Quasicrystals and Poisson summation formula

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Certificate of Examination

This is to certify that the dissertation titled **"Quasicrystals and Poisson summation formula"** submitted by **Iswarya Sitiraju** (Reg. No. MS13104) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Shobha Madan at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Shobha Madan

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Abstract

We say that a set $\Lambda \subset \mathbb{R}^n$ holds a Poisson summation formula in terms of tempered distribution if it supports a measure μ which is a tempered distribution such that its Fourier transform $\widehat{\mu}$ is also a measure.

The aim of my thesis is to understand whether a Poisson summation formula can hold for any uniformly discrete subsets of $\mathbb{R}^n$. If it holds for a set then what will be its characterization. We will see that for the lattice $\mathbb{Z}^n$, a Poisson summation formula holds. Naturally, we can ask whether there are other uniformly discrete sets for which it holds. Initially, Cordoba has investigated this case with some control conditions on Dirac masses. The result was later generalized by Nir Lev and Olevskii recently in 2014.

We begin this report with an introduction on tempered distributions and defining some operations on tempered distributions. We will also explain the well known identity, the Poisson summation formula which holds for a suitable class of functions. Then, we will state and prove Cordoba's first, second result and Nir Lev and Olevskii's result.

One of the key concept used in the proof of Nir Lev and Olevskii' result is 'Meyer sets'. Meyer sets was discovered by Yves Meyer in 1970's. It has applications in Number theory also. We will also explain and understand these sets.

Contents

Chapter 1

Tempered distribution

1.1 Introduction

The **Schwartz space** $\mathcal{S}(\mathbb{R}^n)$ is defined to be the space of all smooth functions on $\mathbb{R}^n$ that are rapidly decreasing at infinity with all their derivatives. That is

$$\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : |x^\alpha(\partial^\beta \phi)(x)| < \infty\}$$

for all $\alpha, \beta \in \mathbb{N}^n$. Let $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ then

$$|\alpha| = \alpha_1 + \ldots + \alpha_n \ ;$$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$$

and

$$\partial^\beta = \partial^{(\beta_1+\beta_2+\ldots+\beta_n)}/\partial x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}.$$

We now define increasing sequence of norms $||\cdot||_N$, where $N \in \mathbb{N}$, as

$$||\phi||_N = \sup_{x \in \mathbb{R}^n, |\alpha|, |\beta| \leq N} |x^\alpha(\partial^\beta \phi)(x)|.$$

Hence for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ we have that $||\phi||_N < \infty$ for every N.

We say that a sequence $\phi_k \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$ whenever $||\phi_k - \phi||_N \to 0$, as $k \to \infty$, for every N.

Now let us define **tempered distribution $\mathcal{S}'$**. Tempered distribution is the space of all complex continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$.

Proposition 1.1. *Suppose μ is a tempered distribution. Then there is a positive integer N and a constant $C > 0$, such that*

$$|\mu(\phi)| \leqslant C||\phi||_N, \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Proof. From the definition of metric it follows that the sets $U_{N,\varepsilon} = \{\phi \in \mathcal{S}(\mathbb{R}^n) : ||\phi||_N < \varepsilon\}$, where $\varepsilon > 0$ and $N \in \mathbb{N}$, forms a basis around $0 \in \mathcal{S}(\mathbb{R}^n)$. Since μ is a tempered distribution, it is continuous at 0. Thus, there exists a neighbourhood $U_{N,\varepsilon}$ around 0 such that $|\mu(\phi)| \leqslant 1$ whenever $\phi \in U_{N,\varepsilon}$. Let $0 < \varepsilon' < \varepsilon$ and consider the Schwartz function $\psi = (\varepsilon'/||\phi||_N)\phi$. We see that $\psi \in U_{N,\varepsilon}$. Therefore,

$$(\varepsilon'/||\phi||_N)|\mu(\phi)| = |\mu(\psi)| \leqslant 1$$

Hence if we let $C = 1/\varepsilon'$ then

$$|\mu(\phi)| \leqslant C||\phi||_N$$

. ∎

Let us look at an example of tempered distributions.

Example 1.1. *Let δ_x be the translate of Dirac delta 'function', where $x \in \mathbb{R}^n$. The 'function' δ_x acts on a Schwartz function ϕ in the following way*

$$\delta_x(\phi) = \phi(x)$$

Clearly, it is a linear functional on $\mathcal{S}(\mathbb{R}^n)$. Let $\phi_k \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$. Hence we get $|\delta_x(\phi_k - \phi)| = |\phi_k(x) - \phi(x)| \to 0$ as $k \to \infty$. Thus, δ_x is continuous.

1.2 Operations on tempered distributions

First we will define the support of a tempered distribution.

Definition 1.2. For a tempered distribution μ we say that μ vanishes in an open set if $\mu(\phi) = 0$, for all Schwartz function ϕ having their support in that open set.

Thus the **support of a tempered distribution** is defined to be the complement of largest open set on which μ vanishes.

Now we will define few operations on tempered distributions.

- We will define the **product** of a slowly increasing smooth function with a tempered distribution. Slowly increasing means that for each α, $(\partial^\alpha\psi)(x) = O(|x|^{N_\alpha})$, for some $N_\alpha > 0$. Let ψ be a slowly increasing C^∞ function and μ be a tempered distribution, then we define the product $\psi \cdot \mu$ by

$$(\psi \cdot \mu)(\phi) = \mu(\psi\phi) \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$$

- A key feature of tempered distributions is that it can be differentiated any number of times. We define the **derivative** $\partial^\alpha\mu$ of a tempered distribution μ as

$$(\partial^\alpha\mu)(\phi) = (-1)^{|\alpha|}\mu(\partial^\alpha\phi), \quad \text{whenever } \phi \in \mathcal{S}(\mathbb{R}^{\mathrm{n}})$$

.

Note that the above two operations on a tempered distribution is again a tempered distribution.

- We extend the notion of **convolution** of appropriate functions to the convolution of Schwartz functions and tempered distributions which is again a smooth function. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ and μ be a tempered distribution then the function $\psi * \mu$ is defined as

$$\psi * \mu(x) = \mu(\tilde{\psi_x})$$

where $\tilde{\psi_x}(y) = \psi(x-y)$.

Proposition 1.3. *Suppose μ is a tempered distribution and $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then $\psi * \mu$ is a slowly increasing smooth function.*

Proof. Let $x_n \to x$ as $n \to \infty$, then $|\tilde{\psi_{x_n}}(y) - \tilde{\psi_x}(y)| = |\psi(x_n - y) - \psi(x-y)| \to 0$ as $n \to \infty$ uniformly in y. And, all the partial derivative of $\tilde{\psi_{x_n}}$ exists and converges uniformly to the corresponding partial derivative of $\tilde{\psi_x}$. Hence $\tilde{\psi_{x_n}} \to \tilde{\psi_x}$ in $\mathcal{S}(\mathbb{R}^n)$. Since μ is continuous we get that

$$\psi * \mu(x_n) = \mu(\tilde{\psi_{x_n}}) \to \mu(\tilde{\psi_x}) = \psi * \mu(x)$$

as $n \to \infty$. Similarly we have

$$(\partial^\alpha(\psi * \mu))(x_n) = \mu(\partial^\alpha(\tilde{\psi_{x_n}})) \to \mu(\partial^\alpha(\tilde{\psi_x})) = (\partial^\alpha(\psi * \mu))(x)$$

as $n \to \infty$.

Since μ is a tempered distribution, there exist $C > 0$ and $N \in \mathbb{N}$ such that $|\mu(\phi)| \leqslant C||\phi||_N$ for $\phi \in \mathcal{S}(\mathbb{R}^n)$. Observe that

$$||\partial^\alpha(\phi)||_N \leqslant ||\phi||_{N+|\alpha|}$$

and

$$||\tilde{\phi_x}||_N \leqslant c(1 + |x|^N)||\phi||_N$$

for $\phi \in \mathcal{S}(\mathbb{R}^n)$. Thus,

$$(\partial^\alpha\psi * \mu)(x) = (-1)^{|\alpha|}\mu(\partial^\alpha\tilde{\psi_x}) \leqslant Cc||\psi||_{N+|\alpha|}(1 + |x|^N) = O(|x|^N).$$

Hence , $\phi * \mu$ is a slowly increasing smooth function. ■

Proposition 1.4. *Let μ be a tempered distribution whose support is A and $\phi \in \mathcal{S}(\mathbb{R}^n)$ has a compact support B. Then the support of $\phi * \mu$ is contained in $A + B$.*

Proof. The support of $\tilde{\psi_x}$ is $x - B$. Thus, for each x such that $\mu(\tilde{\psi_x}) \neq 0$, we must have that $A \cap (x - B) \neq \emptyset$. Let $y \in A \cap (x - B)$, then $x = y + (x - y) \in A + B$. Thus supp$(\psi * \mu) \subseteq$ A + B. ■

- The definition of **Fourier transform** $\widehat{\mu}$ for a tempered distribution μ is

$$\widehat{\mu}(\phi) = \mu(\widehat{\phi}), \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$

Since, $\phi \to \widehat{\phi}$ is a continuous linear mapping and μ is also continuous, we have that $\widehat{\mu}$ is also a tempered distribution.

Chapter 2

The Poisson summation formula and Cordoba's first result

First we will understand what is a Poisson summation formula. And for what kind of functions does it hold. Then we will present the Poisson summation formula in terms of tempered distribution. And then, we will state and prove Cordoba's first result.

2.1 Poisson summation formula

Theorem 2.1. *Suppose that $f \in L^1(\mathbb{R})$, then the series*

$$F(t) = \sum_{n\in\mathbb{Z}} f(t+n)$$

converges in $L^1[0,1]$ and is a period 1 function. The Fourier coefficient is obtained as

$$\widehat{F}(k) = \widehat{f}(k), \quad k \in \mathbb{Z}.$$

In addition if $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty$, then the Fourier series of F converges and we have the almost everywhere equality

$$F(t) = \sum_{n\in\mathbb{Z}} f(t+n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m) e^{2\pi i m t}.$$

Proof. Let $N \in \mathbb{N}$ and $t \in [0,1]$. Consider the partial sums

$$F_N(t) = \sum_{-N}^{N} f(t+n).$$

Then

$$\begin{aligned} ||F_{N+K} - F_N||_{L^1[0,1]} &\leqslant \sum_{N<|n|\leqslant N+K} \int_0^1 |f(t+n)|dt \\ &\leqslant \int_{|x|>N} |f(x)|dx \end{aligned}$$

which tends to 0 as N, K tends to infinity since f is an $L^1(\mathbb{R})$ function. So $F_N \to F \in L^1[0,1]$.

Next, by Dominated convergence theorem we have

$$\begin{aligned} \widehat{F}(k) &= \int_0^1 \left(\sum_{n\in\mathbb{Z}} f(t+n) \right) e^{-2\pi ikt} dt \\ &= \sum_{n\in\mathbb{Z}} \int_0^1 f(t+n) e^{-2\pi ikt} dt \\ &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x) e^{-2\pi ik(x-n)} dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx} dx \\ &= \widehat{f}(k). \end{aligned}$$

If $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty$, then the Fourier series of f convergence uniformly to an $L^1[0,1]$ function g. By the uniqueness of Fourier transform, $g = F$ a.e. Therefore,

$$F(t) = \sum_{n\in\mathbb{Z}} f(t+n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m) e^{2\pi imt} \quad a.e.$$

If f is continuous,

$$F(0) = \sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m)$$

∎

There are many applications of Poisson summation formula. One of the application is Benedicks theorem, which states that

Theorem 2.2. *Let $f \in L^1(\mathbb{R})$ be such that*

$$|\text{supp}(f)||\text{supp}(\widehat{f})| < \infty$$

Then $f = 0$ a.e.

