We will define the colored Jones polynomial $J'_K(N) \in \mathcal{R} = \mathbb{Z}[q, q^{-1}]$ for a knot $K$, where $N$ is a positive integer bigger than or equal to 2.
We will define the colored Jones polynomial $J'_K(N) \in \mathcal{R} = \mathbb{Z}[q, q^{-1}]$ for a knot $K$, where $N$ is a positive integer bigger than or equal to 2.

- $J'_K(2)$ is the ordinary Jones polynomial for $K$.
- If $K$ is the unknot $J'_K(N) = 1$. 
Vu Huynh and Thang Le have the following description of the colored Jones polynomial.

\[
J'_K(N) = q^{(N-1)(\omega(\beta)-m+1)/2} \hat{E}_N \left( \frac{1}{\tilde{\det}_q(I - q \rho'(\gamma))} \right)
\]
Let $\sigma_i, 1 \leq i \leq m - 1$ be the standard generators of the braid group on $m$ strands.

**Definition**

For a sequence $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ of pairs $\gamma_j = (i_j, \epsilon_j)$, $1 \leq i_j \leq m - 1$ and $\epsilon_j = \pm$, let $\beta = \beta(\gamma)$ be the braid

$$
\beta := \sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \ldots \sigma_{i_k}^{\epsilon_k}.
$$

Fix $\gamma$ so that the closure of $\beta(\gamma)$ is the knot $K$. $\omega(\beta)$ is the writhe of $\beta(\gamma)$. 
The Operators

Define operators $\hat{x}$ and $\tau_x$ and their inverses acting on the ring $\mathbb{R}[x^{\pm 1}, y^{\pm 1}, u^{\pm 1}]$:

$$\hat{x}f(x, y, \ldots) = xf(x, y, \ldots), \quad \tau_x f(x, y, \ldots) = f(qx, y, \ldots)$$
The Operators

Define operators \( \hat{x} \) and \( \tau_x \) and their inverses acting on the ring \( \mathcal{R}[x^{\pm 1}, y^{\pm 1}, u^{\pm 1}] \):

\[
\hat{x}f(x, y, \ldots) = xf(x, y, \ldots), \quad \tau_x f(x, y, \ldots) = f(qx, y, \ldots)
\]

Also define \( \hat{y}, \tau_y, \hat{u}, \tau_u \), and their inverses similarly.
The Operators

Define operators $\hat{x}$ and $\tau_x$ and their inverses acting on the ring $\mathbb{R}[x^\pm 1, y^\pm 1, u^\pm 1]$: 

$$\hat{x} f(x, y, \ldots) = xf(x, y, \ldots), \quad \tau_x f(x, y, \ldots) = f(qx, y, \ldots)$$

Also define $\hat{y}$, $\tau_y$, $\hat{u}$, $\tau_u$, and their inverses similarly.

**Definition**

$$a_+ = (\hat{u} - \hat{y} \tau_x^{-1}) \tau_y^{-1}, \quad b_+ = \hat{u}^2, \quad c_+ = \hat{x} \tau_y^{-2} \tau_u^{-1},$$

$$a_- = (\tau_y - \hat{x}^{-1}) \tau_x^{-1} \tau_u, \quad b_- = \hat{u}^2, \quad c_- = \hat{y}^{-1} \tau_x^{-1} \tau_u,$$
Quantum Matrices

Definition

A $2 \times 2$ matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is right quantum if

\[
ac = qca \\
bd = qdb \\
ad = da + qcb - q^{-1}bc
\]
Quantum Matrices

Definition

A 2 × 2 matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is right quantum if

\[
ac = qca \\
bd = qdb \\
ad = da + qcb - q^{-1}bc
\]

An \( m \times m \) matrix is right-quantum if all 2 × 2 submatrices of it are right-quantum.
Quantum Determinants

Definition

If $A = (a_{ij})$ is right-quantum, then the quantum determinant is

$$\det_q(A) := \sum_{\pi \in \text{Sym}(m)} (-q)^{\text{inv}(\pi)} a_{\pi 1,1} a_{\pi 2,2} \cdots a_{\pi m,m}$$

where $\text{inv}(\pi)$ denotes the number of inversions.
Quantum Determinants

In general, $I - A$ is no longer right-quantum.

