Euler characteristic class

- $p: E \longrightarrow B$ a map; for any $x \in B$, the set $p^{-1}(x)$ is the fibre over x. Let $U \subseteq B$ be a nbd. p is said to be trivial on U with fibre F, if $\exists$ $\varphi: p^{-1}(U) \longrightarrow U \times F$ s.t

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\text{proj}_1"] \\ & B & \end{tikzcd}$$

trivialisation homeo
(trivialisation of p over U)

(E, B, F) are top spaces

- A fibre bundle w F is a map $p: E \longrightarrow B$ s.t the set $\{U \underset{\text{open}}{\subseteq} B \mid p \text{ is locally trivial on } U \text{ with fibre } F\}$
- A section of fibre bundle $p: E \longrightarrow B$ is a map $s: B \longrightarrow E$ s.t $p \circ s = id_B$

$\boxed{\text{Thom spaces}}$

- One pt compactification is functorial

- Let $p: E \longrightarrow B$ be a n-dim vector bundle. The fibre bundle $\tilde{p}: \tilde{E} \longrightarrow B$ defined by taking fibrewise 1-pt compactification of E so that the fibre of $\tilde{E}$ above $b \in B$ is a copy of S^n. (produces new section s_∞, section at infinity, given by taking each $b \in B$ to ∞ in its fibre), Thom space of E, ($Th(E)$)

$$Th(E) \overset{\text{def}}{=} \tilde{E} / s_\infty(B)$$

eg. $E \cong B \times \mathbb{R}^n$ (trivial bundle)
$D(E)$ (disk bundle) $S(E)$ sphere bundle

$$Th(E) \cong D(E)/S(E) = B \times D^n / B \times S^{n-1}$$

$$= B \times S^n / B \times \{*\}$$

$$\cong \underbrace{(B_+) \wedge S^n}_{\text{suspension}} \cong \Sigma^n (B_+)$$

($\cdot_+ : Top \longrightarrow Top$ functor adds distinct basept.) just means B with a point

$$\tilde H^{n+i}(Th(E)) \cong H^i(B)$$

- A class $c \in \tilde H^n(Th(E); A)$ is c/a a Thom class with coefficients in A if $\forall\, b \in B$, $c|_{p^{-1}(b)}$ is gen. of $\tilde H_n(S^n, A)$. A bundle with a choice of Thom class with coeff in A is c/a A-oriented vector bundle.

Characteristic classes: defined axiomatically and shown to exist only so the axioms can be comfortably used

Motivation for Euler class: (closely related to Euler characteristic)

Let $p: E \to B$ an n-dim. A-oriented vector bundle with Thom class c ($\in \tilde H^n(Th(E))$).

$$Th(E) \cong D(E)/S(E)$$

induces a long exact seq. in cohom.

$$\cdots \to \tilde H^i(Th(E)) \xrightarrow{j^*} \tilde H^i(D(E)) \longrightarrow \tilde H^i(S(E)) \longrightarrow \tilde H^{i+1}(Th(E)) \to \cdots$$

Recall: $j: D(E) \hookrightarrow Th(E)$

zero section $s_0: B \to D(E)$ is a homotopy equivalence

$$\begin{tikzcd}
\tilde H^i(Th(E)) \arrow[r, "j^*"] & H^i(D(E)) \arrow[r] \arrow[d, "s_0^* = \text{iso}"] & \tilde H^i(S(E)) \\
H^{i-n}(B) \arrow[u, "\text{iso}", "\upsilon_c"'] \arrow[r, dashed, "\upsilon(s_0^* j^* c)"'] & H^i(B) \arrow[ur, "(p|_{S(E)})^*"'] &
\end{tikzcd}$$

cup product is natural

$$\begin{array}{ccc} H^*(D(E)) \otimes H^*(Th(E)) & \xrightarrow{\smile} & H^*(Th(E)) \\ \downarrow 1 \otimes j^* & & \downarrow j^* \\ H^*(D(E)) \otimes H^*(D(E)) & \xrightarrow{\smile} & H^*(D(E)) \end{array}$$