Proof. Let us assume that $f, \widehat{f} \in L^1 \cap C_0(\mathbb{R})$. And by using dilations assume that $|\text{supp(f)}| < 1$.

Let $A := \{x \in \mathbb{R} : f(x) \neq 0\}$ and $B := \{\xi \in \mathbb{R} : \widehat{f}(\xi) \neq 0\}$. Let

$$G(\xi) = \sum_{n\in\mathbb{Z}} \chi_B(\xi + n)$$

where χ_B is the indicator function of the set B. Then

$$\int_0^1 \sum_{n\in\mathbb{Z}} \chi_B(\xi + n) d\xi = |B| < \infty.$$

So $G(\xi)$ is finite almost everywhere. It follows that there exists a subset $E \subseteq [0, 1]$, $|E| = 1$ such that for $\xi \in E$, $(\xi + \mathbb{Z}) \cap B$ is a finite set.

For each $\eta \in E$, define a 1 - periodic function

$$\phi_n(x) = \sum_{n\in Z} f(x + n)e^{-2\pi i\eta(x+n)}.$$

Then $\phi_\eta \in L^1[0, 1]$ and $\widehat{\phi_\eta}(k) = \widehat{f}(\eta + k)$. Since $|B| < \infty$ we have that $\text{supp}(\widehat{\phi_\eta})$ is finite for a.e $\eta \in [0, 1]$. Hence $\sum_{n\in Z} |\widehat{\phi_\eta}(n)| < \infty$ and by Poisson summation formula we get

$$\phi_\eta(x) = \sum_{k\in\mathbb{Z}} \widehat{\phi_\eta}(k)e^{2\pi ixk}.$$

It follows that ϕ_η is a trigonometric polynomial and can have only finitely many zeroes in $[0, 1]$, unless it is identically zero.

On the other hand, note that

$$
\begin{aligned}
|\phi_\eta(x)| &\leqslant \sum_n \chi_A(x+n)|f(x+n)| \\
&\leqslant ||f||_\infty \sum_n \chi_A(x+n).
\end{aligned}
$$

But

$$\int_0^1 \sum_n \chi_A(x+n)dx = |A| < 1.$$

Therefore, $\sum_n \chi_A(x+n) < 1$ i.e. $\sum_n \chi_A(x+n) = 0$ for a positive measure. Hence $\phi_\eta = 0$ for a positive measure. This means that for almost all $\eta \in [0,1]$, $\phi_\eta = 0$, so $\widehat{f}(\eta+n) = 0$ a.e $\eta \in [0,1]$. Hence $\widehat{f} = 0$ a.e. The result can be generalized to L^1 functions since $L^1 \cap C_0$ is dense in Ł^1

■

2.2 Cordoba's first result

For a function f lying in appropriate function space, we have a Poisson summation formula in n - dimension, i.e.

$$\sum_{m \in \mathbb{Z}^n} f(m) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(k).$$

Hence for a Schwartz function, Poisson summation formula holds.

The Poisson Summation Formula in terms of tempered distribution is as follows,

$$\widehat{\delta_{\mathbb{Z}^n}} = \delta_{\mathbb{Z}^n}.$$

If $\phi \in \mathcal{S}(\mathbb{R})$ then

$$\widehat{\delta_{\mathbb{Z}^n}}(\phi) = \sum_{m \in \mathbb{Z}^n} \widehat{\phi}(m)$$

and

$$\delta_{\mathbb{Z}^n}(\phi) = \sum_{m \in \mathbb{Z}^n} \phi(m).$$

Basically, a Poisson summation formula in terms of tempered distribution means that if $\mu = \delta_{\mathbb{Z}^n}$ is a tempered distribution as well as a measure supported by $\mathbb{Z}^n$ then its Fourier transform is also a measure.

Definition 2.3. A set $\Lambda \subset \mathbb{R}^n$ is called **uniformly discrete** if

$$d(\Lambda) := \underset{\lambda,\lambda' \in \Lambda, \lambda \neq \lambda'}{inf} |\lambda - \lambda'| > 0$$

Can we find any other uniformly discrete sets X of $\mathbb{R}^n$ such that if a measure μ supported by X is a tempered distribution and its Fourier transform is also a measure. The next proposition gives the answer.

Proposition 2.4. *If* $\mathrm{A} : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{n}}$ *is an invertible linear transformation and if* $\mathrm{X} = \mathrm{A}\mathbb{Z}^{\mathrm{n}}$ *and* $\mathrm{Y} = (\mathrm{A}^{-1})^{\mathrm{t}}\mathbb{Z}^{\mathrm{n}}$*, then the Poisson summation formula for the sets* X *and* Y *is*

$$\widehat{\delta_{\mathrm{X}}} = \frac{1}{\det(\mathrm{A})}\delta_{\mathrm{Y}} \tag{2.1}$$

Proof. Let ϕ be a Schwartz function on $\mathbb{R}^n$ and $x \in \mathrm{X}$ (i.e. $x = \mathrm{A}z$ for some $z \in \mathbb{Z}^n$), then we have that

$$\begin{aligned}
\widehat{\phi}(x) &= \int \phi(y) e^{-2\pi i <y,x>} dy \\
&= \int \phi(y) e^{-2\pi i <y,\mathrm{A}z>} dy \\
&= \int \phi(y) e^{-2\pi i <(\mathrm{A})^t y,z>} dy
\end{aligned}$$

Since we know that $< y, \mathrm{A}z > = < (\mathrm{A})^t y, z >$. Now by substituting $y = (\mathrm{A}^{\mathrm{t}})^{-1}\mathrm{s}$ we get,

$$\begin{aligned}
\widehat{\phi}(x) &= \frac{1}{\det(\mathrm{A})} \int \phi((\mathrm{A}^{\mathrm{t}})^{-1}s) e^{-2\pi i <s,z>} ds \\
\widehat{\phi}(x) &= \widehat{\psi}(z)
\end{aligned}$$

where $\psi(x) = \phi((\mathrm{A}^t)^{-1}x)$. Therefore, we get

$$\begin{aligned}
\widehat{\delta_{\mathrm{X}}}(\phi) &= \sum_{x\in\mathrm{X}} \widehat{\phi}(x) \\
&= \frac{1}{\det \mathrm{A}} \sum_{z\in\mathbb{Z}^n} \widehat{\psi}(z) \\
&= \frac{1}{\det \mathrm{A}} \sum_{z\in\mathbb{Z}^n} \psi(z) \qquad \text{(by \quad PSF)} \\
&= \frac{1}{\det \mathrm{A}} \sum_{y\in\mathbb{Y}} \phi(y) \\
&= \frac{1}{\det \mathrm{A}} \delta_Y(\phi)
\end{aligned}$$

hence proving the proposition. ■

Are there any sets other than lattices for which a Poisson summation holds? Cordoba tried to investigate this assuming different conditions. His first result is stated below.

Theorem 2.5. *Let* $\mathrm{X} = \{\mathrm{x_k}\}$ *and* $\mathrm{Y} = \{\mathrm{y_k}\}$ *be uniformly discrete subsets of* $\mathbb{R}^n$*, and let* $\{c_k\}$ *be positive real numbers. Let* $\mu_1 = \sum c_k\delta_{x_k}$ *and* $\mu_2 = \sum \delta_{y_k}$ *be tempered distributions. If* $\widehat{\mu_1} = \mu_2$*, then there exists a linear transformation* $\mathrm{A} : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{n}}$ *such that* $\mathrm{X} = \mathrm{A}\mathbb{Z}^{\mathrm{n}}$ *and* $\mathrm{Y} = (\mathrm{A}^{-1})^{\mathrm{t}}\mathbb{Z}^{\mathrm{n}}$ *with* $\det(\mathrm{A}) = 1$*. In particular,* $c_k = 1$*.*

Proof. We will prove this theorem for the case of dimension 1. The theorem in 1 dimension means that $X = \mathbb{Z}$ and $Y = \mathbb{Z}$. So only $\mathbb{Z}$ can satisfy the above hypothesis.
Consider a continuous positive function ϕ with $\mathrm{supp}(\phi) \subset [-1, 1]$, and such that $\phi(0) = 1, \widehat{\phi}(\xi) > 0\,\forall\xi \in \mathbb{R}$, $\widehat{\phi}(0) = 1$ and $\widehat{\phi} \leqslant c(1 + |\xi|)^{-1-\delta}$. For example, let $\phi = (1 - |x|(-1, 1))$. Then $\widehat{\phi}(\xi) = \left(\frac{sin\pi\xi}{\pi\xi}\right)^2$. And ϕ satisfies all the mentioned properties.

1. We claim that under the hypothesis, $0 \in X$ & $0 \in Y$.
   Suppose $0 \notin X$. Now choose $\epsilon_0 > 0$ such that $d(X, 0) > \epsilon_0$. Then for all $0 < \epsilon < \epsilon_0$, $|x_k|/\epsilon > 1$.

$$0 = \sum c_k\phi\left(\frac{x_k}{\epsilon}\right) = \epsilon\sum \widehat{\phi}(\epsilon x_k) > 0,$$

Which gives us the contradiction. By similar arguments we can show that $0 \in Y$.

2. We show that $x_j.y_k \in \mathbb{Z}$ for all j,k. Let,

$$\phi_{a,b,\epsilon}(x) = e^{2\pi iax}\phi(\epsilon^{-1}(x-b)).$$

Then,

$$\widehat{\phi}_{a,b,\epsilon}(\xi) = \epsilon e^{-2\pi ib(\xi-a)}\widehat{\phi}(\epsilon(\xi-a))$$

Fix $b = x_j$ and let $\epsilon < \min\limits_{i\neq k}\{|x_i - x_k|\}$. Then we have,

$$\begin{aligned}\sum_k c_k\phi_{a,x_j,\epsilon}(x_k) &= \sum_k c_k e^{2\pi iax_k}\phi(\epsilon^{-1}(x_k-x_j)) = c_j e^{2\pi iax_j}\\ &= \epsilon\sum_k e^{-2\pi i(y_k-a)x_j}\widehat{\phi}(\epsilon(y_k-a))\end{aligned}$$

Hence,

$$1 = \frac{\epsilon}{c_j}\sum_k e^{2\pi iy_kx_j}\widehat{\phi}(\epsilon(y_k-a)).$$

Taking $x_j = 0$, we also get,

$$1 = \frac{\epsilon}{c_j}\sum_k \widehat{\phi}(\epsilon(y_k-a))$$

and by comparing we get $e^{2\pi iy_kx_j} = 1$ and so $y_k.x_j \in \mathbb{Z}$.