**Definition**

$$\widetilde{\det}_q(I - A) := 1 - C \quad \text{where} \quad C := \sum_{\emptyset \neq J \subset \{1, \ldots, m\}} (-1)^{|J| - 1} \det_q(A_J),$$

where $A_J$ is the $J$ by $J$ submatrix of $A$. 
In general, $I - A$ is no longer right-quantum.

**Definition**

$$\widetilde{\det}_q(I - A) := 1 - C \quad \text{where} \quad C := \sum_{\emptyset \neq J \subset \{1, \ldots, m\}} (-1)^{|J|-1} \det_q(A_J),$$

where $A_J$ is the $J$ by $J$ submatrix of $A$.

Also

$$\frac{1}{\widetilde{\det}_q(I - A)} = \frac{1}{1 - C} = \sum_{n=0}^{\infty} C^n$$
Define matrices which are right quantum

\[
S_+ := \begin{pmatrix} a_+ & b_+ \\ c_+ & 0 \end{pmatrix} \quad S_- := \begin{pmatrix} 0 & c_- \\ b_- & a_- \end{pmatrix}
\]
Operators

Define matrices which are right quantum

\[
S_+ := \begin{pmatrix} a_+ & b_+ \\ c_+ & 0 \end{pmatrix} \quad S_- := \begin{pmatrix} 0 & c_- \\ b_- & a_- \end{pmatrix}
\]

If \( P \) is a polynomial operator in the operators \( a_\pm, b_\pm, \) and \( c_\pm \) then we get a polynomial \( \mathcal{E}_N(P) \in \mathcal{R} \) by having \( P \) act on the constant polynomial 1 and replacing \( x \) and \( y \) with \( q^{N-1} \) and replacing \( u \) with 1.
The Matrix $\rho'(\gamma)$

Associate to each $\sigma_{ij}^{\varepsilon_j}$ the matrix which is the identity except for the $2 \times 2$ minor of rows $i_j, i_j + 1$ and columns $i_j, i_j + 1$ which is replaced by the matrix $S_{\varepsilon_j,j}$.

Here $S_{\pm,j}$ is the same as $S_{\pm}$ with $x, y, u$ replaced by $x_j, y_j, u_j$. 
The Matrix $\rho'(\gamma)$

Associate to each $\sigma_{ij}^{\epsilon_j}$ the matrix which is the identity except for the $2 \times 2$ minor of rows $i_j, i_j + 1$ and columns $i_j, i_j + 1$ which is replaced by the matrix $S_{\epsilon_j,j}$.

Here $S_{\pm,j}$ is the same as $S_{\pm}$ with $x, y, u$ replaced by $x_j, y_j, u_j$.

The matrix $\rho(\gamma)$ is the product of these matrices.
The Matrix $\rho'(\gamma)$

Associate to each $\sigma_{ij}^{\epsilon_j}$ the matrix which is the identity except for the $2 \times 2$ minor of rows $i_j, i_j + 1$ and columns $i_j, i_j + 1$ which is replaced by the matrix $S_{\epsilon_j,j}$.

Here $S_{\pm,j}$ is the same as $S_{\pm}$ with $x, y, u$ replaced by $x_j, y_j, u_j$.

The matrix $\rho(\gamma)$ is the product of these matrices.

The matrix $\rho'(\gamma)$ is $\rho(\gamma)$ with the first row and column removed.
The Trefoil

Take $K$ to be the right-handed trefoil. We can use $\beta = \sigma_1^3$ with 2 strands.
The Trefoil

Take $K$ to be the right-handed trefoil. We can use $\beta = \sigma_1^3$ with 2 strands. Thus $\rho(\gamma) = S_{+,1} S_{+,2} S_{+,3}$ and we get $\rho'(\gamma) = c_1 a_2 b_3$. 
The Trefoil

Take $K$ to be the right-handed trefoil.
We can use $\beta = \sigma_1^3$ with 2 strands.
Thus $\rho(\gamma) = S_{+,1} S_{+,2} S_{+,3}$ and we get $\rho'(\gamma) = c_1 a_2 b_3$. Thus

$$J'_K(N) = q^{N-1} \mathcal{E}_N \left( \frac{1}{1 - q c_1 a_2 b_3} \right) = q^{N-1} \sum_{n=0}^{\infty} \mathcal{E}_N(q^n c_1^n a_2^n b_3^n)$$
The Trefoil

Take $K$ to be the right-handed trefoil.
We can use $\beta = \sigma_1^3$ with 2 strands.
Thus $\rho(\gamma) = S_{+,1} S_{+,2} S_{+,3}$ and we get $\rho'(\gamma) = c_1 a_2 b_3$.