Definition: Let $p: E \longrightarrow B$ be an n-dim vector bundle with Thom class $c \in \tilde{H}^n(Th(E))$. Let $s_0: B \to Th(E)$ be the zero section. The A-Euler class $e(E)$ is a cohomology class

$$e(E) := s_0^* c \in H^n(B; A)$$

As a consequence the above long exact seq becomes

$$\cdots \longrightarrow H^{i-n}(B) \xrightarrow{\smile e(E)} H^i(B) \xrightarrow{(p|_{S(E)})^*} H^i(S(E)) \longrightarrow H^{i-n+1}(B) \to \cdots$$

K/a "Gysin seq" (exists for any sphere bundle)

Lemma 5.3: $p: E \to B$ n dim vector bundle with $e(E) \in \tilde{H}^n(B)$
$f: B' \to B$ any other map, then $f^*(e(E)) = e(f^*(E))$

previously, $p: E \to B$ oriented vector bundle, $c \in \tilde{H}^n(Th(E))$
Then $\exists$ induced map $Th(f): Th(f^*(E)) \longrightarrow Th(E)$
$c \longmapsto$ a thom class for f^*E.

"oriented bundles induce orientation on pull back bundle"

So we may conclude that $\because$ Thom classes are preserved under pullbacks so are Euler classes.

Proposition 5.4: $A = \mathbb{Z}$ or $\mathbb{Z}_2$ B: smooth n mfld TB is A-orientable
then $H^n(B) \cong A$.
For an A-oriented n-dim vector bundle E over B
$e(E) = n[1]$, $n = \#\{$pts in intersection of a generic section $s: B \to E$ and 0 section $s_0\}$
& intersects transversally
(Motivation is not immediate.

Corollary 5.5: If E has a nowhere non-zero section, then $e(E) = 0$ for any coefficients.
generic s
Proof: $s \cap s_0 = \phi$
$e(E) = 0$

In particular, if $\exists$ one bundle with $e(E) \neq 0$, then $\exists$ bundles with no everywhere-non zero section
"Hairy Ball theorem"

Corollary 5.6: $\forall n$ $\exists$ an n-bundle with a nonzero $\mathbb{Z}_2$ Euler class. Consequently for universal bundle γ_n over G_n, $\mathbb{Z}/2$-Euler class is non-zero.
Proof: Hint (Möbius bundle)
a particular example

Computations of Euler classes

Proposition: (commuting with products)
$p: E \to B$ and $p': E' \to B'$ two vector bundles.
Then, $\mathbb{Z}_2$ Euler class for bundle $p \times p': E \times E' \to B \times B'$ is
$$e(E \times E') = e(E) \otimes e(E')$$

First we analyze that it makes sense:
by Künneth's $H^*(B \times B'; \mathbb{Z}_2) \cong H^*(B; \mathbb{Z}_2) \otimes H^*(B'; \mathbb{Z}_2)$

or $e(E\times E') \in \underbrace{H^{\dim E + \dim E'}(B\times B')}_{\text{contains } e(E)\otimes e(E')}$

Proof: Recall that $s_0 : B \to D(E)$ is a homotopy equivalence. $Th(E) = D(E)/S(E)$

$\Rightarrow$ $\cup_c : H^*(B) \longrightarrow H^*(Th(E))$ and
$\cup_{c'} : H^*(B') \longrightarrow H^*(Th(E'))$ are isomorphisms.

They induce,

$$H^*(B\times B') \underset{\text{Künneth}}{\cong} H^*(B)\otimes H^*(B') \xrightarrow{\cup c\otimes c'} H^*(Th(E))\otimes H^*(Th(E')) \underset{\text{Künneth}}{\cong} H^*(Th(E\times E'))$$

$\Rightarrow$ $c\otimes c'$ is a Thom class for $B\times B'$

Recall from last time, $Th(E\times E') \cong Th(E)\wedge Th(E')$

also $e(E) = s_0^* c$ (pullback of thom class along 0-section)

$$\Rightarrow e(E\times E') = s_0^*(c\otimes c')$$
$$= s_0^*(c)\otimes s_0^*(c')$$
$$= e(E)\otimes e(E')$$

Corollary 5.9: $p : E\to B$ and $p' : E' \to B$ two vector bundles. Then, $e(E\oplus E') = e(E)e(E') \in H^*(B)$.