3. Choose $t = x_j$ such that t is minimal and $t > 0$.
Let, $Ax_j = t.1 \implies A = [1]$. Put, $\overline{y_k} = (A^{-1})^t y_k = y_k$. By Poisson summation formula, we get,

$$t\sum \widehat{\phi}(t(y_k-a)) = 1 \tag{2.2}$$

Taking $a = y_{k_0}$ we get,

$$t\widehat{\phi}(0) + t\sum_{k\neq k_0}\widehat{\phi}(t(y_k-y_{k_0})) = 1 \implies t \leqslant 1.$$

Since $ty_k \in \mathbb{Z}$ we can write $y_k = m_k/t$. Integrating both sides of (2.2),

$$\begin{aligned}\int_{\frac{-1}{2}}^{\frac{1}{2}} 1 &= \int_{\frac{-1}{2}}^{\frac{1}{2}} t\sum_k \widehat{\phi}(t(y_k - a))da \\ &= \sum_k \int_{\frac{-1}{2}}^{\frac{1}{2}} t\widehat{\phi}(t(y_k - a))da\end{aligned}$$

Choose $z = t(y_k - a)$, we have

$$\begin{aligned}\int_{\mathbb{R}} \widehat{\phi}(y)dy = \phi(0) = 1 \\ &= \sum_k \int_{t(y_k - \frac{1}{2})}^{t(y_k + \frac{1}{2})} \widehat{\phi}(z)dz \\ &= \int_{\cup[t(y_k - \frac{1}{2}), t(y_k + \frac{1}{2})]} \widehat{\phi}(z)dz.\end{aligned}$$

But since $\widehat{\phi}(u) \geqslant 0$, we must have $\cup[t(y_k - \frac{1}{2}), t(y_k + \frac{1}{2})] = \cup[m_k - t/k, m_k + t/k] = \mathbb{R}$ a.e. Hence, $t = 1$ & $\{y_k\} = \mathbb{Z}$. Now since $x_j y_k \in \mathbb{Z}\ \forall j$ we also have $\{x_k\} = \mathbb{Z}$.

■

Chapter 3

Cordoba's second result

3.1 Introduction

In the previous chapter we have seen that the only uniformly discrete sets which hold a Poisson summation formula with equal Dirac masses are lattices. In other words, if μ is a measure with equal Dirac masses and if $\hat{\mu}$ is also a measure with Dirac masses equal to 1 , then the support of μ has to be a lattice. And it turns out that the masses of μ is also equal to 1. In Cordoba's second result the restriction on Dirac masses is relaxed to some extent. In this chapter we will prove Cordoba's second result.

Before that let us look at an example.

Example 3.1. *Consider the two disjoint lattices $\Lambda_1 = 1/2+2\mathbb{Z}$ and $\Lambda_2 = \mathbb{Z}$ and let $\Lambda = \Lambda_1 \cup \Lambda_2$. Let a_1, a_2 be distinct complex numbers. Now, consider the tempered distribution*

$$\mu = a_1 \sum_{x\in\Lambda_1} \delta_x + a_2 \sum_{y\in\Lambda_2} \delta_y$$

Let us analyze how the fourier transform of μ looks like. Let $\phi \in \mathcal{S}(\mathbb{R})$.

$$\widehat{\mu}(\phi) = \mu(\widehat{\phi}) = a_1 \sum_{x\in\Lambda_1} \widehat{\phi(x)} + a_2 \sum_{y\in\Lambda_2} \widehat{\phi(y)} \tag{3.1}$$

Observe that for the latter sum of R.H.S we can just apply PSF of $\mathbb{Z}$*. Let* $x = 1/2 + 2n$*, then*

$$\begin{aligned}\widehat{\phi}(x) &= \widehat{\phi}(1/2 + 2n)\\ &= (1/2)(\phi(y/2)e^{-\pi iy/2})\widehat{\ }(n)\end{aligned}$$

By applying PSF we get

$$\begin{aligned}\sum_{x\in\Lambda_1}\widehat{\phi(x)} &= \sum_{n\in\mathbb{Z}}(1/2)(\phi(y/2)e^{-\pi iy/2})\widehat{\ }(n) = \sum_{n\in\mathbb{Z}}(1/2)(\phi(n/2)e^{-\pi in/2})\\ &= (1/2)\sum_{m\in\mathbb{Z}/2}\phi(m)e^{-\pi im}\end{aligned}$$

Therefore,

$$\widehat{\mu} = \sum_{m\in\mathbb{Z}/2}(a_1/2)e^{-\pi im}\delta_m + \sum_{n\in\mathbb{Z}}a_2\delta_n. \tag{3.2}$$

We see that supp$(\widehat{\mu}) \subseteq \mathbb{Z} \cup \mathbb{Z}/2$*, which is a uniformly discrete set.*

We can write $\mathbb{Z} = 2\mathbb{Z} \cap (1 + 2\mathbb{Z})$*. Then*

$$\widehat{\mu} = \sum_{m\in\mathbb{Z}/2}\left[(a_1/2)e^{-\pi im} + (a_2/2)(1 + e^{-2\pi im})\right]\delta_m.$$

Hence supp$(\widehat{\mu}) \subset \mathbb{Z}/2$*.*

■

In the above example for a distribution, whose support is union of two disjoint lattices having two distinct masses, has a Fourier transform whose support is again a uniformly discrete set but with different masses. Now we can ask what happens if we consider a distribution with support being finite disjoint union of subsets of $\mathbb{R}^n$ having distinct masses and assume that its Fourier transform is also a measure with different masses. It turns out that the support of such distribution is a finite superpositions of periodic structures. Cordoba has proved that the support has to be a finite disjoint union of translates of full dimensional lattices. Let us state and prove Cordoba's result.

3.2 Preliminaries

This section contains lemmas and statements which are used to prove Cordoba's result.

Definition 3.1. A **ring of sets** is a family of sets closed under unions and set theoretic differences. That is, a family of sets $\mathcal{R}$ is said to be a ring if

1. $\emptyset \in \mathcal{R}$
2. if A, B $\in \mathcal{R}$ then A $\cup$ B $\in \mathcal{R}$
3. if A, B $\in \mathcal{R}$ then A$\backslash$B $\in \mathcal{R}$.

One of the most important lemma that will be used in the proof of Cordoba is by P. Cohen for idempotent measures on a group G. The lemma is stated below without the proof.

Lemma 3.2. *On locally compact abelian groups G*

$$\mu * \mu = \mu \Longleftrightarrow supp(\widehat{\mu}) \in \textit{coset ring of dual group of} \quad \widehat{G}.$$

i.e. $supp(\widehat{\mu})$ *is the finite union of sets of the form* $(\mathrm{x} + \mathrm{H})\backslash(\mathrm{x}_1 + \mathrm{H}_1)\backslash(\mathrm{x}_2 + \mathrm{H}_2)\backslash \ldots \backslash(\mathrm{x}_\mathrm{s} + \mathrm{H}_\mathrm{s})$ *where* $\mathrm{x}, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{s} \in \hat{\hat{\mathrm{G}}}$ *and* $\mathrm{H}, \mathrm{H}_1, \ldots, \mathrm{H}_\mathrm{s} < \hat{\hat{\mathrm{G}}}$.

Before we go to the next lemma let us look into a definition which will be needed in the next chapter also. Observe that any ball of radius 1.5 in $\mathbb{R}$ it intersects the lattice $\mathbb{Z}$. Such a set is called relatively dense set.

Definition 3.3. A set $S \subset \mathbb{R}^n$ is said to be **relatively dense** if there is $R > 0$ such that every ball of radius R in $\mathbb{R}^n$ intersects S.

Any lattice is a relatively dense set.

Lemma 3.4. *Let* $\{x_j\} \subset \mathbb{R}^n$ *be a discrete subset which is not relatively dense, and suppose that* $\mu = \sum a_j \delta_{x_j}$ *is a tempered distribution whose Fourier transform* $\widehat{\mu}$ *can be expressed in the form* $\widehat{\mu} = \sum b_k \delta_{y_k}$ *and satisfies the condition*

$$\sum_{y_\alpha \in Q} |b_\alpha| \leqslant \mathrm{C} \leqslant \infty, \ \textit{for every unit cube } Q \in \mathbb{R}^n \qquad (\star)$$

Then $\mu \equiv 0$.

Proof. A set S is not relatively dense if for every positive number t one can find a ball $B(z_t;t)$ which is disjoint with the set S.

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a Schwartz function whose $supp(\psi) \in \mathrm{B}(0;1)$ and $\widehat{\psi}(0) = 1$. We can choose ψ to be a bump function which has the above mentioned properties. Now, for a fixed y_k consider the function

$$\Phi(x) = e^{(2\pi i y_k . x)}\psi\left(\frac{x - z_t}{t}\right)$$

where $t > 0$ and z_t is such that $\mathrm{B(z_t;t)} \cap \{\mathrm{x_j}\} = \emptyset$. Since $|\frac{x_j - z_t}{t}| > 1 \; \forall \; x_j$ and $supp(\psi) \in \mathrm{B}(0;1)$, we have that

$$\hat{\mu}(\hat{\Phi}) = \mu(\Phi) = 0$$

which gives

$$0 = t^n \sum_{\alpha} b_\alpha \hat{\Psi}(t(y_\alpha - y_k))e^{(2\pi i z_t . (y_\alpha - y_k))}$$

which implies that

$$b_k = \sum_{\alpha \neq k} b_\alpha \hat{\Psi}(t(y_\alpha - y_k))e^{(2\pi i z_t . (y_\alpha - y_k))}$$

This yields,

$$|b_k| \leqslant \sum_{\alpha \neq k} |b_\alpha||\hat{\Psi}(t(y_\alpha - y_k))| \; .$$

Take the limit when t goes to infinity and use the condition ($\star$) to get $b_k = 0$. Now, we have that $\hat{\mu} = 0$ and therefore, $\mu = 0$. ■

Before going to the next lemma, recall that the only discrete subgroups of $\mathbb{R}^n$ are lattices. Also, a subgroup H of $\mathbb{R}^n$ is not discrete if and only if each neighbourhood of 0 in $\mathbb{R}^n$ contains infinitely many elements of H.

Lemma 3.5. *If $A = (x+H)\backslash(x_1+H_1)\backslash(x_2+H_2)\ldots\backslash(x_r+H_r) \neq \emptyset$, is a uniformly discrete set, where $x, x_1, \ldots, x_r \in \mathbb{R}^n$ and $H, H_1, \ldots, H_r < \mathbb{R}^n$. Then H is a discrete group.*

Proof. We will prove this by induction on r.

Let $r = 1$. If $(x + H) \cap (x_1 + H_1) = \emptyset$ then $A = (x + H)$ and H has to be uniformly discrete if A is uniformly discrete.

If $(x + H) \cap (x_1 + H_1) \neq \emptyset$, then there exists elements $h \in H, h_1 \in H_1$ such that $x + h = x_1 + h_1$. We have

$$(x + H) \cap (x_1 + H_1) = x + h + H \cap H_1$$

and

$$A = x + (H \backslash (h + H \cap H_1))$$

Since H is a disjoint union of cosets of $H \cap H_1$, we have that A is uniformly discrete implies that $H \cap H_1$ is discrete.

We claim that H is discrete. If not, then each open ball $B(0; s)$ in $\mathbb{R}^n$ must contain infinite number of points in H. Since $h + H \cap H_1$ is discrete, $B(0, s) \cap (h + H \cap H_1)$ is finite. Therefore, $(A - x) \cap B(0; s)$ is an infinite set. This is a contradiction since A is a uniformly discrete set.

If $r > 1$, then we have that

$$A = (x + H \setminus x_1 + H_1) \cap \ldots \cap (x + H \setminus x_r + H_r) \neq \emptyset$$

A is uniformly discrete implies that at least one of the set $(x + H \setminus x_l + H_l) \neq \emptyset$ is uniformly discrete. Therefore, by the case $r = 1$ we get that H is a discrete subgroup. ■

Lemma 3.6. *Suppose $\mathcal{B} = \{e_1, \ldots, e_m\}$ are linearly independent vectors in $\mathbb{R}^n$. Then for every $r > 0$ the set*

$$D_r = \{v = k_1 e_1 + \ldots + k_m e_m \mid d(v, \mathbb{Z}^n) < r \; ; \; k_j \in \mathbb{Z}, j = 1, \ldots, m\}$$

cannot be contained in a finite union of $(m - 1)$ dimensional planes.

