Define unpointed space and unreduced cohomology.

# Unpointed spaces & unreduced cohomology:

- Functor $\cdot_+ : \mathrm{Top} \longrightarrow \mathrm{Top}_*$ , adds a disjoint basepoint to given space
- For unpointed space $Y$ (pointed spaces are spaces with distinguished basepoint $* \in X$)

$$H^n(Y;A) := \tilde{H}^n((Y)_+;A) = [Y_+, K(A,n)]$$

is called unreduced cohomology of $Y$.

↳ eg. One example of this functor is

$\hat{X}$ = One point compactification of space $X$

& $H^*_c(X) := \tilde{H}^*(\hat{X})$ this is called compactly supported cohomology of $X$.

Introduce two Kunneth formulas

- **Theorem:** (Kunneth Theorem, [Hat02, Theorem 3.18]). *Let $R$ be a commutative ring. There is a group homomorphism, natural in both $X$ and $Y$,*

$$H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \wedge Y;R).$$

- **Theorem:**

Consider a quotient $X/A$ of unpointed spaces. This is naturally pointed by the point $[A]$. The diagonal map $\Delta : X_+ \longrightarrow X_+ \wedge X_x$ induces a map $\Delta : X/A \longrightarrow X_+ \wedge X/A$ ...

induces a map $\Delta : X/A \longrightarrow X_+ \wedge X/A$ s.t.

$$X \setminus A \ni x \longmapsto (x,x)$$

$$A \ni a \longmapsto (a,a)$$ which is the basepoint of $X_+ \wedge X/A$

This map composed with homomorphism from Kunneth theorem defines a product

Thus we can define a cup product

$$\cup : H^*(X;R) \otimes \tilde{H}^*(X/A) \longrightarrow \tilde{H}^*(X_+ \wedge X/A) \xrightarrow{\Delta^*} \tilde{H}^*(X/A)$$

- Define the Thom space and the Thom bundle
- Do example 4.20.

# • Thom spaces & Thom isomorphism:

•<u>Def$^n$</u>: Let $p: E \longrightarrow B$ be an $n$-dim. vector bundle. The fiber bundle $\tilde{p}: \tilde{E} \longrightarrow B$ is defined by taking the fiberwise 1-point compactification of $E$, so that the fiber of $\tilde{E}$ above $b \in B$ is a copy of $S^n$.

This produces a new section $s_\infty$, the section at infinity given by $b \in B \longmapsto$ point at infinity in its fiber.

The <u>Thom space</u> of $E$ is defined by

$$Th(E) := \tilde{E} / s_\infty(B)$$

or

Let $p: E \longrightarrow B$ be a vector bundle. Define a metric $\mu: E \to \mathbb{R}$ s.t. restriction of $\mu$ to each fiber is a <u>positive definite quadratic form</u>. This has two imp. fiber bundles sitting inside it:

① Disk bundle: Let $D(E) = \{e \in E \mid \mu(e) \leq 1\}$

then $p^{-1}(b) \cong D^n \quad \forall b \in B$

② Sphere bundle: Let $S(E) = \{e \in E \mid \mu(e) = 1\}$

then $p^{-1}(b) \cong S^{n-1} \quad \forall b \in B$

Then

$$Th(E) \cong D(E)/S(E)$$

pointed space

$\hookrightarrow$ eg. $E \cong B \times \mathbb{R}^n$ since its a trivial bundle

$\Rightarrow D(E) \cong B \times D^n$ & $S(E) \cong B \times S^{n-1}$

$$\frac{B \times S^n \sqcup \{*\} \times S^n}{B \times \{*\} \cup \{*\} \times S^n}$$

$\Rightarrow Th(E) = B \times D^n / B \times S^{n-1} \cong B \times S^n / B \times \{*\} \cup \{*\} \times S^n$

$\cong (B_+) \wedge S^n$

$*$ is disjoint from B

$\cong \Sigma^n (B_+)$

$\Rightarrow \tilde{H}^{n+i}(Th(E)) \cong \tilde{H}^{n+i}(\Sigma^n(B_+))$

$\cong \tilde{H}^i(B_+) \quad (\because \tilde{H}^i(X) = \tilde{H}^{i+1}(\Sigma X))$

$\cong H^i(B)$ (from defn of homology of unpointed space)

suspension destroys cup products $\Rightarrow$ this is not a ring

By kunneth thm,

$\tilde{H}^*(S^n) \otimes \tilde{H}^*(B_+) \cong \tilde{H}^*(\Sigma^n B_+)$

$\tilde{H}^*(S^n) \otimes H^*(B) \cong H^*(\Sigma^n B)$

$\Rightarrow H^i(Th(E)) \cong H^{i-n}(B)$ this iso. is given by multiplying by generator in $\tilde{H}^n(S^n)$

4 Prove lemma 4.22 and corollary 4.23

Thom spaces work well with products of vector bundles.

