Bott Periodicity

Mo. cohomology theory, singular cohomology,

$\subseteq$

K-theory

We want to KO, KU spectrum for K-theory,

· $GU_n = BU(n)$

understand this

means we need to understand $U(n)$.

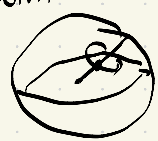
$U(n) \curvearrowright S^{2n-1} \subseteq \mathbb{C}^n$.

stabilizer of a point,

is $U(n-1)$ - rotations of sphere

w.r.t axis through the point

and the origin.



$G_n = BO(n)$ $\quad V_n(\mathbb{C}^{n}) \to V_n($[illegible]$)$

$V_n(\mathbb{R}^k) \longrightarrow V_{n+1}(\mathbb{R}^{k+1})$

$EO(n)$ ← contractible.

$\downarrow$

$BO(n) = EO(n)/O(n)$.

$O(n) \to V_n$ $\quad$ iso on homotopy groups

$\downarrow$ $\quad V_n \to *$.

$G_n = V(n)/O(n)$

$\uparrow \lim$

$S^{2n-1} = \dfrac{U(n)}{U(n-1)}$ $\quad | \quad$ $S^{n-1} = \dfrac{O(n)}{O(n-1)}$

$U(n-1) \hookrightarrow U(n) \longrightarrow S^{2n-1}$ (fiber sequence)

[illegible]

$\cdots \to \pi_{i+1}(S^{2n-1}) \longrightarrow \pi_i U(n-1) \longrightarrow \pi_i U(n) \longrightarrow \pi_i(S^{2n-1}) \longrightarrow \cdots$

$\pi_i\, U(n-1) \cong \pi_i\, U(n)$ for $i < 2n-2$.

$U = \operatorname{colim}_n U(n)$

$O = \operatorname{colim}_n O(n)$

Homotopy groups of unitary groups are stable.

Bott periodicity:

(a) $\pi_i U = \pi_{i+2} U$

(b) $\pi_i O = \pi_{i+8} O$.

original proof are geometric and complex

this more may deep

The idea of the proof is to construct $\Phi : BU \to \Omega SU$.

this a weak equivalence.

$SU = \operatorname{colim} SU_n$.

This is where the book shines.

$1 \to S(n) \longrightarrow U(n) \xrightarrow{\det} U(1) \longrightarrow 1$ exact sequence

$|\det| = 1$

$\det = 1$ $\quad SU(n) \trianglelefteq U(n) \quad U(n) \cong SU(n) \rtimes U(1)$.

apply colim

$1 \longrightarrow SU \longrightarrow U \longrightarrow S^1 \longrightarrow 1$

its left exact because its filtered colimit.

Modulo the proof existence of the Bott map, the proof is done.

Proof of Bott Periodicity:

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_h"] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

$(X, *) \quad (S^1, \cdot) \longrightarrow (X, *)$.

$c(S^1) = *$

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & EU \arrow[d] \\ * \arrow[r] & BU \end{tikzcd} \quad \text{upto homotopy} \rightsquigarrow \quad \begin{tikzcd} \Omega BU \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & BU \end{tikzcd}$$

$U \simeq \Omega BU \xrightarrow{\Omega\Phi} \Omega^2 SU \cong \Omega_0^2 U$.

want to show $\Omega^2 SU \simeq \Omega^2 U$.

$$\begin{tikzcd}
\Omega SU \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & \Omega U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & * \arrow[d] & \\
* \arrow[r] & \Omega S^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & SU \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & * \arrow[d] \\
& * \arrow[r] & U \arrow[r] & S^1
\end{tikzcd}$$

$\Omega SU \longrightarrow \Omega U \longrightarrow \Omega S^1$ h fiber sequence

$\to \pi_i(\Omega^2 SU) \longrightarrow \pi_i(\Omega^2 U) \longrightarrow \pi_i(\Omega^2 S^1) \longrightarrow \pi_{i-1}(\Omega^2 SU)$.

$\to \pi_{i+2}(SU) \longrightarrow \pi_{i+2}(U) \longrightarrow \pi_{i+2}(S^1) \xrightarrow{0} \pi_i(\Omega\ )$

$\underline{i+2 > 1}$

$i > -1, i \geq 0$.

$\pi_i(\Omega SU) \cong \pi_i(\Omega U)$. $\pi_0(\Omega SU) = \pi_1$

Using this compute homotopy groups of U

$\pi_i(U) \cong \pi_i(\Omega^2 U)$.