Proof. The lemma says that the set of all points of the lattice generated by $\mathcal{B}$ which is at a distance of less than r from the lattice $\mathbb{Z}^n$ cannot be contained in finite union of $(m - 1)$ dimensional planes of $\mathbb{R}^n$. Proof is by induction on m and using the theorem of Hermann Weyl stating that $(k\theta_1, \ldots, k\theta_s)$ is uniformly distributed modulo 1 if $\{1, \theta_1, \ldots, \theta_s\}$ are linearly independent over the rationals.

Case : $m = 1$. If e_1 has all rational coordinates, then consider $k \in \mathbb{Z}$ such that ke_1 has all integer coordinates and belong to $\mathbb{Z}^n$. Hence, for any $r > 0$, D_r has infinitely many points. If e_1 has $s \geqslant 1$ distinct irrational coordinates, by Weyl's theorem $(k\theta_1, \ldots, k\theta_s)$ is equidistributed modulo 1 and hence dense in $[0,1]^s$. By this we can conclude that D_r has infinitely many points.

Case : $m > 1$. Let it be true for $m-1$. Fix k_m, then $\{v = k_1e_1 + \ldots + k_me_m\}$ is contained in an $m-1$ dimensional plane. Now, by the first case there are infinitely many k_m such that $d(k_m, \mathbb{Z}^n) < r$ for each $r > 0$. This concludes that D_r cannot be contained in a finite union of $(m-1)$ dimensional planes. ■

3.3 Cordoba's Result

Definition 3.7. Let $\widehat{\mathbb{R}^n}$ be the dual of the group $\mathbb{R}^n$ with compact open topology. Consider the space $\widehat{\mathbb{R}^n}$ with discrete topology and call it $\widehat{\mathbb{R}^n_d}$. The dual of $\widehat{\mathbb{R}^n_d}$ is called **Bohr compactification** of $\mathbb{R}^n$ and denoted by $b\mathbb{R}^n$.

Note that the $b\mathbb{R}^n$ is compact with respect to compact open topology and $\mathbb{R}^n$ is dense $b\mathbb{R}^n$. Recall Banach- Alaoglu theorem and Riesz representation theorem which are stated below.

Theorem 3.8 (Banach- Alaoglu)**.** *The closed unit ball with respect weak* topology is compact for a Banach space B.*

Theorem 3.9 (Riesz representation theorem)**.** *Let X be a compact hausdorff space. Let $C(X)$ be a linear space of all continuous real valued functions X with supremum norm and Radon(X) be a linear space of signed Radon measures on X with total variation as its norm. Then $(C(X))^* \cong$ Radon(X), where $\cong$ is an isometric isomorphism of linear spaces.*

Cordoba's result has been introduced earlier in this chapter. Now we will state and prove the result.

Theorem 3.10. *Let the set $\Lambda = \overset{N}{\underset{j=1}{\cup}} \Lambda_j$ be a finite disjoint union of subsets of $\mathbb{R}^n$ such that Λ is a uniformly discrete set*

Given distinct complex numbers $\{a_j\}_{j=1,\ldots,N}$ *consider the tempered distribution*

$$\mu = \sum_{j=1}^{N} a_j \sum_{x \in \Lambda_j} \delta_x$$

Assume that the Fourier transform $\hat{\mu}$ *can also be expressed in the form of*

$$\hat{\mu} = \sum_{\alpha} b_\alpha \delta_{y_\alpha}$$

and satisfies the property

$$\sum_{y_\alpha \in Q} |b_\alpha| \leqslant \mathrm{C} \leqslant \infty, \ \textit{for every unit cube } Q \in \mathbb{R}^n. \tag{*}$$

Then we have that each set Λ_j *is a finite disjoint union of translates of* $n-$ *dimensional lattices.*

.

Proof. We will prove this theorem for the case when $N = 2$.

Let Ψ be the function which was chosen in lemma 3.4. For each positive integer M, consider the measure,

$$\nu_m = \frac{1}{M^n} \Psi\Big(\frac{\cdot}{M}\Big).\hat{\mu} = \frac{1}{M^n} \sum_{\alpha} b_\alpha \Psi\Big(\frac{y_\alpha}{M}\Big) \delta_{y_\alpha}.$$

The condition (*) implies that for each M, ν_m is a measure of finite total variation and is bounded uniformly on M: $||\nu_m|| \leqslant c < \infty$.

There is a natural extension of ν_m as a finite measure $\bar{\nu}$ in $b\mathbb{R}^n$ i.e. the restriction of $\bar{\nu}_m$ to $\mathbb{R}^n$ is same as ν_m. By Riesz representation theorem and Banach Alaoglu's theorem, there is a subsequence which we shall also denote by $\{\bar{\nu_m}\}$, which converges to a finite measure $\bar{\nu}$ in *weak** topology.

We have that

$$\begin{aligned}
\hat{\bar{\nu}}(\zeta) &= \lim_{M \to \infty} \hat{\bar{\nu_M}}(\zeta) = \lim_{M \to \infty} \hat{\Psi}(M\cdot) * \mu(\zeta) \\
&= \lim_{M \to \infty} [a_1 \sum_{x \in \Lambda_1} \hat{\Psi}(M(\zeta - x)) + a_2 \sum_{x \in \Lambda_2} \hat{\Psi}(M(\zeta - x))].
\end{aligned}$$

Therefore by Riemann-Lebesgue lemma we get

$$\hat{\bar{\nu}}(\zeta) = \begin{cases} a_1, & \text{if } \zeta \in \Lambda_1 \\ a_2, & \text{if } \zeta \in \Lambda_2 \\ 0, & \text{if } \zeta \notin \Lambda. \end{cases}$$

As a next step, we claim that for each Λ_j, there is a finite measure μ_j on $b\mathbb{R}^n$ satisfying

$$\hat{\mu_j}(\zeta) = 1, \text{ if } \zeta \in \Lambda_j; \quad \hat{\mu_j}(\zeta) = 0, \text{ if } \zeta \notin \Lambda_j.$$

To prove this, consider $\bar{\nu_1} = \bar{\nu} * \bar{\nu} - a_1\hat{\nu}$.

$$\hat{\bar{\nu}}_1(\zeta) = \hat{\bar{\nu}}^2 - a_1\hat{\bar{\nu}}(\zeta) = \begin{cases} a_2^2 - a_1a_2, & \text{if } \zeta \in \Lambda_2 \\ 0, & \text{if } \zeta \notin \Lambda_2. \end{cases}$$

Therefore the measure $\mu_2 = \frac{1}{a_2^2 - a_1a_2}\bar{\nu}_1$ satisfies the above mentioned properties for Λ_2. Similarly, take $\bar{\nu_2} = \bar{\nu} * \bar{\nu} - a_2\bar{\nu}$ and the measure $\mu_1 = \frac{1}{a_1^2 - a_1a_2}\bar{\nu}_2$ for Λ_1

Observe that μ_1 and μ_2 both are idempotent measure whose supports are Λ_1 and Λ_2 respectively. Therefore we can apply P. Cohen's theorem to conclude that Λ_1 and Λ_2 belong to the coset ring of $\mathbb{R}^n$ with discrete topology which is the dual group of $b\mathbb{R}^n$.

That is, each Λ_j is of the form

$$A = (x + H)\backslash(x_1 + H_1)\backslash(x_2 + H_2)\ldots\backslash(x_r + H_r) \neq \emptyset$$

and by 24 4.1, H is a discrete subgroup of $\mathbb{R}^n$ (with usual topology), which is a lattice. Furthermore, $\exists$ elements $h_j \in H_j$ such that

$$A = x + \{H\backslash(h_1 + H \cap H_1)\backslash\ldots\backslash(h_r + H \cap H_r)\}.$$

Note the fact that if $G_1 < G_2$ where G_2 is a discrete and if G_1 and G_2 have same dimension as lattices then $G_2 =$ finite union of cosets of G_1.

Using this fact we can write

$$A = x + \{(y_1 + K_1) \cup \ldots \cup (y_s + K_s)\} \backslash A_1$$

where $K_1 \ldots K_s$ are subgroups of H having same lattice dimension as H and A_1 is contained finite union of hyperplanes.

Consider the identity: for any sets X_1, X_2, Y_1, Y_2

$$(X_1 \backslash Y_1) \cup (X_2 \backslash Y_2) = X_1 \cup X_2 \backslash \, [({X_2}^c \cap X_1 \cap Y_1) \cup ({X_1}^c \cap X_2 \cap Y_2) \cup (Y_1 \cap Y_2)]. \quad (*)$$

(*) yields that each Λ_j can be written in the form

$$\Lambda_j = (x_1 + H_1) \ldots (x_l + H_l) \cup B_1 \backslash \, B_2$$

where H_j are n-dimensional lattices of $\mathbb{R}^n$ and the discrete sets B_1, B_2 are contained in finite union of hyperplanes.

Let us express each Λ_j as disjoint union of translated lattices.

Suppose that $(x_l + H_l) \cap (x_k + H_k) \neq \emptyset$, then there exists elements $h_l \in H_l, h_k \in H_k$ such that $x_l + h_l = x_k + h_k$, that is:

$$(x_l + H_l) \cap (x_k + H_k) = x_l + h_l + H_l \cap H_k$$
$$(x_l + H_l) \cup (x_k + H_k) = x_l + h_l + H_l \cup H_k$$

We claim that $_j$ is uniformly discrete implies that $H_l \cap H_k$ has dimension n. If it is not so, we can assume that $H_l = \mathbb{Z}^n$ after an application of an invertible linear transformation T.

$$H_k = \{m_1 e_1 + \ldots + m_n e_n \mid m_i \in \mathbb{Z}\}$$

where $e_i = (\theta^i_1, \ldots, \theta^i_n)$ and at least one of the θ^i_j is not rational. If all them are rationals, then $H_k \cap \mathbb{Z}^n$ is n dimensional which is contrary to our assumption.

But, an application of 24 4.2 yields that the set

$$\{x \in H_k \mid \exists y \in H_l, \; 0 < d(x, y) < r\}$$

cannot be contained in finite union of hyperplanes. Therefore, $H_l \cup H_k \backslash B$ cannot be a uniformly discrete set for any set B which is contained in a finite union of hyperplanes. This is a contradiction, since it tells us that Λ_j is not uniformly discrete.

Further, if $H_l \cap H_k \neq \emptyset$ and has dimension n, then $H_l \cup H_k =$ finite disjoint union of cosets of $H_l \cap H_k \neq \emptyset$. After a finite applications of this procedure we can write each Λ_j as

$$\Lambda_j = (z_1 + Y_1) \cup \ldots \cup (z_m + Y_m) \cup C_1 \setminus C_2$$

where the n dimensional lattices $(z_k + Y_k)$ are disjoint and the discrete sets C_1, C_2 are contained in finite union of hyperplanes.

Now, consider the sets

$$\Lambda_j^* = (z_1 + Y_1) \cup \ldots \cup (z_m + Y_m)$$

Define the measure $\mu_1 = a_1 \sum_{x \in \Lambda_1^*} \delta_x + a_2 \sum_{x \in \Lambda_2^*} \delta_x$. By using the Poisson summation for lattices mentioned in previous chapter, we have that $\hat{\mu_1}$ is a measure satisfying the condition (*). In that case $\mu - \mu_1$ is also a measure whose Fourier transform satisfies (*). The support of $\mu - \mu_1$ is contained in a finite union of hyperplanes and hence is not relatively dense. By lemma 3.4 conclude that $\mu = \mu_1$ and we must have $\Lambda_j = \Lambda_j^*$ for $j = 1, 2$. This concludes the proof. ∎

For the case when $N > 2$ the same steps can be followed for the proof.

