{0}{e^x} = 0.$

Let $f(y) = \left(1+\frac{x}{y}\right)^{-y} = e^{-y\ln\left(1+\frac{x}{y}\right)}.$

Then $f'(y) = e^{-y\ln\left(1+\frac{x}{y}\right)}\left[-y\left(\frac{1}{1+\frac{x}{y}}\left(-\frac{x}{y^2}\right)\right)+\ln\left(1+\frac{x}{y}\right)\right]$

$f'(y) > 0$ for $x>0$ & $y>0$.

Hence strictly increasing and $\displaystyle\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{-n} = e^{-x}$

$\therefore$ $f_n(x) \leq e^{-x}\sin\left(\frac{x}{n}\right) \leq e^{-x}$

$\therefore$ By LDCT.

$$\lim_{n\to\infty}\int_0^\infty\left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right)dx = 0$$

$\blacksquare$

III. 19 (a) $\displaystyle\lim_{n\to\infty}\int_0^n \frac{\sin x}{1+nx^2}\,dx.$

$\displaystyle\chi_{[0,n)}\frac{\sin x}{1+nx^2} \leq \frac{1}{1+nx^2} \leq \frac{1}{1+x^2} \in L^1([0,\infty))$

$$\therefore \lim_{n\to\infty}\int_0^\infty \frac{\sin x}{1+nx^2}\,dx = \int_0^\infty \lim_{n\to\infty}\frac{\sin x}{1+nx^2}\,dx$$

$$= \int_0^\infty 0\,dx = 0.$$

(b) $\lim_{n\to\infty} \int_0^{e^n} \frac{x}{1+nx^2}\,dx$.

It is Reimann integrable on $[0, e^n]$.

Let $nx^2 = y \Rightarrow 2nx\,dx = dy$

$$\int_0^{ne^{2n}} \frac{dy}{2n(1+y)} = \frac{1}{2n} \ln(1+y) \Big|_0^{ne^{2n}}$$

By L' Hospital's rule.

$$\lim_{n\to\infty} \frac{1}{2}\left(\left(\frac{1}{1+ne^{2n}}\right)(ne^{2n}/2) + e^{2n})\right)$$

$$= \lim_{n\to\infty} \frac{e^{2n}(2n+1)}{2(1+ne^{2n})} = \lim_{n\to\infty} \frac{(2n+1)}{2(e^{-2n}+n)}$$

$$= \lim_{n\to\infty} \frac{n(2+1/n)}{n(\frac{e^{-2n}}{n}+1)2} = \frac{2}{2} = 1.$$

III. 25

$$\int_0^1 f_n(x)\,dx = \int_0^{1/n} n^2x + \int_{1/n}^{2/n} (2n-n^2x) + \int_{2/n}^1 0$$

$$= \frac{n^2x^2}{2}\Big|_0^{1/n} + 2nx\Big|_{1/n}^{2/n} - \frac{n^2x^2}{2}\Big|_{1/n}^{2/n}$$

$$= \frac{n^2}{2n^2} + 4 - 2 - \frac{n^2 4}{n^2 2} + \frac{n^2}{n^2 2}$$

$$= \frac{1}{2} + 4 - 4 + \frac{1}{2} = 1$$

$$\therefore \int_0^1 f_n(x)g(x)\,dx - g(0) = \int_0^1 (f_n(x)g(x) - g(0)f_n(x))\,dx$$

$$= \int_0^1 f_n(x)(g(x) - g(0))\,dx$$

Since $g$ is continuous $\Rightarrow$ $g$ uniformly continuous on $[0,1]$.

Let $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $|g(x) - g(y)| < \varepsilon$ $\forall$ $|x-y| < \delta$.

Choose $n_0$ s.t. $\frac{2}{n_0} < \delta$. Then $\forall$ $n \geq n_0$.

Then $\int_0^1 f_n(x)\,(g(x) - g(0)) \leq \int_0^{2/n} f_n(x)\,\varepsilon \leq \varepsilon$.

Hence $\lim_{n \to \infty} \int_0^1 f_n(x)\,(g(x) - g(0)) = 0$.