Proof: Recall: $\cup : H^*(B)\otimes H^*(B) \longrightarrow H^*(B\times B)$
$$a\otimes b \longmapsto a\cup b = p^*(a)\cup p'^*(b)$$

Consider diagonal map $\Delta : B \longrightarrow B\times B$
$$b \longmapsto (b,b)$$

$$a\cup b = \Delta^*(a\otimes b)$$

as earlier $e(E\times E') = e(E)\otimes e(E')$ for bundles $p\times p'$ over $B\times B$

Now $E \oplus E'$ is pullback of $E \times E'$ along the Δ and euler class is natural under pullback.

so $e(E \oplus E') = \Delta^* e(E \times E') = \Delta^*(e(E) \otimes e(E'))$
$= e(E)\, e(E')$

Theorem 5.10 (Poincaré Hopf) It is possible to comb a closed Riemannian $\mathbb{Z}$-oriented mfld iff $\chi(M) = 0$.

Proof beyond scope rn

Cohomology of Grassmanians

Recall $V_n(\mathbb{R}^k) = \{(v_1, \dots v_n) \mid v_i \in \mathbb{R}^k, v_j \perp v_j, |v_i| = 1\}$
$\subseteq (S^{k-1})^n$

$(v_1 \dots v_n) \sim (w_1 \dots w_n)$ if they generate the same space

$$G_n(\mathbb{R}^k) = V_n(\mathbb{R}^k)/\sim$$

- space of n-planes in $\mathbb{R}^k$
- $G_n = \bigcup_{k=1}^{\infty} G_n(\mathbb{R}^{k+1})$
- $G_n(\mathbb{R}^k)$ Hausdorff and dim $n(k-n)$

- We work with $\mathbb{Z}_2$ coefficients.

Lemma 5.11 (str. of G_n and G_{n-1})
Let $S(\gamma_n)$ be the unit sphere bundle in γ_n
Then $S(\gamma_n) = \{(v, w) \in \mathbb{R}^\infty \times G_n \mid v \in w, \|v\| = 1\}$
and the map $p' : S(\gamma_n) \longrightarrow G_{n-1}$ is a h.e.
$$(v, w) \longmapsto v^\perp \cap w$$

Recall (γ_n)
Universal bundle γ_{nk} (rank n) is a bundle $p : \gamma_{nk} \longrightarrow G_n(\mathbb{R}^k)$
$\gamma_{nk} = \{(w, v) \mid w \in G_n(\mathbb{R}^k), v \in w \subseteq \mathbb{R}^k\} \subseteq G_n(\mathbb{R}^k) \times \mathbb{R}^k$
$p : \gamma_{nk} \longrightarrow G_n(\mathbb{R}^k)$ projection onto the first coordinate
when $k = \infty$ $\gamma_{nk} := \gamma_n$.

Proof idea:

$$p' : S(\gamma_n) \longrightarrow G_{n-1}$$
$$(v, w) \longmapsto v^{\perp} \cap w$$

a $w \in G_{n-1}$ $\dim(w) = n-1$, fibre of p' contains v s.t. $v \perp w$, $\|v\| = 1$ $\operatorname{span}(v, w) \in G_n$

exactly the choice of $v \perp w$ in $\mathbb{R}^\infty$ $\Rightarrow$ this is S^∞

$\therefore$ $p' : S(\gamma_n) \longrightarrow G_{n-1}$ is a fibre bundle with fibre S^∞

S^∞ contractible

$$\begin{tikzcd} S(\gamma_n) \arrow[rr, "\varphi"] \arrow[dr, "p'"'] & & S(\gamma_n) \times S^\infty \arrow[dl, "\mathrm{proj}_1"] \\ & G_{n-1} & \end{tikzcd}$$

infact we get

$$S^\infty \hookrightarrow S(\gamma_n) \xrightarrow{p'} G_{n-1}$$

long exact seq. in homotopy

$$\cdots \to \underset{\substack{\parallel \\ 0}}{\pi_k(S^\infty)} \longrightarrow \pi_k(S(\gamma_n)) \xrightarrow{\boxed{\cong}} \pi_k(G_{n-1}) \longrightarrow \underset{\substack{\parallel \\ 0}}{\pi_{k-1}(S^\infty)} \cdots$$

$\Rightarrow$ p' is a h.e.