As shown in the example **2.1**, since $\Lambda_1 \cup \Lambda_2$ is a uniformly discrete set we were able to write it as union of cosets of a common lattice. But is it true in general. Cordoba's theorem only concludes that each Λ_j is a finite disjoint union of translates of $n-$dimensional lattices. But J.C. Lagarias says something more.

He remarks that since Λ is a uniformly discrete set then there is a common lattice L such that each of the Λ_j is a union of cosets of common lattice L. Indeed, since disjoint union of such translates $(L_1 + a_1) \cup (L_2 + a_2)$ cannot be uniformly discrete unless both can be written as a finite union of cosets of a common full rank L. This follows from Kronecker's theorem in Diophantine approximation.

Let us look at it in the dimension 1 case. Kronecker's theorem says that given any real x, any irrational θ and any $\epsilon > 0$, there exists integers h and $k > 0$ such that

$$|k\theta - h - x| < \epsilon.$$

Without loss of generality assume that $L_1 = \mathbb{Z}$ and let $L_2 = \beta\mathbb{Z}$. If $\beta = p/q$, a rational number, then L_1 and L_2 can be written as a union of cosets of a common lattice, i.e. $L_1 = p\mathbb{Z} \cup \ldots \cup ((p-1) + p\mathbb{Z})$ and $L_2 = p\mathbb{Z} \cup (p/q + p\mathbb{Z}) \cup \ldots \cup ((q - 1/q)p + p\mathbb{Z})$. Hence nothing to prove.

Let β be an irrational then $(L_1 + a_1) \cup (L_2 + a_2)$ cannot be written as union of cosets of a common lattice. Let $x = a_1 - a_1$ and $\theta = \beta$. By Kronecker's lemma, for every $\epsilon > 0$ there exists $h, k \in \mathbb{Z}$ and $k > 0$ such that

$$|k\beta - h - a_1 + a_2| < \epsilon \implies |(k\beta + a_2) - (h + a_1)| < \epsilon$$

where $(k\beta + a_2) \in (a_2 + L_2)$ and $(h + a_1) \in (a_1 + L_1)$. This proves that $(L_1 + a_1) \cup (L_2 + a_2)$ is not uniformly discrete.

What happens if we relax all the conditions on Dirac masses. Nir Lev and Olevskii have ivestigated this hypothesis. We will see their result in the next chapter. ∎

Chapter 4

Quasicrystals and Poisson summation formula

4.1 Introduction

In 2014 Nir Lev and Alexander Olevskii characterized the measures on $\mathbb{R}$ for which both their support and spectrum are uniformly discrete. A similar result was obtained for positive measures in $\mathbb{R}^n$. But we will only understand the former characterization. We will also study an important object known as “Meyer sets”.

Definition 4.1. Spectrum of a tempered distribution μ denoted by $spec(\mu)$ is the support of its Fourier transform.

Let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set. Consider a complex measure μ on $\mathbb{R}^n$ supported on Λ:

$$\mu = \sum_{\lambda \in \Lambda} \mu(\lambda)\delta_\lambda,\ \mu(\lambda) \neq 0,\ d(\Lambda) > 0. \tag{4.1}$$

Assume that μ is a tempered distribution and its Fourier transform is also a measure supported by a uniformly discrete set S:

$$\widehat{\mu} = \sum_{s \in S} \widehat{\mu}(s)\delta_s \tag{4.2}$$

The set S is the spectrum of the measure μ.

Theorem 4.2. *Let μ be a measure on $\mathbb{R}$ satisfying 4.1 and 4.2. Then the support Λ is contained in a finite union of translates of a certain lattice. The same is true for the dual lattice S.*

Before we proceed we will need some preliminaries.

Notation: Denote $B_r(x) := \{y \in \mathbb{R}^n : |y| < r$. And for $B_r(0)$ we simply denote it by B_r.

By a "distribution" we mean a tempered distribution on $\mathbb{R}^n$. By a "measure" we mean a complex, locally finite measure which is also a tempered distribution.

Lemma 4.3. *Let μ be a measure in $\mathbb{R}^n$ supported by a uniformly discrete set Λ. Then μ is a tempered distribution if and only if*

$$|\mu(\lambda)| \leqslant C(1 + |\lambda|^N),\ \lambda \in \Lambda$$

for some positive constants C and N.

Proof. Let ϕ be a Schwartz function such that $supp(\phi) \subset B_\delta$ where $\delta = d(\Lambda)$ and $\phi(0) = 1$. Let μ be a tempered distribution. Then there exists constants $B, N > 0$ and $N \in \mathbb{Z}$ such that $|\mu(\phi)| \leqslant B||\phi||_N,\ \forall \phi \in \mathcal{S}(\mathbb{R}^n)$. Let $\phi_\lambda = \phi(x - \lambda)$. Then we get that

$$||\phi_\lambda||_N \leqslant C'(1 + |\lambda|)^N ||\phi||_N.$$

And we also have that $(1 + |\lambda|)^N \leqslant \mathcal{O}(1 + |\lambda|^N)$. Therefore, combining all these we get

$$|\mu(\lambda)| = |\mu(\phi_\lambda)| \leqslant BC'(1 + |\lambda|)^N ||\phi||_N \leqslant C(1 + |\lambda|^N).$$

Conversely, let $|\mu(\lambda)| \leqslant C(1 + |\lambda|^N)$, $\lambda \in \Lambda$. Let $\phi_n \to \phi$ in Schwartz space. Then,

$$\begin{aligned} |\mu(\phi_n - \phi)| &\leqslant \sum_{\lambda \in \Lambda} |\mu(\lambda)||\phi_n(\lambda) - \phi(\lambda)| \\ &\leqslant C \sum_{\lambda \in \Lambda} (1 + |\lambda|^N)|\phi_n(\lambda) - \phi(\lambda)| \\ &\leqslant C' \sum_{\lambda \in \Lambda} ||\phi_n - \phi||_N \end{aligned}$$

As $n \to \infty$ the R.H.S of the above inequality tends to zero. Hence μ is a tempered distribution. ∎

Lemma 4.4. *Let μ be a measure in $\mathbb{R}^n$ satisfying 4.1 and 4.2. Then*

$$\sup_{\lambda \in \lambda} |\mu(\lambda)| < \infty. \tag{4.3}$$

Proof. Let ϕ be a Schwartz function such that $\widehat{\phi}(0) = 1$ and $supp(\widehat{\phi}) \subset B_\delta$, where $\delta = d(\Lambda) > 0$. Then

$$|\mu(\lambda)| = |\mu(\widehat{\phi})| = |\widehat{\mu}(\phi)| \leqslant \sum_{s \in S} |\phi(s)||\widehat{\mu}(s)|.$$

By Lemma 4.3 there are constants C, N such that $|\widehat{\mu}(s)| \leqslant C(1 + |s|^N)$. Then

$$|\mu(\lambda)| \leqslant \sum_{s \in S} |\phi(s)| C(1 + |s|^N) \leqslant \int_{\mathbb{R}^n} |\phi(x)| C(1 + |x|^N) dx \tag{4.4}$$

Since ϕ is a Schwartz function the R.H.S of 4.4 converges.Thus proving the lemma. ∎

Definition 4.5. Let $\Lambda \subset \mathbb{R}^n$. Then the upper and lower uniform densities are defined respectively to be

$$D^+(\Lambda) := \limsup_{R \to \infty} \sup_{x \in \mathbb{R}^n} \frac{\#(\Lambda \cap B_R(x))}{|B_R|}$$
$$D^-(\Lambda) := \liminf_{R \to \infty} \inf_{x \in \mathbb{R}^n} \frac{\#(\Lambda \cap B_R(x))}{|B_R|}$$

The following version of density is also needed.

$$D_\#(\Lambda) := \liminf_{R \to \infty} \frac{\#(\Lambda \cap B_R)}{|B_R|}$$

Clearly we have that $D^-(\Lambda) \leqslant D^\#(\Lambda) \leqslant D^+(\Lambda)$. If Λ is a uniformly discrete set then the above densities are finite. Let $d(\Lambda) = \delta$. Then the open balls of radius $\delta/2$ around each $\lambda \in \Lambda \cap B_R(x)$ are all disjoint for any $x \in \mathbb{R}^n$ and for any $R > 0$. They lie inside the ball of radius $R + \delta/2$ around x. This gives us the volume bound $\#\Lambda \cap B_R(x) \leqslant (\frac{2R}{\delta} + 1)^n$ uniformly for all x. Thus

$$\sup_{x \in \mathbb{R}^n} \frac{\#\Lambda \cap B_R(x)}{|B_R|} \leqslant C\left(\frac{2}{\delta} + \frac{1}{R}\right)^n$$

for some constant $C > 0$. Hence,

$$\limsup_{R\to\infty} \sup_{x\in\mathbb{R}^n} \frac{\#\Lambda \cap B_R(x)}{|B_R|} \leqslant C\left(\frac{2}{\delta}\right)^n < \infty.$$

Threfore, all the other densities will also be finite.

Also $D_{\#}$ and D^- are super additive because lim inf is super additive and D^+ is sub additive because lim sup is sub additive.

Clearly we have that all the above mentioned densities are translation invariant.

4.2 Delone and Meyer sets

Recall the definition of relatively dense set from the previous chapter.

Definition 4.6. $\Lambda \subset \mathbb{R}^n$ is called **Delone set** if Λ is a uniformly discrete set and relatively dense set.

This means that the atoms of a Delone should not be too close to each other as well as they cant be too far from others. For example any lattice is a Delone set since it is uniformly discrete as well as relatively dense set.

Definition 4.7. $\Lambda \subset \mathbb{R}^n$ is called a **Meyer set** if Λ is a Delone set and there is a finite set F such that $\Lambda - \Lambda \subset \Lambda + F$.

Meyer has termed the set Λ with the above defined property as 'Quasicrystals', but other mathematicians use the term 'Meyer sets' itself. Any lattice Γ is also a Meyer set since $\Gamma - \Gamma = \Gamma$. For a lattice we can take $F = 0$.

Lagarias has proved that Λ is a Meyer set if and only if $\Lambda - \Lambda$ is a uniformly discrete set. Nir Lev and Olevskii prove a stronger result stated below:

Lemma 4.8. *Let $\Lambda \subset \mathbb{R}^n$ be a delone set, such that $D^+(\Lambda - \Lambda) < \infty$. Then Λ is a Meyer set.*

Proof. Without loss of generality we may assume that $0 \in \Lambda$. Since by translating Λ to $\Lambda - x$ leaves $\Lambda - \Lambda$ unchanged and F is replaced by $F + x$, for any $x \in \mathbb{R}^n$.

Since Λ is a Delone set and $D^+(\Lambda - \Lambda) < \infty$, we can fix $R > 0$ such that every ball of radius R intersects Λ and for all $r \geqslant R$ we have,

$$\sup_{x \in \mathbb{R}^n} \#(\Lambda - \Lambda) \cap B_r(x) \leqslant M. \tag{$\star$}$$

Let $h \in \Lambda - \Lambda$ i.e. $h = y - x$ for some $x, y \in \Lambda$. Choose a sequence $x_0, x_1, \ldots, x_s$ such that $x_0 = x, x_s = 0, |x_i - x_{i+1}| < R$. Let $y_i = x_i + h$, then we have that $y_0 = y, y_s = h, |y_i - y_{i+1}| < R$. Choose $p_i, q_i \in \Lambda$ such that $|p_i - x_i| < R, |q_i - y_i| < R (0 \leqslant i \leqslant s)$. We can always find such p_i, q_i since Λ is a relatively dense set. Let us take $p_0 = x, q_0 = y$ and $p_s = 0$. Consider the set $F_1 = (\Lambda - \Lambda) \cap B_{3R}$. We have that

$$|p_i - p_{i+1}| \leqslant |p_i - x_i| + |x_i - x_{i+1}| + |p_{i+1} - x_{i+1}| < 3R$$

Similarly, $|q_i - q_{i+1}| < 3R$. Therefore, $p_i - p_{i+1}, q_i - q_{i+1} \in F_1 (0 \leqslant i \leqslant s)$. We see that F_1 is a finite set by ($\star$).