■

III.27 If $f_n \in L^1([0,1])$ for each $n \in \mathbb{N}$ and

$\sum_{n \in \mathbb{N}} \int_0^1 |f_n(x)|\,dx < \infty$ then P.T.

$\sum_{n \in \mathbb{N}} f_n(x) < \infty$ a.e. and.

$$\int_0^1 \sum_{n \in \mathbb{N}} f_n(x) = \sum_{n \in \mathbb{N}} \int_0^1 f_n(x)\,dx.$$

Proof:- By Tonelli's theorem.

$$\sum_{n \in \mathbb{N}} \int_0^1 |f_n(x)|\,dx = \int_0^1 \sum_{n \in \mathbb{N}} |f_n(x)|\,dx < \infty$$

$$\Rightarrow \sum_{n \in \mathbb{N}} f_n(x) \leq \sum_{n \in \mathbb{N}} |f_n(x)| < \infty \quad \text{a.e.}$$

Let $F(x,n) = f_n(x) \in L^1([0,1] \times \mathbb{N})$ with Lebesgue measure on $[0,1]$ and Counting measure on $\mathbb{N}$.

Hence by Fubini :

$$\int_0^1 \sum_{n\in\mathbb{N}} f_n(x)\,dx = \sum_{n\in\mathbb{N}} \int_0^1 f_n(x)\,dx .$$

$\blacksquare$

III.29 Let $f \geq 0$ be Integrable.

$$\int_0^1 (f(x))^n\,dx = \int_0^1 f(x)\,dx \qquad \forall\, n.$$

Let $E = \{x : |f(x)| > 1\}$

$\Rightarrow \quad (f(x))^n \leq (f(x))^{n+1}$ for $x \in E$.

Hence by MCT.

$$\lim_{n\to\infty} \int_E (f(x))^n\,dx = \int_E \lim_{n\to\infty} (f(x))^n$$

we have that $\lim_{n\to\infty} (f(x))^n = \infty$,

and $\int_E (f(x))^n \leq \int_0^1 (f(x))^n = \int_0^1 f(x)\,dx < \infty \quad \forall\, n.$

Hence we have that $\mu(E) = 0$.

$\therefore \quad 0 \leq f(x) \leq 1$ a.e.

Now $\int_0^1 f(x)^2 = \int_0^1 f(x)\,dx \Rightarrow \int_0^1 f(x)(1-f(x)) = 0.$

$\because \quad f(x) \geq 0$ and $1-f(x) \geq 0$

$\Rightarrow \quad f(x)(1-f(x)) = 0$ a.e.

$\therefore$ $f(x)=0$ or $f(x)=1$ a.e.

Let $A = f^{-1}(\{1\})$ $\Rightarrow$ $f(x) = \chi_A(x)$.

where $A$ is measurable set.

III. 35 Let $p(x_1, x_2) \neq 0$. Let $S = \{x : p(x_1,x_2) = 0\}$.

Then $(\ell \times \ell)(S) = 0$ needs to be proven.

let $F(x_1,x_2) = \chi_S = \begin{cases} 1 & \text{if } p(x_1,x_2)=0 \\ 0 & \text{otherwise} \end{cases}$.

Then by Tonelli's theorem

$$\int_{\mathbb{R}} \int_{\mathbb{R}} F(x_1,x_2)\,dx_1\,dx_2 = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} F(x_1,x_2)\,dx_1 \right) dx_2.$$

~~Sta~~. For fixed $x_2$, $\{x_1 : P_{x_2}(x_1) = P(x_1,x_2) = 0\}$

is a finite set. (Fundamental theorem of Algebra).

$\Rightarrow$ $F_{x_2}(x_1) = \begin{cases} 1 & \text{if } P_{x_2}(x_1)=0 \\ 0 & \text{otherwise} \end{cases}$.

$$\therefore \int_{\mathbb{R}} \left( \int_{\mathbb{R}} F(x_1,x_2)\,dx_1 \right) dx_2 = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} F_{x_2}(x_1)\,dx_1 \right) dx_2$$

$$= \int_{\mathbb{R}} 0 \, dx_2 = 0$$

III.20 $f \in L^1((0,\infty))$ and $\int_{(0,\infty)} x|f(x)| < \infty$.

