Birman-Craggs-Johnson Homomorphism of the Torelli group

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**Mapping Class Group Basics**

**Definition**

The **Mapping Class Group** of a surface $S$ is the group of orientation preserving automorphisms of $S$ up to isotopy, denoted $\text{Mod}(S)$.

**Definition**

A **Dehn Twist** about a simple closed curve (scc), $c$, is denoted $T_c$.

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**Fact:** $\text{Mod}(S)$ is finitely generated by Dehn Twists about scc’s.
The Torelli Subgroup

Definition

The **Torelli subgroup**, \( \mathcal{I} \), is the subgroup of \( \text{Mod}(S) \) that acts trivially on \( H_1(S, \mathbb{Z}) \).

Figure: BP map \( T_c T_d^{-1} \)

Definition

A **bounding pair map** (BP) is opposite twists about two disjoint, non-separating, homologous scc’s in \( S \).
More on the Torelli Subgroup

Other elements in $\mathcal{I}$ are twists about separating curves.

![Separating curve](image)

**Figure:** Separating curve

**Fact:** $\mathcal{I}$ is generated by Dehn twists about separating curves and BP maps.

**Definition**

The **Johnson Kernel**, $\mathcal{K}$, is the subgroup generated by twists about separating curves.
Commutator Pairs

Definition

A **commutator pair** is \([T_c, T_d]\) where \(c\) and \(d\) are simple closed curves with \(\hat{i}(c, d) = 0\) and \(i(c, d) = 2\).

[Figure: Commutator Pair \([T_c, T_d]\)]

Commutator pairs are in \(\mathcal{I}\) because:

\[
[T_c, T_d] = (T_c)(T_d T_c^{-1} T_d^{-1}) = T_c T_{T_d(c)}^{-1}
\]

Curves \(c\) and \(T_d(c)\) are homologous

\[\Rightarrow T_c T_{T_d(c)}^{-1} \in \mathcal{I}\]
Questions about $\mathcal{I}$

- **Known:** $\mathcal{I}$ is finitely generated by BP maps when $g \geq 3$. The order of the set is exponential in the genus.

- **Question:** What are other generating sets?

- **Question:** Can we find smaller ones?

- **Question:** What is the subgroup generated by commutator pairs? Is it $\mathcal{I}$?

- **Question:** Is $\mathcal{I}$ finitely presentable?
The Rochlin Invariant

Definition

Let $W$ be a homology sphere and $X$ be a simply connected parallelizable 4-manifold so that $W = \partial X$.

1. Such an $X$ always exists
2. signature($X$) is divisible by 8
3. \( \frac{\text{signature}(X)}{8} \mod 2 \) is independent of $X$.
4. \( \Rightarrow \) it is an invariant of $W$, called the **Rochlin Invariant**.
(1978) The **Birman-Craggs homomorphisms** are a collection of homomorphisms

$$\rho_h : \mathcal{I} \rightarrow \mathbb{Z}_2$$

defined as follows:

1. Choose an embedding \( h : S \hookrightarrow S^3 \) and identify \( S \) with \( h(S) \).
2. For \( f \in \mathcal{I} \), split \( S^3 \) along \( S \) and reglue the two pieces using \( f \) creating \( W(h, f) \).
3. Since \( f \) acts trivially on \( H_1(S, \mathbb{Z}) \)
   \( \Rightarrow \) 3-manifold \( W(h, f) \) is a homology sphere.
   \( \Rightarrow \) Rochlin invariant \( \mu(h, f) \in \mathbb{Z}_2 \) is defined.
4. Fix \( h \) (the embedding) then

$$\rho_h(f) = \mu(h, f)$$
It turns out that $\rho$ is not very sensitive to the embedding $h$.

- Any surface $S \subset S^3$ has an associated bilinear Seifert Linking Form $L(\alpha, \beta)$ for $\alpha, \beta \in H_1(S, \mathbb{Z})$.

Then

1. Take the bilinear Seifert Linking form $L(\alpha, \beta)$
2. Use $\mathbb{Z}_2$ coefficients
3. Set $\alpha = \beta$

This gives a mod 2 self-linking form $\omega_h$ on $H_1(S, \mathbb{Z}_2)$

Given an embedding $h : S \hookrightarrow S^3$ and identifying $S$ with $h(S)$ we induce a self-linking form $\omega_h$ on $H_1(S, \mathbb{Z}_2)$
BC Homomorphism Facts

**Theorem**

\[ \rho_1 = \rho_2 \iff h_1 \text{ and } h_2 \text{ induce the same mod 2 self-linking form on } S. \]

- \[ \Rightarrow \rho_h \text{ is only dependent on the quadratic form } \omega \text{ induced by } h, \text{ so we could call it } \rho_\omega. \]
- These forms are completely classified and yield the following facts.