Set $h_i = q_i - p_i$. Then

$$h_i - h_{i+1} = (q_i - q_{i+1}) - (p_i - p_{i+1}) \in F_1 - F_1$$

Also

$$|h_i - h| = |(q_i - y_i) - (p_i - x_i)| < 2R,$$

hence

$$h_i \in V(h) := (\Lambda - \Lambda) \cap (h + B_{2R}).$$

Again by ($\star$), we obtain $\#V(h) \leqslant M$. Thus the sequence $h_0, h_1, \ldots, h_s$ has at most M distinct values. Hence the sum

$$h_0 - h_s = (h_0 - h_1) + (h_1 - h_2) + \ldots + (h_{s-1} - h_s)$$

can be expressed as a sum of at most $M - 1$ terms. Each of the term $(h_i - h_{i+1})$ is an element of $F_1 - F_1$. And $h_0 - h_s \in F$ where

$$F := \{\sum_{j=1}^{N} v_j \,|\, v_j \in F_1 - F_1,\ N \leqslant M - 1\}$$

Hence

$$h = h_0 = h_0 + (q_s - h_s) = q_s + (h_0 - h_s) \in \Lambda + F.$$

This proves that $\Lambda - \Lambda \subset \Lambda + F$, so Λ is a Meyer set. ■

Remark 4.9. The similar arguments can be used to prove the Lagarias' result. Instead of bounding the number of atoms in $V(h)$ using density we can use the fact the $\Lambda - \Lambda$ is a Delone set. That is, let $r' = d(\Lambda - \Lambda)$ then the open balls of radius $\frac{r'}{2}$ around each $w \in V(h)$ are all disjoint and lie inside the ball of radius $2R + \frac{r'}{2}$ around h. This gives us the volume bound $\#V(h) \leqslant \left(\frac{4R}{r'} + 1\right)^n$. Rest of the arguments are same as the above.

The key concept in the proof of theorem 4.2 is "model sets". Meyer introduced model sets in 1972, constructed using "cut and project" method.

Let Γ be a lattice in $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ $(m \geqslant 0)$. Let p_1, p_2 denote the projections onto $\mathbb{R}^n, \mathbb{R}^m$, respectively. Choose Γ such that p_1 restricted to Γ is injective and $p_2(\Gamma)$ is dense in $\mathbb{R}^m$. Let Ω be a bounded set in $\mathbb{R}^m$.

Definition 4.10. Under the above assumption, the model set M defined by Γ and Ω is the set

$$\mathfrak{M}(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega) := \{p_1(\gamma) : \gamma \in \Gamma, p_2(\gamma) \in \Omega\}.$$

Note that in the case when $m = 0$, $\mathbb{R}^m$ is taken to be $\{0\}$ and the model set we get is just a lattice in $\mathbb{R}^n$.

Lemma 4.11. $\Lambda \subset \mathbb{R}^n$ *is a relatively dense set if and only if there exists a compact set* K *such that* $\Lambda + K = \mathbb{R}^n$*.*

Proof. This is easy to see. If Λ is a relatively dense set then there exists $R > 0$ such that for any $x \in \mathbb{R}^n$, $B_R(x) \cap = \emptyset$. Let $K = \overline{B_R}$ and let $y \in \mathbb{R}^n$. Then there exists a $\lambda \in \Lambda$ such that $|y - \lambda| < R$. Hence $y \in \overline{B_R(\lambda)}$.

Conversely, there exists an $R > 0$ such that $K \subset_R (z)$ for some $z \in \mathbb{R}^n$. Hence for any $y \in \mathbb{R}^n$, there exists $\lambda \in \Lambda$ such that $y \in \lambda + K$. Thus, $\lambda \in B_R(y)$ and the lemma is proved ■

Next, we can ask questions about the structure of Model sets. Are atoms of models sets spaced very closely or are sparsely spread. The next proposition addresses this.

Proposition 4.12. *Any model set* $M = \mathfrak{M}(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega)$ *is a Delone set.*

Proof. Consider the set $M_r := (M - M) \cap B_r$, where $r > 0$. To prove that M is a uniformly discrete set, it is enough to show that $M_r = \{0\}$, for small enough r. Indeed since, $M_r = \{0\}$ for small r implies that 0 is not a limit point in $\Lambda - \Lambda$. This tells us that there does not exist any distinct $\lambda_1, \lambda_2 \in M$ such that $|\lambda_1 - \lambda_2| < r$.

For each $r > 0$, $K_r := \overline{B_r} \times \overline{\Omega} - \overline{\Omega}$ is a compact set in $\mathbb{R}^n \times \mathbb{R}^m$. For small enough r, $K_r \cap \Gamma = \{0\}$. Otherwise, 0 would be limit point in Γ which is not possible. For such r, if $\lambda_1, \lambda_2 \in M$ (there exists unique λ_1', λ_2' such that $(\lambda_1, \lambda_1'), (\lambda_2, \lambda_2') \in \Gamma$ and $\lambda_1', \lambda_2' \in \Omega$) is such that $|\lambda_1 - \lambda_2| < r$, then $(\lambda_1 - \lambda_2, \lambda_1' - \lambda_2') \in K_r \cap \Gamma$. So, $\lambda_1 = \lambda_2$, which proves that $M_r = \{0\}$. Thus, M is a uniformly discrete set.

Let us prove that M is a relatively dense set. All we need to find is a compact set C such that $C + \Lambda = \mathbb{R}^n$. Let C_1 and C_2 be compact subset of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively such that, $\Gamma + C_1 \times C_2 = \mathbb{R}^n \times \mathbb{R}^m$. Since $p_2(\Gamma)$ is dense in $\mathbb{R}^m$, $p_2(\Gamma) + (-\Omega) = \mathbb{R}^m$ and this forms a cover for the compact set C_2. Hence there exists a finite set $F \subset \Gamma$ such that $C_2 \subset \bigcup_{f \in F} (p_2(f) + (-\Omega))$. Let $C = C_1 - p_1(F)$. We have $\mathbb{R}^n \times \mathbb{R}^m = \Gamma + C_1 \times (p_2(F) + (-\Omega)) = \Gamma + (C_1 - p_1(F)) \times (-\Omega) = \Gamma + C \times (-\Omega)$. Now each $(x, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ can be written as

$$(x, 0) = \gamma + (c, -\omega), \quad \gamma \in \Gamma, c \in C, -\omega \in -\Omega.$$

Hence, $(x - c, \omega) = \gamma$ and $x \in c + \Lambda$, since $p_1(\gamma) \in \Lambda$. Therefore, $C + \Lambda = \mathbb{R}^n$. ■

It turns out that model sets are Meyer set, proof of which is skipped. But let us understand this from an example in dimension 1.

Example 4.1. Let α, β be distinct irrational numbers. Consider the invertible linear transformation A, where

$$A = \begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix}$$

The image of $\mathbb{Z}^2$ under this linear transformation is the lattice Γ which has the property that p_1 restricted to Γ is one-one and the $p_2(\Gamma)$ is dense in $\mathbb{R}$.

Let us take α such that $\frac{1}{k+1} < \alpha < \frac{1}{k}$ where $k \in \mathbb{N}$ and $\beta = \frac{-1}{\alpha}$. Now let us take $\Omega = [0, 1]$. And let

$$M = \mathfrak{M}(\mathbb{R} \times \mathbb{R}, \Gamma, \Omega).$$

By Proposition 4.12, we have that M is a Delone set. Let us prove that its a Meyer set, i.e. we need to find a finite set F such that $M - M \subset M + F$.

Let us look at the elements of M. We have that $n + m\alpha \in M$ where $n, m \in \mathbb{Z}$, if $n + m\beta \in \Omega$. That is, for $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ such that $(n-1)\alpha \leqslant m \leqslant n\alpha$, we have $n + m\alpha \in M$. That is

$$M = \{n + \lfloor n\alpha \rfloor \alpha : (n-1)\alpha \leqslant \lfloor n\alpha \rfloor\}$$

Let us look at the each interval $(i, i+1)$, $i \in \mathbb{Z}$. Observe that there are at least k integers and at most $k+1$ consecutive integers such that $n\alpha \in (i, i+1)$. Hence every integer i will occur as the coefficient of α such that $i = \lfloor l\alpha \rfloor$ and $(l-1)\alpha \leqslant \lfloor l\alpha \rfloor$ for some $l \in \mathbb{Z}$. Hence $\forall i \in \mathbb{Z}$ there exists $l \in \mathbb{Z}$ such that $l + i\alpha \in M$.

Also, if $l + \lfloor l\alpha \rfloor \alpha \in M$, then the next succeeding element of M is either $l + k + (\lfloor l\alpha \rfloor + 1)\alpha$ or $l + k + 1 + (\lfloor l\alpha \rfloor + 1)\alpha$.

Now let us look at the elements of $M - M$. Let $z \in M - M$ then $z = x - y$ where $x = n + \lfloor n\alpha \rfloor \alpha$ and $y = m + \lfloor m\alpha \rfloor \alpha$ be elements on M. Look at the coefficient of α in $x - y$. Hence there exists an integer l such that $\lfloor n\alpha \rfloor - \lfloor m\alpha \rfloor = \lfloor l\alpha \rfloor$. By using the following inequalities

$$\lfloor n\alpha \rfloor - 1 \leqslant (n-1)\alpha \leqslant \lfloor n\alpha \rfloor$$
$$\lfloor m\alpha \rfloor - 1 \leqslant (m-1)\alpha \leqslant \lfloor m\alpha \rfloor$$
$$(l-1)\alpha \leqslant \lfloor l\alpha \rfloor \leqslant l\alpha$$

we have that

$$l - k - 1 \leqslant n - m \leqslant l + k + 1$$

Now we take cases where $n - m = l + j$ and $-k - 1 \leqslant j \leqslant k + 1$. Then we have that

$$l + j + \lfloor l\alpha \rfloor \alpha = j + (l + \lfloor l\alpha \rfloor \alpha) \in M + F$$

where

$$F := \{-k-1, -k, -k+1, \ldots, 0, \ldots, k, k+1\}$$

Hence $M - M \subset M + F$ and M is a Meyer set.

$\blacksquare$

R. V. Moody has characterized Meyer sets in his paper "*Meyer sets and their duals*". The following theorem gives a characterization of Meyer sets, proof of which can be referred from Moody's paper.

Theorem 4.13. *Let Λ be a Delone set in $\mathbb{R}^n$. Then the following are equivalent:*
(i) Λ is a Meyer set;
(ii) There exists a model set M and a finite set F such that $\Lambda \subset M + F$.

The next lemma is also stated without proof.

Lemma 4.14. *Let $M = \mathfrak{M}(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega)$ be a model set in $\mathbb{R}^n$, and suppose that the boundary of Ω is a set of lebesgue measure zero in $\mathbb{R}^m$. Then*

$$D^-(M) = D^+(M) = \frac{mes(\Omega)}{det(\Gamma)}$$

Lemma 4.15. *Let $M = \mathfrak{M}(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega)$ be a model set and F be a finite set in $\mathbb{R}^n$. Then there is another model set $M' = \mathfrak{M}(\mathbb{R}^n \times \mathbb{R}^m, \Gamma', \Omega')$ and a finite set F' such that*

$$M + F \subset M' + F', \quad p_1(\Gamma') \cap \mathbb{Z}[F'] = \{0\}, \quad \Gamma \subset \Gamma'$$

where $\mathbb{Z}[F']$ is the additive group generated by the elements of F'.