Let $g(y) = \int_{(0,\infty)} e^{-xy} f(x)dx$.

We will show that the integrand $F(x,y) = e^{-xy} f(x)$ satisfies the three properties of dependence on parameter theorem. Hence $g$ is differentiable.

1. $y \mapsto e^{-xy} f(x)$ is clearly differentiable $\forall x$.

2. We know that $|e^{-xy}| \leq 1 \quad \forall x, y > 0$

$\therefore e^{-xy} f(x) \in L^1 \quad \forall y$.

3. $\frac{\partial \phi}{\partial y} = e^{-xy}(-x)f(x)$

$\left|\frac{\partial \phi}{\partial y}\right| = |xf(x)| e^{-xy} \leq |xf(x)| \in L^1(0,\infty)$.

$\therefore g$ is differentiable at every $y \in (0,\infty)$.

III.44 Let $f, h \in L^1(\mathbb{R})$. and

$f * h(x) = \int_{\mathbb{R}} f(x-y) h(y) dy$.

Then $|f*h(x)| \leq \int_{\mathbb{R}} |f(x-y)| |h(y)| dy$

$$\Rightarrow \int_{\mathbb{R}} |\phi * h(x)|\,dx \le \int_{\mathbb{R}} \int_{\mathbb{R}} |\phi(x-y)|\,|h(y)|\,dy\,dx.$$

By Tonelli

$$\le \int_{\mathbb{R}} |h(y)| \left( \int_{\mathbb{R}} |\phi(x-y)|\,dx \right) dy$$

$$\le \left( \int_{\mathbb{R}} |h(y)|\,dy \right) \left( \int_{\mathbb{R}} |\phi(x)|\,dx \right)$$ as lebesgue measure is translation invariant.

$$\le \|h\|_1 \, \|\phi\|_1 .$$

$\therefore$ $\phi * h(x) \in L^1(\mathbb{R}) \Rightarrow \phi * h(x) < \infty$ a.e.

$\blacksquare$

# Questions on $L^p$-spaces

III.47 Need to show that $L^q(X) \subseteq L^p(X)$ if $1 \le p \le q \le \infty$ and $\mu(X) < \infty$.

If $q = \infty$. Then $\left(\int_X |f|^p\right)^{1/p} \le \|f\|_\infty \, \mu(X)^{1/p} < \infty$.

$\therefore L^\infty(X) \subseteq L^p(X)$.

If $q \neq \infty$, then let $r = q/p$.

By Hölder's inequality

$$\int_X |f|^p \, d\mu = \int_X |f|^p . 1 \, d\mu \le \left(\int_X (|f|^p)^{q/p}\right)^{p/q} \left(\int_X 1 \, d\mu\right)^{1-p/q}$$

$$= \left(\int |f|^q\right)^{p/q} \mu(X)^{1-p/q}$$

$$\therefore \left(\int_X |f|^p\right)^{1/p} \le \left(\int |f|^q\right)^{1/q} \mu(X)^{1/p - 1/q}.$$

Hence if $f \in L^q \Rightarrow f \in L^p$. ■

V.20 We have that $f \geq 0$ and $0 < \mu(X) < \infty$.

$$\|f\|_\infty = \sup_{c > 0} \{ c : \mu(\{x : |f(x)| \geq c\}) \neq 0 \}.$$

By previous question we have that

$$\left(\int_X |f|^p\right)^{1/p} \le \|f\|_\infty \, \mu(X)^{1/p}.$$

We use the fact that if $a>0$ then $a^{1/p} \longrightarrow 1$ as $p \to \infty$.

$$\therefore \lim_{p\to\infty} \|f\|_p \leq \|f\|_\infty .$$

To prove the other way, choose $\varepsilon > 0$.

Let $S_\varepsilon = \{x : |f(x)| \geq \|f\|_\infty - \varepsilon\}$.