$\Rightarrow \pi_i(U) \cong \pi_{i+2}(U)$

π_0, π_1 is sufficient to know.

$\pi_i(U(n-1)) \cong \pi_i U(n)$ $i < 2n-2$.

$\pi_i(U(1)) \cong \pi_i U(2)$ $\underline{i < 2}$.

$U(2) \cong S^1 \times SU(2) \cong S^1 \times S^3$, $\pi_0 = 0$, $\pi_1 = \mathbb{Z}$, $\pi_{even} = 0$, $\pi_{odd} = \mathbb{Z}$.

Constructing the Bott Map

$k<k'$, $\quad \mathbb{C}^k \xhookrightarrow{i_{kk'}} \mathbb{C}^{k'}$

fix k, n and $\theta \in [0, 2\pi]$

$$\mathbb{C}^k \times \mathbb{C}^n \xrightarrow{\alpha^\theta_{k,n}} \mathbb{C}^k \times \mathbb{C}^n$$

$$(z_1, z_2) \longmapsto (z_1 e^{i\theta}, z_2 e^{-i\theta})$$

$\alpha^\theta_{k,n} \in U(k+n)$, $\quad \alpha_{k,n} : S^1 \longrightarrow U(k+n)$.

$$\alpha_{k,n} : S^1 \longrightarrow U(k+n) \Rightarrow \alpha_{k,n} \in \Omega U(k+n)$$

$$\tilde{\Phi}_{k,n} : U(k+n) \longrightarrow \Omega SU(k+n)$$
$$T \longmapsto \left(\theta \longmapsto T \circ \alpha^\theta_{k,n} \circ T^{-1} \circ (\alpha^\theta_{k,n})^{-1}\right)$$

Suppose $T = T_k \times T_n \in U(k) \times U(n)$, Then

$$\tilde{\Phi}_{k,n}(T_k \times T_n) = \begin{pmatrix} T_k e^{i\theta} T_k^{-1} e^{-i\theta} & 0 \\ 0 & T_n e^{-i\theta} T_n^{-1} e^{i\theta} \end{pmatrix}$$
$$= I$$

$\tilde{\Phi}$ takes any $T \in U(k) \times U(n)$ to the trivial loop.

$$\phi_{k,n} : \frac{U(k+n)}{U(k) \times U(n)} \longrightarrow \Omega SU(k+n)$$

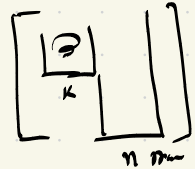
$$i_{kk'} \times i_{nn'} : \mathbb{C}^k \times \mathbb{C}^n \hookrightarrow \mathbb{C}^{k'} \times \mathbb{C}^{n'}$$
$$\rightsquigarrow$$
$$U(k+n) \longrightarrow U(k'+n')$$
$$\rightsquigarrow$$

$$\begin{tikzcd}
\frac{U(k+n)}{U(k) \times U(n)} \arrow[r] \arrow[d, "\phi_{k,n}"'] & \frac{U(k'+n')}{U(k') \times U(n')} \arrow[d] \\
\Omega SU(k+n) \arrow[r] & \Omega SU(k'+n')
\end{tikzcd}$$

note this: naturality

Lemma $\operatorname{Colim}_{k} \dfrac{U(k+n)}{U(k)\times U(n)} \cong Gr_n(\mathbb{C}^\infty)$

Pf: $V_n(\mathbb{C}^{k+n}) \cong U(k+n)/U(k)$



$$\operatorname{Colim}_{k\to\infty} \frac{U(k+n)}{U(k)\times U(n)} \cong \operatorname{Colim}_{k\to\infty} \frac{V_n(\mathbb{C}^{k+n})}{U(n)}$$

$$\cong \operatorname{Colim}_{k} Gr_n(\mathbb{C}^{k+n})$$

$$\cong Gr_n(\mathbb{C}^\infty)$$

$$\frac{U(k+n)}{U(k)\times U(n)} \xrightarrow{\phi_{k,n}} \Omega SU(k+n)$$

$$\begin{tikzcd}
\operatorname{Colim}_{k,n} \dfrac{U(k+n)}{U(k)\times U(n)} \arrow[r, "\operatorname{Colim}\phi_{k,n}"] \arrow[d, phantom, "\cong"] & \operatorname{Colim}_{k,n} \Omega SU(k+n) \arrow[d, phantom, "\cong"] \\
\operatorname{Colim}_{n} Gr_n(\mathbb{C}^\infty) \arrow[d, phantom, "="] & \Omega SU \\
\operatorname{Colim}_{n} BU(n) = BU \arrow[ur, "\phi"] &
\end{tikzcd}$$

BOH map.