Proof. Let $f_1, \ldots, f_s$ be the elements of F. The vector space $V = \mathbb{Q}[F]$ generated by elements of F is of finite dimension over $\mathbb{Q}$. Let $U := V \cap \mathbb{Q}[p_1(\Gamma)]$, a linear subspace of V. Let W be any linear subspace of V such that $V = U + W$. Then each f_i has a unique representation as $f_i = u_i + w_i$, where $u_i \in U, w_i \in W$.

Since $U \subset \mathbb{Q}[p_1(\Gamma)]$ we have $f_i = (\frac{r_i}{q_i})p_1(\gamma_i*)$, where $\frac{r_i}{q_i} \in \mathbb{Q}, \gamma_i* \in \Gamma$ for $1 \leqslant i \leqslant s$. Let q be the largest among $q_1, \ldots, q_s$. Then we can write $f_i = p_1(\gamma_i/q)$. Define,

$$\Gamma' := (1/q)\Gamma, \quad \Omega' := \bigcup_{i=1}^{s}(\Omega + p_2(\gamma_i/q)), \quad F' = \{w_1, \ldots, w_s\}.$$

We see that Γ' is a lattice in $\mathbb{R}^{n+m}$, the restriction of p_1 to Γ' is injective (if $p_1(x/q) = p_1(y/q)$ then $p_1(x) = p_1(y)$ which implies $x = y$), and $p_2(\Gamma')$ is dense in $\mathbb{R}^m$. The set Ω' is bounded in $\mathbb{R}^m$ and F' is a finite set in $\mathbb{R}^n$. Also, clearly $\Gamma \subset \Gamma'$.

Let M' be the model set defined by Γ' and Ω'. We need to show that $M + F \subset M' + F'$. If $\lambda \in M + F$ then $\lambda = p_1(\gamma) + f_j$, $\gamma \in \Gamma$ and $p_2(\gamma) \in \Omega$. Further,

$\lambda = p_1(\gamma + \gamma_j/q) + w_j$. We get this because $f_j = p_1(\gamma_j/q) + w_j$. Set $\gamma' = \gamma + \gamma_j/q$, then $\gamma' \in \Gamma'$ and $p_2(\gamma') \in \Omega'$. Hence

$$\lambda = p_1(\gamma') + w_j \in M' + F'.$$

Finally, observe that the set $p_1(\Gamma') \cap \mathbb{Z}[F'] \subset \mathbb{Z}[w_1, \ldots, w_s] \subset W$. Also, if y is an element of $p_1(\Gamma') \cap \mathbb{Z}[F']$, then clearly $y \in Q(p_1(\Gamma))$ and

$$y = \sum_{i=1}^{s} n_i w_i = \sum_{i=1}^{s} n_i (f_i - u_i) \in U$$

. Therefore, $p_1(\Gamma') \cap \mathbb{Z}[F'] = \{0\}$ and the lemma is proved. ∎

For the case when $m = 0$ the above lemma reduces to:

Lemma 4.16. *Let L be a lattice in $\mathbb{R}^n$. Then there is another lattice L' and a finite set F' such that*

$$L + F \subset L' + F', \quad L' \cap \mathbb{Z}[F'] = \{0\}, \quad L \subset L'.$$

4.3 Proof of Theorem 4.2

Step 1 :

Our first step is to prove that Λ is a Delone set. We already assumed that Λ is a uniformly discrete set. All we have to prove is that it is a relatively dense set. For this we will need a lemma which will be stated without the proof.

Lemma 4.17. *Given $a > 0$ there is an R depending on a such that, if a measure ν is supported by a uniformly discrete set Q in $\mathbb{R}$, $d(Q) > a$, and if $\widehat{\nu}$ vanishes on a ball of radius R, then $\nu = 0$.*

Let $a = d(S)$. If Λ is not a relatively dense set, then $\exists z \in \mathbb{R}$, $\forall r > 0$ such that $B_r(z) \cap \Lambda = \emptyset$. By applying the previous lemma to $Q = S, \nu = \widehat{\mu}$ we get that $\widehat{\mu} = 0$, which implies $\mu = 0$. We arrive at a contradiction.

Thus, Λ is a Delone set.

Step 2 :

Next, we will prove that Λ is a Meyer set.

Notation. For $h = \lambda_1 - \lambda_2 \in \Lambda - \Lambda$, denote

$$\Lambda_h := \Lambda \cap (\Lambda - h) = \{\lambda \in \Lambda : \lambda + h \in \Lambda\}.$$

Clearly Λ_h is a non-empty subset of Λ since $\lambda_2 \in \Lambda_h$.

Let μ be a measure in $\mathbb{R}$ satisfying the conditions 4.1 and 4.2. For each $\in \Lambda$ define a new measure

$$\mu_h := \sum_{\lambda\in\Lambda} \mu(\lambda)\mu(\lambda+h)\delta_\lambda. \tag{4.5}$$

Observe that it is a non-zero measure with $supp(\mu_h) = \Lambda_h$. Hence 4.5 becomes

$$\mu_h := \sum_{\lambda\in\Lambda_h} \mu(\lambda)\mu(\lambda+h)\delta_\lambda. \tag{4.6}$$

By Lemma 4.4, $\sup_{\lambda\in\Lambda_h} |\mu(\lambda)\mu(\lambda+h)| < C$ for some constant $C > 0$. Hence by Lemma 4.3 μ_h is a tempered distribution.

Lemma 4.18. *Let* $a := d(S) > 0$*. Then we have* $\mathrm{spec}(\mu_\mathrm{h}) \cap \mathrm{B_a} \subset \{0\}$.

Proof. Let $\phi \in \mathcal{S}(\mathbb{R})$ such that $\mathrm{supp}(\phi) \subset \mathrm{B_a}\backslash\{0\}$. Now consider the measure

$$\nu = (\phi * \widehat{\mu}) \cdot \widehat{\mu}$$

Then the $\mathrm{supp}(\nu) \subset (\mathrm{S} + (\mathrm{B_a}\backslash\{0\})) \bigcap \mathrm{S}$ which is an empty set. Hence $\nu = 0$. Now consider $\widehat{\nu}$

$$\begin{aligned}
0 = \widehat{\nu} &= (\widehat{\phi} \cdot \mu) * \mu \\
&= \sum_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda} \widehat{\phi}(\lambda)\mu(\lambda)\mu(\lambda')\delta_{\lambda-\lambda'} \\
&= \sum_{h\in\Lambda-\Lambda}\left[\sum_{\lambda\in\Lambda_h} \widehat{\phi}(\lambda)\mu(\lambda)\mu(\lambda+h)\right]\delta_h \\
&= \sum_{h\in\Lambda-\Lambda} \mu_h(\widehat{\phi})\delta_h \\
&= \sum_{h\in\Lambda-\Lambda} \widehat{\mu_h}(\phi)\delta_h.
\end{aligned}$$

It follows that for every $h \in \Lambda - \Lambda$ we have

$$\widehat{\mu_h}(\phi) = 0.$$

Thus $\mathrm{spec}(\mu_{\mathrm{h}})$ is disjoint from $B_a \backslash \{0\}$. ■

The above Lemma implies that $\widehat{\mu_h}$ vanishes on the open interval $(0, a)$. This is called a spectral gap.

Definition 4.19. A measure μ is said to have a spectral gap of size $a > 0$ if the Fourier transform $\widehat{\mu}$ vanishes on a ball of radius a.

The following proposition gives a necessary condition for which a uniformly discrete set Λ to support a measure has a spectral gap in dimension 1.

Proposition 4.20. *Let $\Lambda \subset \mathbb{R}$ be a uniformly discrete set, $d(\Lambda) = \delta > 0$. Assume that Λ supports a non-zero measure μ, such that $\widehat{\mu}$ vanishes on an open interval $(0, a)$ for some $a > 0$. Then*

$$D_{\#}(\Lambda) \geqslant c(a, d),$$

where $c(a, d) > 0$ depends only on a and δ.

To prove this we need the next lemma.

Lemma 4.21. *Let Λ be a finite set contained in $(-R, R) \backslash (-\delta, \delta)$, where $d(\Lambda) = \delta > 0$, $R \geqslant 1$, and let $a > 0$. There is $c(a, \delta) > 0$ such that if $(\#\Lambda)/2R < c(a, \delta)$ then one can find a Schwartz function ϕ with the following properties:*

$$\phi(0) = 1, \quad \phi(\lambda) = 0, \quad spec(\phi) \subset (0, a), \quad \sup_{|x|>R} |\phi(x)| \leqslant 1.$$

Proof. Assume that the number of points in Λ is even. Let $n = \frac{\#\Lambda}{2}$ and $\varepsilon = n/R$. Define the polynomial

$$P(z) = \prod_{\lambda \in \Lambda} \frac{z - e^{i\pi\lambda/R}}{1 - e^{i\pi\lambda/R}}$$

Then $P(1) = 1$. Using the fact that for $|x| \leqslant 1$

$$|\sin(\frac{\pi \mathrm{x}}{2})| = \sin|\frac{\pi \mathrm{x}}{2}| \geqslant |\mathrm{x}|$$

we have

$$\max_{|z=1|}|P(z)| \leqslant \prod_{\lambda \in \Lambda} \frac{2}{2\mathrm{sin}|\frac{\pi\lambda}{2\mathrm{R}}|} \leqslant \prod_{\lambda \in \Lambda} \frac{R}{|\lambda|}.$$

If Λ is the set $\{j\delta : 1 \leqslant |j| \leqslant n\}$ then right hand side is maximized. And also using the fact that $n! \geqslant (\frac{n}{e})^n$ we get

$$\max_{|z=1|}|P(z)| \leqslant \frac{R^{2n}}{\delta^{2n}(n!)^2} \leqslant \left(\frac{eR}{\delta n}\right)^{2n} = \left(\frac{e}{\delta\varepsilon}\right)^{2\varepsilon R}.$$

Given $a > 0$, we choose a Schwartz function ψ satisfying

$$\mathrm{spec}(\psi) \subset (0, \mathrm{a}), \quad \psi(0) = 1, \quad \eta := \sup_{|\mathrm{x}| \geqslant 1}|\psi(\mathrm{x})| < 1.$$

Let ϕ be a Schwartz function such that such that $\mathrm{spec}(\phi) \subset (0, \mathrm{a})$, $\phi(0) = 1$ and let r be such that $\sup_{|x| \geqslant r}|\phi(x)| < 1$. Hence take $\psi = \phi(rx)$ which has all the required properties. Set

$$\varphi(x) := P(e^{i\pi x/2}).(\psi(x/R))^{\lfloor R\rfloor+1}. \tag{4.7}$$

Then φ is a Schwartz function, $\varphi(0) = 1$, $\varphi(\lambda) = 0$ for $\lambda \in \Lambda$. In terms of tempered distribution we have that $\widehat{\varphi} = \widehat{P(e^{i\pi x/2})} * \widehat{\psi(x/R)} * \ldots * \widehat{\psi(x/R)}$ ($\lfloor R\rfloor + 1$ times).We can write

$$P(e^{i\pi x/2}) = a_{2n}e^{2ni\pi x/R} + a2n - 1e^{(2n-1)i\pi x/R} + \ldots + a_0$$

If $\phi \in \mathcal{S}(\mathbb{R})$ then

$$\widehat{P}(\phi) = P(\widehat{\phi}) = a_{2n}\phi(n/R) + a_{2n-1}\phi((2n-1)n/2R) + \ldots + a_0\phi(0).$$

Hence, the spectrum of $P(e^{i\pi x/2})$ is contained in $[0, \varepsilon]$. While the spectrum of second factor of 4.7 is contained in $(0, a/2)$, since spectrum of convolution is added. Hence, if $\varepsilon < a/2$ then $\mathrm{spec}(\varphi) \subset (0, \mathrm{a})$. Finally, we have

$$\sup_{|x| \geqslant R}|\varphi(x)| \leqslant \left[\gamma\left(\frac{e}{\delta\varepsilon}\right)^{2\varepsilon}\right]^R$$

If ε is sufficiently small depending on a, δ then the expression inside the square brackets is smaller than one. Hence the lemma is proved.