By definition we have that $\mu(S_\varepsilon) \neq 0$.

$$\therefore \left(\int_X |f(x)|^p\right)^{1/p} \geq \left(\int_{S_\varepsilon} |f(x)|^p dx\right)^{1/p} \geq \left(\int_{S_\varepsilon} (\|f\|_\infty - \varepsilon)^p d\mu\right)^{1/p}$$

$$\geq (\|f\|_\infty - \varepsilon)\, \mu(X)^{1/p}$$

$$\therefore \lim_{p\to\infty} \|f\|_p \geq \|f\|_\infty - \varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow \|f\|_\infty \leq \lim_{p\to\infty} \|f\|_p$$

Hence $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.

Qual Exam ~~Aug 2019~~

Q, 10, (a)

8) Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. ~~If~~ $f \in L^p$, $g \in L^q[0,1]$ and $fg \in L^r[0,1]$. Show that

$$\|fg\|_r \leq \|f\|_p \|g\|_q .$$

Proof :- we have that $\frac{r}{p} + \frac{r}{q} = 1 \Rightarrow \frac{1}{p/r} + \frac{1}{q/r} = 1$

Then by Hölder's inequality

$$\int \|fg\|^r = \int_0^1 |f|^r |g|^r \le \left( \int_{[0,1]} (|f|^r)^{p/r} \right)^{r/p} \left( \int_{[0,1]} (|g|^r)^{q/r} \right)^{r/q}$$

$$\le \|f\|_p^r \|g\|_q^r$$

$\therefore$ $\|fg\|_r \le \|f\|_p \|g\|_q$.

Qualifying exam Aug 2019

Q10) b). Let $X = \mathbb{N}$, $\mu$ be counting measure and $1 \le p \le q < \infty$.

Let $\|(a_n)\|_p \le 1 \Rightarrow \sum |a_n|^p \le 1$

i.e. $|a_n| \le 1 \quad \forall n \in \mathbb{N}$, since $p \le q$

$$\Rightarrow |a_n|^q \le |a_n|^p$$

$$\therefore \sum |a_n|^q \le \sum |a_n|^p \le 1$$

Hence $\|\{a_n\}\|_q \le 1$ for $\|\{a_n\}\|_p \le 1$.

Now let $b_n = \dfrac{a_n}{\|a_n\|_p}$ then

$\|\{b_n\}\|_p \le 1 \Rightarrow \|\{b_n\}\|_q \le 1$.

$\Rightarrow \left\| \dfrac{\{a_n\}}{\|a_n\|_p} \right\|_q \le 1 \Rightarrow \|\{a_n\}\|_q \le \|\{a_n\}\|_p$.

Q. E
Aug 2019
Q9

Let $\|f\| = \|f\|_p + \|f\|_r$.

We will prove its a norm on $L^p \cap L^r$.

1) If $f=0 \Rightarrow \|f\|=0$

If $\|f\|=0 \Rightarrow \|f\|_p = 0$ and $\|f\|_r = 0$.

$\Rightarrow f=0$ a.e.

2) $\|cf\| = \|cf\|_p + \|cf\|_q = |c| \left( \|f\|_p + \|f\|_q \right) = |c| \, \|f\|$

3) $\|f+g\| = \|f+g\|_p + \|f+g\|_q \leq \|f\|_p + \|f\|_q + \|g\|_p + \|g\|_q$

$\leq \|f\| + \|g\|$.

Hence a norm.

Let $\{f_n\}$ be a cauchy sequence. Then $f_n \to g$ in $L^p$ and $f_n \to h$ in $L^r$.

$\Rightarrow f_n \to g$ in measure and $f_n \to h$ in measure

$\mu\{|g-h| \geq \varepsilon\} \leq \mu\{x : |g-f_n| + |f_n - h| \geq \varepsilon\}$.

$\leq \mu\{x : |g-f_n| \geq \frac{\varepsilon}{2}\} + \mu(\{x : |f_n - h| \geq \frac{\varepsilon}{2}\}) \to 0$.