- **lemma:** For vector bundles $E \longrightarrow B$ & $E' \longrightarrow B'$

$$Th(E \times E') \cong Th(E) \wedge Th(E')$$

Pf: Observe that

$$D(E\times E') \cong D(E)\times D(E') \quad \text{product}$$

Restricting on $S(E\times E')$, we get

$$S(E\times E') \cong (S(E)\times D(E')) \cup (D(E)\times S(E'))$$

$$\Rightarrow Th(E\times E') = D(E\times E')/S(E\times E')$$
$$\cong D(E)\times D(E') \big/ (S(E)\times D(E') \cup (D(E)\times S(E')))$$
$$\cong \underbrace{(D(E)/S(E))}_{\text{pointed space with point } [S(E)]} \wedge (D(E')/S(E')) = Th(E)\wedge Th(E')$$

- Corollary: $Th(E\oplus \varepsilon^k) \cong Th(E)\wedge S^k$

Pf: $E\oplus\varepsilon^k \cong E\times \varepsilon^k$

we think of $\varepsilon^k$ as a bundle over a point

$$\Rightarrow Th(\varepsilon^k) \cong D(\varepsilon^k)/S(\varepsilon^k) \cong D^k/S^{k-1} \cong S^k$$

Introduce Thom Class defn 4.24.

$$B \xrightarrow{S_0} E \hookrightarrow \hat{E} \cong Th(E)$$

$S_0$: zero section; $\hat{E}$: compactificat^n of each fiber

$\exists$ no project^n map $Th(E)\to B$, but $B$ sits nicely inside $Th(E)$.

i.e. $\forall b\in B$ $\quad F_b \cong S^n \hookrightarrow Th(E)$

This induces

$$\tilde{H}^n(Th(E);A) \longrightarrow \tilde{H}^n(S^n)$$

called restriction of $u$ to the fiber above $b$.

**Definition 4.24.** A class $c \in \widetilde{H}^n(\mathrm{Th}(E); A)$ is called a *Thom class with coefficients in A* if for every $b \in B$ the restriction of $c$ to the fiber above $b$ is a generator for $\widetilde{H}^n(S^n; A)$.

The usual cases of interest are when $A = \mathbb{Z}/2$ and when $A = \mathbb{Z}$. When $A = \mathbb{Z}$ this is often simply called a *Thom class.*

A bundle with a choice of Thom class with coefficients in $A$ is called an *A-oriented vector bundle.*

↳ doesn't always exist. eg. cohomology of Mobius bundle with coe. $\mathbb{Z}$.

Thom classes are natural w.r.t. pullbacks.

**Proposition 4.25.** *Let $p\colon E \longrightarrow B$ be an oriented vector bundle, let $c \in \widetilde{H}^n(\mathrm{Th}(E))$ be the chosen Thom class, and let $f\colon B' \longrightarrow B$ be any map. There is an induced map $\mathrm{Th}(f)\colon \mathrm{Th}(f^*(E)) \longrightarrow \mathrm{Th}(E)$ which takes $c$ to a Thom class for $f^*E$.*

pf :

$$\begin{tikzcd}
& \mathrm{Th}(f^*E) \arrow[r, "f'^*"] & \mathrm{Th}(E) & \\
& \widetilde{f^*E} \arrow[u] \arrow[r, "\tilde{f}'"] & \tilde{E} \arrow[u] & \\
p^{-1}(b') & f^*(E) \arrow[u] \arrow[r, "f'"] \arrow[d, "p'"] & E \arrow[u] \arrow[d, "p"] & p^{-1}(f(b')) \\
& b' \in B' \arrow[r, "f"] & B \quad f(b') &
\end{tikzcd}$$

$B' \times_B E$

same as fibers

$d: B \xrightarrow{S_0} E \hookrightarrow \tilde{E} \longrightarrow \tilde{E}/S_\infty(B) = \mathrm{Th}(E)$

...

**Theorem 4.26** (Thom Isomorphism Theorem). *Let $c$ be a Thom class with coefficients in $A$ for the $n$-dimensional bundle $p\colon E \longrightarrow B$. The homomorphism*

$$\Phi\colon H^i(B;\mathbb{Z}/2) \longrightarrow \widetilde{H}^{i+n}(\mathrm{Th}(E);\mathbb{Z}/2) \qquad b \longmapsto p^*(b) \smile c$$

*is an isomorphism for all $i$.*

- " $\cup c$ " :

$\Delta: E_+ \longrightarrow E_+ \wedge E_+$ induces $\mathrm{Th}(E) \longrightarrow E_+ \wedge \mathrm{Th}(E)$

by kunneth theorem,

$$H^*(E) \otimes \widetilde{H}^*(\mathrm{Th}(E)) \xrightarrow{\cup} \widetilde{H}^*(\mathrm{Th}(E))$$

( $p^* : H^*(B) \longrightarrow H^*(E)$ is an iso ?)

precomposing with $p^* \otimes 1$

$H^*(B) \otimes \widetilde{H}^*(\mathrm{Th}(E)) \longrightarrow \mathrm{Th}^*(\mathrm{Th}(E))$

$\phi$ = restriction of this map where 2nd coordinate is set to be $c$.