Thus

$$J'_K(N) = q^{N-1} E_N \left( \frac{1}{1 - q c_1 a_2 b_3} \right) = q^{N-1} \sum_{n=0}^{\infty} E_N(q^n c_1^n a_2^n b_3^n)$$

To apply $q^n c_1^n a_2^n b_3^n$ to the constant polynomial 1, recall

$$a_+ = (\hat{u} - \hat{y} \tau_x^{-1}) \tau_y^{-1}, \quad b_+ = \hat{u}^2, \quad c_+ = \hat{x} \tau_y^{-2} \tau_u^{-1},$$
The Trefoil

Take $K$ to be the right-handed trefoil. We can use $\beta = \sigma_1^3$ with 2 strands. Thus $\rho(\gamma) = S_{+,1} S_{+,2} S_{+,3}$ and we get $\rho'(\gamma) = c_1 a_2 b_3$. Thus

$$J'_K(N) = q^{N-1} \mathcal{E}_N \left( \frac{1}{1 - qc_1 a_2 b_3} \right) = q^{N-1} \sum_{n=0}^{\infty} \mathcal{E}_N(q^n c_1^n a_2^n b_3^n)$$

To apply $q^n c_1^n a_2^n b_3^n$ to the constant polynomial 1, recall

$$a_+ = (\hat{u} - \hat{y} \tau^{-1}_x) \tau^{-1}_y, \quad b_+ = \hat{u}^2, \quad c_+ = \hat{x} \tau^{-2} \tau^{-1}_u,$$

So we get

$$q^n x_1^n (u_2 - y_2)(u_2 - q^{-1} y_2) \ldots (u_2 - q^{-n+1} y_2) u_3^{2n}$$
The Trefoil

Take $K$ to be the right-handed trefoil. We can use $\beta = \sigma_1^3$ with 2 strands. Thus $\rho(\gamma) = S_{+,1}S_{+,2}S_{+,3}$ and we get $\rho'(\gamma) = c_1a_2b_3$. Thus

$$J'_K(N) = q^{N-1}\mathcal{E}_N\left(\frac{1}{1-qc_1a_2b_3}\right) = q^{N-1}\sum_{n=0}^{\infty}\mathcal{E}_N(q^n c_1^n a_2^n b_3^n)$$

To apply $q^n c_1^n a_2^n b_3^n$ to the constant polynomial 1, recall

$$a_+ = (\hat{u} - \hat{y}\tau_x^{-1})\tau_y^{-1}, \quad b_+ = \hat{u}^2, \quad c_+ = \hat{x}\tau_y^{-2}\tau_u^{-1},$$

So we get

$$q^n x_1^n (u_2 - y_2)(u_2 - q^{-1}y_2) \ldots (u_2 - q^{-n+1}y_2) u_3^{2n}$$

$$J'_K(N) = q^{N-1}\sum_{n=0}^{\infty} q^n (1-q^{N-1})(1-q^{N-2}) \ldots (1-q^{N-n}).$$
Walks

We will define a walk along the braid $\beta$ from $i$ to $j$ as follows: Begining at the $i$-th strand, follow the braid along a strand starting at the bottom, until you reach an over crossing. At an over crossing there is a choice to either continue along the strand or jump down to the strand below and continue following along the braid. Continue to the top of the braid ending at the $j$-th strand.
We will define a walk along the braid $\beta$ from $i$ to $j$ as follows: Begining at the $i$-th strand, follow the braid along a strand starting at the bottom, until you reach an over crossing. At an over crossing there is a choice to either continue along the strand or jump down to the strand below and continue following along the braid. Continue to the top of the braid ending at the $j$-th strand.