$\therefore \mu\{|g-h| \geq \varepsilon\} = 0 \quad \forall \varepsilon > 0$

$\Rightarrow g = h$ a.e.

Hence $f_n \to g$ in $L^p$ and $L^r$. since

$\|f_n - g\| = \|f_n - g\|_p + \|f_n - g\|_r$.

# Some Problems from Analysis Q.E.

Jan 2022

Q 3). $\mu_\phi(A) = \int_A \phi \, d\mu$. is a positive measure as $\phi \geq 0$.

Let $g = \chi_B$ then

$\int_X \chi_B \, d\mu_\phi = \int_B d\mu_\phi = \mu_\phi(B) = \int_B \phi \, d\mu = \int_X \chi_B \phi \, d\mu$

$\therefore$ If $\phi_n$ are simple functions we have

$$\int_X \phi_n d\mu_\phi = \int_X \phi_n \phi \, d\mu.$$

Suppose $g \geq 0$ then $\exists$ $\{\phi_n\}$ simple functions s.t.

$\phi_n \nearrow g$ a.e. By MCT

$$\lim_{n\to\infty} \int_X \phi_n d\mu_\phi = \lim_{n\to\infty} \int \phi_n \phi \, d\mu \quad (\because \phi_n\phi \geq 0 \text{ \& } \phi_n\phi \leq \phi_{n+1}\phi)$$

$$\int_X \lim_{n\to\infty} \phi_n d\mu_\phi \qquad \int_X \lim_{n\to\infty} \phi_n \phi \, d\mu$$

$$\int_X g \, d\mu_\phi = \int_X g\phi \, d\mu.$$

Suppose $g \in L^1(X, \mu_\phi)$ then write $g = g_+ - g_-$ and

$$\int_X g \, d\mu_\phi = \int_X g_+ - g_- \, d\mu_\phi = \int_X g_+ \phi \, d\mu - \int_X g_- \phi \, d\mu$$

$$= \int_X g\phi \, d\mu.$$

Hence $g \in L^1(\mu_\phi) \Rightarrow \phi g \in L^1(\mu)$.

$\blacksquare$

Q5) $X \to$ countable , $\mu$ - counting measure

Let $f_n : X \to \mathbb{R}$ converge uniformly to $f$.
Then $\forall\, \varepsilon > 0$ $\exists\, n_0$ s.t. $|f_n(x) - f(x)| < \varepsilon \ \forall\, x \in X,$
$\forall\, n \geq n_0.$

$$\Rightarrow \quad \mu\{x : |f_n(x) - f(x)| \geq \varepsilon\} = 0 \quad \forall\, n \geq n_0.$$

Hence $f_n \to f$ in measure.

Conversely, let $f_n \to f$ in measure.
Let $\varepsilon \geq 0$ , $c > 0$ then $\exists\, n_0$ s.t.
$$\#\{x : |f_n(x) - f(x)| \geq c\} < \varepsilon.$$

Choose $\varepsilon < 1$, then $\exists\, n_0$ s.t.
$$\#\{x : |f_n(x) - f(x)| \geq c\} < 1 \quad \forall\, n \geq n_0$$
Since for counting measure $\#$ is either $0$ or $1$
Hence $\#\{x : |f_n(x) - f(x)| \geq c\} = 0 \quad \forall\, n \geq n_0$ and $\forall\, n \geq n_0.$
$$\Rightarrow \quad |f_n(x) - f(x)| < c \quad \forall\, x \in X$$

Hence $f_n \to f$ uniformly.

Q8 a) Let $f_n \to f$ uniformly, $\mu(X) < \infty$. Then $\forall\, \varepsilon > 0$ $\exists$ $n_0$ s.t.

$|f_n(x) - f(x)| < \varepsilon \quad \forall\, n \geq n_0, \ \forall\, x \in X$

$\Rightarrow \int_X |f_n - f| \leq \int_X \varepsilon\, d\mu < \varepsilon \cdot \mu(X) \quad \forall\, n \geq n_0$

Since $\mu(X) < \infty$, implies that $f_n \to f$ in $L^1$

$$\int_X |f|\,d\mu = \int_X |f - f_{n_0} + f_{n_0}|\,d\mu \leq \int_X |f - f_{n_0}| + \int_X |f_{n_0}|$$

$$\leq \varepsilon \cdot \mu(X) + \int_X |f_{n_0}| < \infty.$$

$\therefore f \in L^1$.