- If $E$ is trivial, we saw that Thom class exist
- suppose $B = U \cup V$ ( $U, V$ are open)

& Thom iso. holds for $E|_U$, $E|_V$ & $E|_{U\cap V}$

Let Thom classes associated to above. : $c_U, c_V, c_{U\cap V}$

since Thom iso. holds for $E|_{U\cap V}$, for $i=-1$;

$$\widetilde{H}^{-1}(U\cap V) \longrightarrow \widetilde{H}^{n-1}(\mathrm{Th}(E|_{U\cap V}))$$
$$\begin{array}{ccc} \| & & \| \\ 0 & & 0 \end{array}$$

We therefore have a Mayer–Vietoris sequence, which begins:

$$0 \longrightarrow \widetilde{H}^n(\mathrm{Th}(E)) \hookrightarrow \widetilde{H}^n(\mathrm{Th}(E|_U)) \oplus \widetilde{H}^n(\mathrm{Th}(E|_V)) \xrightarrow{u-v} \widetilde{H}^n(\mathrm{Th}(E|_{U\cap V})) \longrightarrow \cdots .$$

The inclusion $\mathrm{Th}(E|_{U\cap V}) \longrightarrow \mathrm{Th}(E|_U)$ induces a homomorphism

$$\widetilde{H}^n(\mathrm{Th}(E|_U)) \longrightarrow \widetilde{H}^n(\mathrm{Th}(E|_{U\cap V}));$$

$$\begin{tikzcd} \mathrm{Th}(E|_{U\cap V}) \arrow[r] \arrow[d, "i"'] & \mathrm{Th}(E|_V) \arrow[d] \\ \mathrm{Th}(E|_U) \arrow[r] & \mathrm{Th}(B) \end{tikzcd}$$

$c_V$ $c_{U\cap V}$ $c_U$

since the theorem holds for $E|_{U\cap V}$ and Thom classes are preserved under pullbacks (and are usnique because of the specified orientation), the image of $c_U$ is $c_{U\cap V}$. Thus the image of $c_U \oplus c_V$ in the middle term is 0; since the sequence is exact there is a unique element $c \in \widetilde{H}^n(\mathrm{Th}(E))$ which hits $c_U \oplus c_V$.

? $\mathbb{Z}/2$ coef.

The Mayer–Vietoris sequence above receives a homomorphism from the Mayer–Vietoris sequence

$$\cdots \longrightarrow \widetilde{H}^{i-1}(E|_{U\cap V}) \longrightarrow \widetilde{H}^i(E) \longrightarrow \widetilde{H}^i(E|_U) \oplus \widetilde{H}^i(E|_V) \longrightarrow \widetilde{H}^i(E|_{U\cap V}) \longrightarrow \cdots$$

induced by cupping with the appropriate Thom class. By the induction hypothesis this is an isomorphism on all terms other than the terms $\widetilde{H}^i(E)$; by the five lemma these must also be isomorphisms, and the proof of this case is complete.

dumb

$$\begin{tikzcd}[column sep=small] \to \widetilde{H}^{i-1}(E|_U)\oplus\widetilde{H}^{i-1}(E|_V) \arrow[r] \arrow[d, "\cong"] & \widetilde{H}^{i-1}(E|_{U\cap V}) \arrow[r] \arrow[d, "\cong"] & \widetilde{H}^i(E) \arrow[r] \arrow[d] & \widetilde{H}^i(E|_U)\oplus\widetilde{H}^i(E|_V) \arrow[r] \arrow[d, "\cong"] & \widetilde{H}^i(E|_{U\cap V}) \arrow[d, "\cong"] \\ \to \widetilde{H}^{n+i-1}(\mathrm{Th}(E|_U))\oplus\widetilde{H}^{n+i-1}(\mathrm{Th}(E|_V)) \arrow[r] & \widetilde{H}^{n+i-1}(\mathrm{Th}(E|_{U\cap V})) \arrow[r] & \widetilde{H}^{n+i}(\mathrm{Th}(E)) \arrow[r] & \widetilde{H}^{n+i}(\mathrm{Th}(E|_U))\oplus\widetilde{H}^{n+i}(\mathrm{Th}(E|_V)) \arrow[r] & \widetilde{H}^{n+i}(\mathrm{Th}(E|_{U\cap V})) \end{tikzcd}$$