$\blacksquare$

(The text says "p' is a h.e. follows from the fact that any fibre bundle with connected base and CW str. on base and fibre has the type of a CW complex).

Def^n: Let $\eta = \tau^*$; $H^*(G_n) \longrightarrow H^*(G_n)$ ring hom. induced by $\underbrace{\tau : G_{n-1} \longrightarrow G_n}$ (Example 3.17)

$$G_n(\mathbb{R}^k) \longrightarrow G_{n+1}(\mathbb{R}^{k+1})$$

by adding to each n-plane the same new coordinate.

(computing cohomology)

Idea is to use induction with inducting step coming from η.

lemma 5.11 induces a construction that simplifies the computation.

($E \cong B \times \mathbb{R}$ ϵ^1 : trivial line bundle (vector bundle of rank 1))

Lemma 5.13: Pick a map $G_{n-1} \xrightarrow{f} S(\gamma_n)$ homotopy inverse to p'. Then, composition $G_{n-1} \xrightarrow{f} S(\gamma_n) \xrightarrow{p'} G_n$ classifies the bundle $\gamma_n \oplus \epsilon^1$. It is hence homotopic to τ and induces homomorphism η on cohomology.

Proof: First part

$$E = \{(\eta, v, w) \in G_{n+1} \times \mathbb{R}^\infty \times \mathbb{R}^\infty \mid v, w \in \eta\}$$

$\pi : E \longrightarrow S(\gamma_{n+1})$ forgetting the η-coordinate

$\downarrow$ $(\eta, v, w) \longmapsto (v, w)$

is a $(n+1)$ bundle on $S(\gamma_{n+1})$, all vectors w in the same $(n+1)$ plane η as v (unit vector)

We wish to show that E is isomorphic to pullback of $\gamma_n \oplus \epsilon^1$ along $p' : S(\gamma_{n+1}) \to G_n$ and pullback of γ_{n+1} along $p : S(\gamma_{n+1}) \to G_{n+1}$.

$$\begin{tikzcd}
S(\gamma_{n+1}) \arrow[d, "p'"'] \arrow[dr, "p"] & \\
G_n \arrow[r, "\tau"'] & G_{n+1}
\end{tikzcd}$$

diagram commutes upto homotopy

- just by def^n (construc^n) of E, it is pullback of $\gamma_{n+1} \to G_{n+1}$ along $p : S(\gamma_{n+1}) \to G_{n+1}$.

- write $E = E^\perp \oplus E''$
  - $E^\perp$: subbundle where $w \parallel v$
  - E'' : " " $w \perp v$.

from lemma 5.11 $p' : S(\gamma_n) \to G_n$
$(v, w) \mapsto v^\perp \cap w$

$\Rightarrow$ $E^\perp$ is n-dimensional and corresponds

to pullback of γ_n along $p' : S(\gamma_n) \longrightarrow G_n$.
and E'' is just the trivial line bundle

concluding, $E \cong E'' \oplus E^{\perp} \cong \varepsilon^1 \oplus p'^* \gamma_n$

By direct application of classification theorem of vector bundles, two maps that pull back γ_{n+1} and $\gamma_n + \varepsilon^1$ to isomorphic bundles must be homotopic

- $S(\gamma_{n+1}) \xrightarrow{p} G_{n+1}$
- $S(\gamma_{n+1}) \longrightarrow G_n \xrightarrow{\tau} G_{n+1}$

$\Rightarrow$ $G_n \xleftarrow{p} S(\gamma_{n+1}) \xleftarrow{f} G_{n-1}$ and $G_{n-1} \xrightarrow{\tau} G_n$
$\simeq$

Theorem 5.14: $H^*(G_n; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \dots w_n]$ with $|w_i| = i$.

Idea: induction on Gysin seq. for universal bundle $\gamma_n \longrightarrow G_n$

Theorem 5.16: Let GU_n be Grassmannian of a n-dim subspace of $\mathbb{C}^\infty$. Then,

$$H^*(GU_n) \cong \mathbb{Z}[c_1, \dots, c_n] \text{ with } |c_i| = 2i$$

Corollory: $\eta : H^*(G_n) \longrightarrow H^*(G_{n-1})$
$w_n \longmapsto 0$
mapps other generators accordingly.