(b) Let $f_n = \frac{1}{n} \chi_{[-n,n]}$.

Then $\int_{\mathbb{R}} f_n\, d\mu = \frac{1}{n} \cdot 2n = 2 < \infty$.

Now $f_n \to 0$ uniformly as $\forall\, \varepsilon > 0$ let $n_0$ be s.t.

$\frac{1}{n_0} < \varepsilon$ then

$\left| \frac{1}{n_0} \chi_{[-n_0,n_0]}(x) \right| \leq \frac{1}{n_0} < \varepsilon$

$\because \frac{1}{n} \leq \frac{1}{n_0} \quad \forall\, n \geq n_0$

$\Rightarrow f_n \to 0$ ~~$\forall\, x$~~ uniformly.

~~By~~ But $\int_{\mathbb{R}} |f_n - 0| = 2 \not\to 0$ ~~for all~~

$\therefore f_n \not\to 0$ in $L^1(\mathbb{R})$

Aug 2021

**Q9)** :- Let $X, Y$ be Banach spaces.

Let $z = cx + y$ then

$$T^t(\phi)(cx+y) = \phi(T(cx+y)) = \phi(cT(x) + T(y))$$

$$= c\,\phi(T(x)) + \phi(T(y)) = c\,T^t(\phi)(x) + T^t\phi(y).$$

Clearly $T^t(\phi)(x) \in \mathbb{C}$. and $T^t\phi$ is linear.

Let $x_n \to 0$ then $T^t\phi(x_n) = \phi(Tx_n)$

$\because$ T is bounded $\Rightarrow$ $Tx_n \to 0$, $\because \phi \in Y^*$

$\phi(Tx_n) \to 0$

$\Rightarrow$ $T^t\phi(x_n) \to 0$. Hence continuous.

$\therefore$ $T^t\phi \in X^*$ $\forall\ \phi \in Y^*$.

To prove $T^t$ is linear. $T^t(c\phi + \psi)(x) = (c\phi + \psi)(Tx)$

$= c\,T^t\phi(x) + T^t\psi(x)$

$$\|T^t\| = \sup_{\|\phi\| \le 1} \|T^t\phi\| = \sup_{\|\phi\| \le 1} \sup_{\|x\| \le 1} |T^t\phi(x)|$$

$$= \sup_{\|\phi\| \le 1} \sup_{\|x\| \le 1} |\phi(T(x))| \le \sup_{\substack{\|\phi\| \le 1 \\ \|x\| \le 1}} \|\phi\|\,\|T(x)\|. \le \|T\|.$$

Hence $T^t$ is bounded and $\|T^t\| \le \|T\|$

~~Jan 2021~~

Aug 2020

Part II
Q 3

$X = Y = [0,1]$.

$\mathcal{M} = \mathcal{N} = \mathcal{B}[0,1]$.

$\mu \to$ lebesgue measure $\quad \nu =$ counting measure

i) $\int_Y \int_X 1_D^{(x,y)} d\mu \, d\nu$

$$1_{D_y}(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \int_X 1_D(x,y)\, d\mu = \int_X 1_{D_y}(x)\, d\mu = 0.$$

$$\therefore \int_Y \int_X 1_D(x,y)\, d\mu\, d\nu = 0.$$

$$\int_X \left( \int_Y 1_D(x,y)\, d\nu \right) d\mu = \int_X \left( \int_Y 1_{D_x}(y)\, d\nu \right) d\mu.$$

i.e. $1_{D_x}(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$

$$\therefore \int_Y 1_{D_x}(y)\, d\nu = \sum_{\substack{\text{for all} \\ y \in [0,1]}} 1_{D_x}(y) = 1.$$

$$\therefore \int_{[0,1]} 1\, d\mu = 1.$$

ii) Fubini's theorem fails as $([0,1], \nu)$ with counting measure is not $\sigma$-finite. $\blacksquare$

Aug 2019
Q4

$f_n = n^{-1/p} \chi_{[0,n)}$ $\quad 1 \le p < \infty$.