Each walk is given a weight defined as follows:
At crossing $j$:
- If the walk jumps down, assign $a_{\epsilon_{j},j}$
- If the walk follows the lower strand, assign $b_{\epsilon_{j},j}$
- If the walk follows the upper strand, assign $c_{\epsilon_{j},j}$

The weight of the walk is the $q$ times the product of the weights of the crossings.
For $J \subset \{1, \ldots, m\}$ and $\pi$ a permutation of $J$, a $\pi$-walk is a collection of walks from $j$ to $\pi(j)$ for $j \in J$. 
For $J \subset \{1, \ldots, m\}$ and $\pi$ a permutation of $J$, a $\pi$-walk is a collection of walks from $j$ to $\pi(j)$ for $j \in J$.

The weight assigned to a $\pi$-walk is the $(-1)^{|J|-1}(-q)^{\text{inv}(\pi)}$ times the product of the weights of the walks in the collection.

Cody Armond
Quantum Determinant for the Colored Jones Polynomial
Theorem

In Huynh and Le’s Theorem

\[ J'_K(N) = q^{(N-1)(\omega(\beta)-m+1)/2} \mathcal{E}_N \left( \frac{1}{\det_q(I - q^{\rho'}(\gamma))} \right) \]

\[ = q^{(N-1)(\omega(\beta)-m+1)/2} \sum_{n=0}^{\infty} \mathcal{E}_N(C^n) \]

the polynomial \( C \) is the sum of the sum of the weights of \( \pi \)-walks for \( J \subset \{2, \ldots, m\} \).
Part 1:

Claim: The matrix $q \rho(\gamma)$ has entries $a_{i,j} = \text{the sum of the weights of the walks from } j \text{ to } i \text{ along } \beta(\gamma)$.

Use induction on the length of $\gamma$: Obvious for $\gamma = \emptyset$ since $\rho(\gamma)$ is the identity matrix and $\beta(\gamma)$ is the identity in the braid group.
For the inductive step let’s look at a specific example:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a_{+,1} & b_{+,1} & 0 \\
0 & c_{+,1} & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4}
\end{pmatrix}
\]
For the inductive step let’s look at a specific example:

\[
\begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & a_{+,1} & b_{+,1} & 0 \\
    0 & c_{+,1} & 0 & 0 \\
    0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
    a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\
    a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\
    a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\
    a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \\
\end{pmatrix}
\]

The \( j \)-th column becomes:

\[
\begin{pmatrix}
    a_{1,j} \\
    a_{+,1} a_{2,j} + b_{+,1} a_{3,j} \\
    c_{+,1} a_{2,j} \\
    a_{4,j} \\
\end{pmatrix}
\]
Part 2: Apply $\tilde{\det}_q(I - q^{\rho'}(\gamma))$

- Removing 1st row and column corresponds to removing 1 from starting and ending positions.
Part 2: Apply $\tilde{\det}_q(I - q^{\rho'}(\gamma))$

- Removing 1st row and column corresponds to removing 1 from starting and ending positions.
- $(-1)^{|J|-1}\det_q(A_J)$ is the sum of the weights of the $\pi$-walks where $\pi$ is a permutation of $J$.
Sketch of Proof

Part 2: Apply $\tilde{\det}_q(I - q\rho'(\gamma))$

- Removing 1st row and column corresponds to removing 1 from starting and ending positions.
- $(-1)^{|J|-1}\det_q(A_J)$ is the sum of the weights of the $\pi$-walks where $\pi$ is a permutation of $J$
- $C = \sum_{\emptyset \neq J \subset \{2, \ldots, m\}} (-1)^{|J|-1}\det_q(A_J)$ is the sum of all $\pi$-walks along $\beta(\gamma)$

Cody Armond
Lemma

The weights of $\pi$-walks which traverse a point on the braid more than once appear in cancelling pairs.
Conjecture

If $K$ is an alternating knot, then

$$J'_K(N) = \pm \sum_{i=1}^{N} a_{N,i} q^{L_N+i-1} + \ldots \pm \sum_{i=1}^{N} b_{N,i} q^{U_{N-i}+1}$$

and $a_{N,i} = a_{M,i}$ for $i \leq N \leq M$ and $b_{N,i} = b_{M,i}$ for $i \leq N \leq M$. 
A Theorem on Positive Braids

Theorem

If $K$ is the closure of the braid $\beta$ where $\beta$ has all positive crossings, then

$$J'_K(N) = q^{L_N} + \sum_{i=L_N+N}^{U_N} a_i q^i$$

Moreover, $L_N = (N - 1)(\omega(\beta) + m - 1)/2$