**Facts about BC-homomorphisms:**

1. The number of BC-homomorphisms \( \mathcal{I}_{g,1} \to \mathbb{Z}_2 \) is \( 2^{2g} \).
2. \( \{ \rho_\omega \} \) span a \( \mathbb{Z}_2 \) vector space of dimension

\[
\binom{2g}{3} + \binom{2g}{2} + \binom{2g}{1} + \binom{2g}{0}.
\]
Two ways to think of BC-homomorphisms

1. Fixing the embedding $h$ we get $\rho_h : \mathcal{I} \rightarrow \mathbb{Z}_2$ where

$$\rho_h(f) := \mu(h,f) = \mu(\omega,f)$$

In a sense BC-homomorphisms are $\mu(\omega,-) : \mathcal{I} \rightarrow \mathbb{Z}_2$

2. Fixing $f \in \mathcal{I}$, consider $\sigma_f : \Psi \rightarrow \mathbb{Z}_2$ where

$$\sigma_f(\omega) := \mu(\omega,f)$$

So we have $\mu(-,f) : \Psi \rightarrow \mathbb{Z}_2$
Combining BC-homomorphisms

In a sense $\sigma$ is the "dual" version of $\rho$.

$$\rho_\omega(f) = \mu(\omega, f) = \sigma_f(\omega)$$

Let $C = \bigcap \ker \rho_\omega$, then

$$\sigma_f = 0 \iff f \in C$$

Then we can define $\sigma$ as follows:

$$\sigma : \mathcal{I} \to \{ \text{vector space of functions } \Psi \to \mathbb{Z}_2 \}$$

Note: $\ker \sigma = C$. 
Boolean Polynomials

Construct from $H_1(S, \mathbb{Z}_2)$ a $\mathbb{Z}_2$-algebra $B$ such that:

1. $B$ is commutative with unity
2. Generated by $\bar{a}$ where for nonzero $a \in H_1(S, \mathbb{Z}_2)$, get function

   $$\bar{a} : \Omega \rightarrow \mathbb{Z}_2$$

   $$\bar{a}(\omega) = \omega(a)$$

3. $\bar{a}^2 = \bar{a}$ $\forall$ $a \neq 0$ in $H_1(S, \mathbb{Z}_2)$.
4. $(a + b) = \bar{a} + \bar{b} + a \cdot b$ where $a \cdot b \in \mathbb{Z}_2 \subset B$.

This shows the connection with quadratic forms:

$$\omega : H_1(S, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$$

$$\omega(a + b) = \omega(a) + \omega(b) + a \cdot b$$

Let $B_k$ be the vector space of all elements of degree at most $k$. 
Johnson (1980) combined all BC homomorphisms into one surjective homomorphism

\[ \sigma : \mathcal{I}_{g,1} \rightarrow B_3 \]

Note:

1. For \( h \in \text{Mod}(S_{g,1}) \) and \( f \in \mathcal{I} \)

\[ \sigma(hfh^{-1}) = h(\sigma(f)) \]

2. BC homomorphisms generate \( B_3^* = \text{Hom}(B_3, \mathbb{Z}_2) \)
Computation of BCJ for genus $k$ separating curve:

Choose a symplectic basis $\{a_1, \cdots, a_k, b_1, \cdots, b_k\}$ for the subsurface bounded by $c$.

$\sigma(T_c) = \sum_{i=1}^{k} \bar{a}_i \bar{b}_i$

Johnson showed this is independent of choice of symplectic basis.
Computation for BCJ of BP map $T_c T_d^{-1}$:

Choose a symplectic basis $\{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ for the subsurface bounded by $c$ and $d$.

\[ \sigma(T_c T_d^{-1}) = \sum_{i=1}^{k} \bar{a}_i \bar{b}_i (1 - \bar{c}) \]

Again this is independent of choice of basis.

**Figure:** $\sigma(T_c T_d^{-1}) = (\bar{a}_1 \bar{b}_1 + \bar{a}_2 \bar{b}_2) (1 - \bar{c})$
A **separating commutator pair** is a commutator pair where at least one curve is separating.

Separating commutator pairs are in $\mathcal{K}$ because $c$ and $T_d(c)$ are both separating curves.
Do separating commutator pairs generate $\mathcal{K}$?

- BCJ will help us.
- **Known:** $\mathcal{K}$ maps onto $B_2$.
- **Question:** What is the image of separating commutator pairs under BCJ?

**Theorem (Childers)**

The image of the subgroup generated by separating commutator pairs under $\sigma$ is $\langle 1, \overline{a_i}, \overline{b_i}, \overline{a_i}b_j, \overline{a_i}b_i + \overline{a_j}b_j | 1 \leq i, j \leq g, i \neq j \rangle$

**Sketch of proof:**

$$
\sigma(T_c T_d T_c^{-1} T_d^{-1}) = \sigma(T_c) + \sigma(T_d T_c^{-1} T_d^{-1})
$$

$$
= \sigma(T_c) + T_d \cdot \sigma(T_c^{-1})
$$

$$
= \left\{ \sigma(T_c) \right\} + \underbrace{T_d \cdot \sigma(T_c)}_{\text{has symplectic term } \overline{a_i}b_i \text{ and symplectic term } (T_d(a_i))(T_d(b_i))}
$$

So the symplectic terms come in pairs.
Special Commutator Pairs

- Consider a regular neighborhood of a commutator pair.
- It is a lantern (i.e., $S_{0,4}$).
- If one of the boundary components is separating, then it is a **special commutator pair**.

![Diagram of Special Commutator Pair $[T_c, T_d]$](image)

**Figure:** Special Commutator Pair $[T_c, T_d]

- Then $[T_c, T_d]$ can be rewritten as $[A, C]$ where $A, C \in \mathcal{I}$.
- So $\sigma([T_c, T_d]) = 0$

**Question:** What commutator pairs are in the kernel of BCJ?
References


Putman, Andrew, *An infinite presentation of the Torelli group*, to appear in GAFA