■

Let us now prove proposition 4.20

Proof. We will prove this by contradiction. Assume that $D_{\#}(\Lambda) < c(a, \delta)$, where $c(a, \delta)$ is given by Lemma 4.21. We will show this implies $\mu = 0$.

It will be enough to prove the claim for finite measures. The general case can be reduced to this one by multiplying μ with a Schwartz function ϕ, such that $|\phi| > 0$ and *spec*$(\phi) \subset (-a/2, 0)$. Consider a Schwartz function ψ such that *supp*$(\psi) \subset (-a/4, 0)$. Let $\phi = \widehat{\psi * \psi}$, which gives the required properties. Then $\phi\mu$ is a non-zero, finite measure (by Lemma 4.3) supported by Λ. Since $\widehat{\phi\mu} = \widehat{\phi} * \widehat{\mu}$ and support of convolution is added up and thus has a spectral gap $(0, a/2)$.

It will be enough to consider the case when $0 \in \Lambda$ and to prove $\mu(0) = 0$ by translating of μ and Λ, since $D_{\#}(\Lambda - \lambda) = D_{\#}(\Lambda)$.

Let $\Lambda_j := \Lambda \cap (-R_j, R_j) \backslash \{0\}$. Choose a sequence $R_j \to \infty$ such that $(\#\Lambda_j)/(2R_j) < c(a, \delta)$. Such a sequence can be chosen since lim inf is limit of a non-decreasing sequence. Let ϕ_j be the function given by the Lemma 4.21 with $\Lambda = \Lambda_j$ and $R = R_j$. Since $\widehat{\mu}$ vanishes on $(0, a)$ we have

$$\widehat{\mu}(\widehat{\phi_j}) = 0$$

We also have that

$$\widehat{\mu}(\widehat{\phi_j}) = \mu(\phi_j) = \mu(0) + \sum_{|\lambda| \geqslant R_j} \phi_j(\lambda)\mu(\lambda).$$

It follows that

$$|\mu(0)| \leqslant \sum_{|\lambda| \geqslant R_j} |\mu(\lambda)|$$

The right hand side of the above inequality tends to 0 as $j \to 0$ since μ is a finite measure. Hence $\mu(0) = 0$ which concludes that $\mu = 0$. ∎

Since μ_h has a spectral gap on $(0, a)$, by above proposition we have that

$$D_{\#}(\Lambda_h) \geqslant c, \quad h \in \Lambda - \Lambda, \tag{4.8}$$

where the constant $c > 0$ depends on $a = d(S)$ and $\delta = d(\Lambda)$.

Now let us prove that $D^+(\Lambda - \Lambda) < \infty$. If we establish this then by Lemma 4.8, we get that Λ is a Meyer set.

Lemma 4.22. *Let Λ be a uniformly discrete set in $\mathbb{R}$. Suppose there is $c = c(\Lambda) > 0$ such that $D_{\#}(\Lambda_h) > c$ for every $h \in \Lambda - \Lambda$. Then $D^+(\Lambda - \Lambda) < \infty$.*

Proof. Let $x \in \mathbb{R}$. Suppose that $h_1, h_2, \ldots, h_n$ are distinct vectors belonging to the set $\Lambda - \Lambda \cap B_\delta(x)$, where $\delta = d(\Lambda)/2$. We claim that $\Lambda_{h_i} \cap \Lambda_{h_j} = \emptyset$, $(i \neq j)$. Let $\lambda \in \Lambda_{h_i} \cap \Lambda_{h_j}$ and since Λ is a uniformly discrete set

$$h_i - h_j = (\lambda + h_i) - (\lambda + h_j) \in (\Lambda - \Lambda) \cap B_{2\delta} = \{0\}$$

which is not possible. Hence $\Lambda_{h_1}, \ldots, \Lambda_{h_n}$ are pairwise disjoint subsets of Λ. Since $D_{\#}$ is super additive, it follows that

$$D_{\#}(\Lambda) \geqslant \sum_{j=1}^{N} D_{\#}(\Lambda_{h_j}) \geqslant cN.$$

This shows that

$$\sup_{x \in \mathbb{R}} \#(\Lambda - \Lambda) \cap B_\delta(x) \leqslant D_{\#}(\Lambda)/c$$

Since lim sup is the limit of non-increasing sequence. Hence,

$$D^+(\Lambda - \Lambda) \leqslant \frac{D_{\#}(\Lambda)}{c|B_\delta|} < \infty.$$

∎

Thus by applying Lemma 4.13 gives that Λ is a Meyer set.

Step 3 :

All there is left to show is that Λ is contained in a finite union of translates of some lattice.

Since Λ is a Meyer set, by Theorem 4.13 there is a model set $M = \mathfrak{M}(\mathbb{R} \times \mathbb{R}^m, \Gamma, \Omega)$ and a finite set F such that $\Lambda \subset M + F$. Then we would be done if we show that M is a lattice in $\mathbb{R}$, i.e. $m = 0$.

Lemma 4.23. *Let Λ be a Meyer set in $\mathbb{R}$. Suppose there is $c = c(\Lambda)$ such that*

$$D^+(\Lambda_h) > c \tag{4.9}$$

for every $h \in \Lambda - \Lambda$. Then Λ is contained in a finite union of translates of some lattice.

Proof. (i) Let $M = \mathfrak{M}(\mathbb{R} \times \mathbb{R}^m, \Gamma, \Omega)$ and a finite set F be such that $\Lambda \subset M + F$. By Lemma 4.16 we may suppose that

$$p_1(\Gamma) \cap \mathbb{Z}[F] = \{0\}. \tag{4.10}$$

Let $\lambda \in \Lambda$ is such that

$$\lambda = p_1(\gamma(\lambda_1)) + \theta(\lambda_1) = p_1(\gamma(\lambda_2)) + \theta(\lambda_2)$$

where $\gamma(\lambda_1), \gamma(\lambda_2) \in \Gamma,\ p_2(\gamma(\lambda_1)), p_2(\gamma(\lambda_2)) \in \Omega,\ \theta(\lambda_1), \theta(\lambda_2) \in F$. Since the restriction of p_1 to Γ is injective and by condition 4.10 it follows that

$$\gamma(\lambda_1) = \gamma(\lambda_2), \quad \theta(\lambda_1) = \theta(\lambda_2).$$

Hence λ admits a unique representation as

$$\lambda = p_1(\gamma(\lambda)) + \theta(\lambda), \quad \gamma(\lambda) \in \Gamma, p_2(\gamma(\lambda)) \in \Omega, \theta(\lambda) \in F \tag{4.11}$$

(ii) Let $h \in \Lambda - \Lambda$, and suppose that $\lambda_1, \lambda_2 \in \Lambda_h$. Denote

$$\lambda_j' := \lambda_j + h, \quad j = 1, 2$$

Then from 4.11 we have that

$$h = \lambda_j' - \lambda_j = p_1(\gamma(\lambda_j') - \gamma(\lambda_j)) + (\theta(\lambda_j') - \theta(\lambda_j)), \quad j = 1, 2.$$

By the condition 4.10 and since the restriction of p_1 to Γ is injective, we must have

$$\gamma(\lambda_1') - \gamma(\lambda_1) = \gamma(\lambda_1') - \gamma(\lambda_1).$$

Thus we obtain that to each $h \in \Lambda - \Lambda$ there corresponds an element $H(h) \in \Gamma$ such that

$$\gamma(\lambda + h) - \gamma(\lambda) = H(h), \quad \forall \lambda \in \Lambda_h. \tag{4.12}$$

(iii) Let $E := \{p_2(\gamma(\lambda)) : \lambda \in \Lambda\}$. The E is a bounded set in $\mathbb{R}^m$ since $E \subset \Omega$. Given any $\delta > 0$, we can choose a vector $\zeta \in E - E$ such that $|\zeta|^2 > diam(E)^2 - \delta^2$. Observe that

$$E - E = \{p_2(H(h)) : h \in \Lambda - \Lambda\},$$

hence $\zeta = p_2(H(h))$ for some $h \in \Lambda - \Lambda$. Let us fix such an h.

Now suppose that $\lambda_1, \lambda_2 \in \Lambda_h$. By 4.12 we have that

$$H(h) = \gamma(\lambda_j + h) - \gamma(\lambda_j), \quad j = 1, 2$$

This yields

$$\zeta = p_2(H(h)) = p_2(\gamma(\lambda_j + h) - \gamma(\lambda_j)) \quad j = 1, 2.$$

By the parallelogram law we have that

$$\begin{aligned} |\zeta|^2 + |p_2(\gamma(\lambda_2) - \gamma(\lambda_1))|^2 &= \frac{1}{2}(|p_2(\gamma(\lambda_2 + h) - \gamma(\lambda_1))|^2 + |p_2(\gamma(\lambda_1 + h) - \gamma(\lambda_2))|^2) \\ &\leqslant (diam(E))^2. \end{aligned}$$

This yields us that

$$|p_2(\gamma(\lambda_2) - \gamma(\lambda_1))|^2 < \delta.$$

Denote $E(h) := \{p_2(\gamma(\lambda)) : \lambda \in \Lambda_h\}$. We conclude that for any given $\delta > 0$ one can find $h \in \Lambda - \Lambda$ such that diam(E(h)) $< \delta$.

(iv) Let $h \in \Lambda - \Lambda$ and $\delta > 0$ be such that $diam(E(h)) < \delta$. We may find an open ball $B_\delta(z)$ such that $E(h) \subset B_\delta(z)$. Consider the model set

$$M' = \mathcal{M}(\mathbb{R} \times \mathbb{R}^m, \Gamma, B_\delta(z)).$$

Then we have that $\Lambda_h \subset M' + F$. Since D^+ is sub-additive and invariant under translations, this yields that

$$D^+(\Lambda_h) \leqslant (\#F)(D^+(M')).$$

By applying Lemma 4.14 we get

$$D^+(\Lambda_h) \leqslant (\#F)\left(\frac{Vol(B_1)\delta^m}{det(\Gamma)}\right).$$

If $m \geqslant 1$, then we may find elements $h \in \Lambda - \Lambda$ with $D^+(\Lambda_h)$ arbitrarily small, which is a contradiction to 4.9. Therefore m has to be 0 and hence M a lattice. Since $\Lambda \subset M + F$, this concludes the proof. ■

Further remarks

Cordoba, Nir Lev and Olevskii have assumed in the hypothesis that both support and spectrum of a tempered distribution μ are uniformly discrete sets. But Nir Lev and Olevskii have further investigated the case when support is uniformly discrete but the spectrum is just a discrete closed set and the case when both of them are just discrete closed sets.

In the former case they proved that the spectrum also has to be uniformly discrete. In the latter case, it turns out the support contains only finitely many elements of any arithmetic progression.

It is also interesting to know that an Israeli scientist named Dan Shechtman recieved Nobel prize for the discovery of Quasicrystals in chemistry. But in mathematical terms, it was already discovered by Yves Meyer in 1970's known as Meyer sets.

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