H-space:

generalization of topological groups.

(X, μ, e) : $\mu : X \times X \longrightarrow X$

$\mu(-, e) \simeq id_X$

$\mu(e, -) \simeq id_X$.

· Homotopy associative,

$$\mu(-, \mu(-,-)) \simeq \mu(\mu(-,-), -)$$

$f : X \longrightarrow Y$ (Hmap of H space) if

$$\begin{tikzcd} X \times X \arrow[r, "\mu"] \arrow[d, "(f,f)"'] & X \arrow[d, "f"] \\ Y \times Y \arrow[r, "\mu"] & Y \end{tikzcd}$$

commutes upto homotopy.

Example.

· Topological group.

· ΩX, μ concatination, e = constant loop.

• G be abelian discrete group.

BG, is homotopy associative H-space.

$B : Gp \longrightarrow Top$ functor that respects products,

$$G \times G \xrightarrow{\mu} G$$

$$BG \times BG \cong B(G \times G) \xrightarrow{B\mu} BG.$$

$e \in G$, $\{e\} \xrightarrow{i} G$,

$Be \xrightarrow{i} BG$.

$a \in BG$.

$$B\mu(a \simeq i(Be)) \cong (a', e) \xrightarrow[\simeq]{\mu} a' = \underline{Ba}.$$

- BU is homotopy associative,

$$\mathbb{C}^\infty \oplus \mathbb{C}^\infty \longrightarrow \mathbb{C}^\infty$$
$$(a_1, a_2 \dots) \quad (b_1, b_2 \dots) \qquad (a_1, b_1, a_2, b_2 \dots)$$

$$U \times U \longrightarrow U$$ ← block diagonal map.

$$BU \times BU \longrightarrow BU$$ ←

- S^0, S^1, S^3, S^7 are only spheres which are H-spaces.

- X be H-space,

$$C_*(X) \otimes C_*(X) \cong C_*(X \times X)$$
Quasi-iso

we can define,

$$\times \; H_i(X) \oplus H_j(X) \longrightarrow H_{i+j}(X \times X)$$
$$[a] \oplus [b] \longmapsto [a \otimes b]$$

$$H_*(X) \otimes H_*(X) \xrightarrow{\times} H_*(X \times X) \xrightarrow{\mu_*} H_*(X)$$

when H space-structure on X is homotopy associative, then this operation on $H_*(X)$ is associative

$(H_*(X), \mu_*)$ ← called Pontrjagin Ring of X.

BU, SU are H-spaces.

<u>Lemma</u>:

ϕ is an H map of H-spaces.

$$\begin{array}{ccc} U(k+n)\times U(k'+n') & \xrightarrow{\Phi_{k,n}\times\Phi_{k',n'}} & \Omega SU(k+n)\times\Omega SU(k'+n') \\ \text{dim}\downarrow & & \downarrow \Omega \text{diag.} \\ U(k+k'+n+n') & \longrightarrow & \Omega SU(k+k'+n+n') \end{array}$$

$$\begin{array}{ccc} \frac{U(k+n)}{U(k)\times U(n)}\times\frac{U(k'+n')}{U(k')\times U(n')} & \xrightarrow{\Phi_{k,n}\times\Phi_{k',n'}} & \Omega SU(k+n)\times\Omega SU(k'+n') \\ \text{dim}\downarrow & \text{take } \underline{\text{colim}} & \downarrow \Omega \text{diag.} \\ \frac{U(k+k'+n+n')}{U(k+k')\times U(n+n')} & \longrightarrow & \Omega SU(k+k'+n+n') \end{array}$$

<u>Whitehead theorem (H space version)</u>

$f: X\to Y$ map of connected spaces.

f is an H-map of H-spaces.

$f_*: H_i(X,\mathbb{Z})\longrightarrow H_i(Y,\mathbb{Z})$ iso $\forall i$,

then f is a weak equivalence.

→ we will show

ϕ_* on homology level iso.

Real Bott periodicity.

$O = \operatorname{colim} O(n)$, $\pi_k(O) = \pi_{k+8}(O)$.