ⓐ Choose $\varepsilon > 0$, let $n_0$ be s.t.

$$\frac{1}{(n_0)^{1/p}} < \varepsilon.$$

Then $|f_n(x)| \le \left|\frac{1}{n^{1/p}}\right| < \varepsilon \quad \forall n \ge n_0 \quad \forall x.$

$\because$ for $n \ge n_0$ $\quad (n_0)^{1/p} \le (n)^{1/p}$.

Hence $f_n \to 0$ uniformly

ⓑ $f_n \to 0$ in measure. $f_n \to 0$ in measure.

Since $f_n \to 0$ uniformly $\Rightarrow f_n \to 0$

ⓒ $\left(\int |f_n|^p\right)^{1/p} = \left(\int_{[0,n]} \left(\frac{1}{(n)^{1/p}}\right)^p\right)^{1/p} = (1)^{1/p} = 1$

$\not\to 0 = \int 0$ as $n \to \infty$.

Hence $f_n \not\to 0$ in $L^p$.

Jan 2019

Q1) a) $f : X \to \bar{\mathbb{R}}$ measurable.

Let $A = \{x : |f(x)| < \infty\}$. By the question $\mu(A) > 0$.

Let $\mu(A) = c$, $\quad B_n = \{x : f(x) \le n\} \subseteq A$.

$$A = \bigcup_{1}^{\infty} B_n$$

we have $B_1 \subseteq B_2 \subseteq \dots$ and $\mu(B_1) \le \mu(B_2) \le \dots$

$\therefore \lim_{n\to\infty} \mu(B_n) = \mu(A) - c$

i.e. $\forall\, \varepsilon > 0 \;\exists\, n_0$ s.t. $|\mu(B_{n_0}) - c| < \varepsilon$

$\Rightarrow c - \mu(B_{n_0}) < \varepsilon$ i.e. $c - \varepsilon < \mu(B_{n_0})$.

choose $\varepsilon$ s.t. $0 < c - \varepsilon$.

$\therefore \exists\, \mu(B_{n_0}) > 0$ and for $x \in B_{n_0}$ $|f(x)| \le n_0$

Hence $f$ is bounded.

(b) $f \in L^1(X, \mu)$. Let $A_n = \{x : |f(x)| \geq n\}$. then

$$\int_X f \geq \int_{A_n} f \geq \int_{A_n} n \, d\mu = n\mu(A_n)$$

$$\Rightarrow \mu(A_n) \le \frac{\int_X f}{n} < \infty. \quad \forall n.$$

Since $A_n$ are measurable, $\therefore$

$$\{x : f(x) \neq 0\} = \bigcup_1^\infty A_n$$

Hence the set on LHS is $\sigma$-finite

Q3) $f \in L^1$, $\phi \in C_c^\infty$. Let $F(x,y) = f(y)\phi(x-y)$

a) then $f * \phi(x) = \int f(y)\,\phi(x-y)$.

i) $x_i \mapsto F(x,y)$ is clearly differentiable for each $x_i$ $1 \le i \le n$.

ii) Let $x_i = 0$ then $f(y)\,\phi(x-y) \in L^1$

As $\int_{\mathbb{R}} |f(y)|\,|\phi(x-y)| \le \sup|\phi(y)| \int |f(y)| < \infty$.

(3) $\dfrac{\partial F}{\partial x_i} = \dfrac{f(y)\,\partial g^{\varphi}(x-y)}{\partial x_i}$

$\left|\dfrac{\partial F}{\partial x_i}\right| \le |f(y)| \sup \left|\dfrac{\partial g}{\partial x_i}\right| \le M|f(y)| \in L^1$

Hence $\dfrac{\partial f * g^{\varphi}(x)}{\partial x_i} = \int \dfrac{\partial F}{\partial x}(x,y)\,dy = \int f(y)\dfrac{\partial g^{\varphi}}{\partial x_i}(x-y)$.

∎

811) b

Let $\phi(t) = \nu([0,t)) < \infty$.

$$\int_{[0,\infty)} \mu(\{x : |f(x)| > t\})\,d\nu(t) = \int_{[0,\infty)} \int_X \chi_{\{x : |f(x)| > t\}}(x)\,d\mu(x)\,d\nu(t)$$

By Tonelli's theorem

$$= \int_X \int_{[0,\infty)} \chi_{\{x : |f(x)| > t\}}\,dt\,d\mu(x)$$

$$= \int_X \nu\{t : t < |f(x)|\}\,d\mu(x)$$

$$= \int_X \phi(|f(x)|)\,d\mu(x).$$

∎

Q) :- ⓐ Let $(X, \mathcal{A}, \mu)$ be finite space. $f : X \to \overline{\mathbb{R}}$ measurable and $f < \infty$ a.e. S.T. $\forall \varepsilon > 0$ $\exists$ a set $E \in \mathcal{A}$ s.t. $\mu(E^c) < \varepsilon$ and $f$ is bounded on $E$

ⓑ Give an eg. that shows one can not always find a null set $E^C$ s.t. $f$ bounded on $E$.

ⓒ Give eg. showing that the statement is not true for $\mu(X) = \infty$.

$\Rightarrow$ ⓐ Let $E_k = \{x : |f(x)| \leq k\}$.

Then $E_k^c = \{x : |f(x)| > k\}$

we have $E_1^c \supseteq E_2^c \supseteq \cdots$

$\therefore \lim\limits_{k \to \infty} \mu(E_k^c) = \mu\left(\bigcap\limits_{k=1}^{\infty} E_k^c\right)$

$= \mu(\{x : |f(x)| = \infty\}) = 0.$

$\therefore$ $\exists$ $k_0$ s.t.
$\mu(E_{k_0}^c) < \varepsilon$

and on $E_k$ $f$ is bounded.

ⓑ. Let $f = \sum\limits_{n=1}^{\infty} n \chi_{\left(\frac{1}{n+1}, \frac{1}{n}\right]}$.

$f$ is measurable.

Suppose $E$ be a measurable set s.t. $f$ is bounded on $E$ then $E \subseteq E_k$ for some $k$.

$\Rightarrow \quad \mu(E^c) \geq \mu(E_k^c)$

$E_k^c = \{x : |f(x)| > k\} = (0, \frac{1}{k+1}]$

$\therefore \quad \mu(E_k^c) = \frac{1}{k+1} > 0$

$\Rightarrow \quad \mu(E^c) > 0.$

(c) Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x$. Then $f < \infty$ a.e.

Suppose $E$ is s.t. $f$ is bounded i.e.

$|f(x)| \leq M \Rightarrow E \subseteq E_M = [-M, M]$.

$\therefore \quad \mu(E^c) \geq \mu(E_m^c) = \infty$.

$\therefore \nexists$ a ~~set~~ such a set $E$. ∎

Analysis Bank II.15

we have that $f^{-1}(\alpha) = \{x : f(x) = \alpha\}$

$= \{x : f(x) \leq \alpha\} \cap \{x : f(x) \geq \alpha\}$

$= f^{-1}([-\infty, \alpha]) \cap f^{-1}([\alpha, +\infty])$.

Since $[-\infty, \alpha)$ is measurable set $\Rightarrow [-\infty, \alpha)^c = [\alpha, \infty]$ is also measurable. $\therefore f^{-1}([\alpha, \infty])$ is measurable.

Now $[-\infty, \alpha] = \cancel{\bigcup_{n=1}^{\infty}(-\infty, \alpha + \frac{1}{n})} \bigcap_{n=1}^{\infty} [-\infty, \alpha + \frac{1}{n})$.

$\therefore f^{-1}([-\infty, \alpha]) = \bigcap_{n=1}^{\infty} f^{-1}([-\infty, \alpha + \frac{1}{n}))$ is a measurable set

Hence $f^{-1}(\alpha)$ is a measurable set ∎