Define unpointed space and unreduced cohomology.

• unpointed spaces & unreduced cohomology:

Functor

- + : Top -> Top * adds a disjoint basepoint to given space
- For unpointed space Y (pointed spaces are spaces with distinguished basepoint $* \in X$) $H^{n}(Y;A) := H^{n}((Y)_{+};A) = [Y_{+}, K(A,n)]$

is called unreduced cohomology of Y.

Leg. One example of this functor is

&= One point compactification of space X

2 $H^*_c(X) := H^*(\hat{X})$ this is called compactly supported cohomology of X.

Introduce two Kunneth formulas

● Theorem: (Kunneth Theorem, [Hat02, Theorem 3.18]). Let R be a commutative ring. There is a group homomorphism, natural in both X and Y,

$$H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \wedge Y;R).$$

• Theorem:

Consider a quotient X/A of unpointed spaces. This is naturally pointed by the point [A]. The diagonal map $\Delta: X_+ \longrightarrow X_+ \wedge X_x$ includes a mass $A: X_+ \longrightarrow X_+ \wedge X_x$

 $(x,x) \longleftrightarrow (x,x)$ $A \ni a \longmapsto (a_1a)$ which is the besepoint of X+ NX/A This map composed with homomorphism from kunneth theorem defines a product Thus we can define a cup product $U: H^*(X;R) \otimes H^*(X/A) \longrightarrow H^*(X_+ \wedge X/A) \xrightarrow{\Delta *} H^*(X/A)$ Define the Thom space and the Thom bundle So example 4.20. Thom spaces & Thom isomorphism: • Defⁿ: Let $p: E \longrightarrow B$ be an n-dim. Vector bundle. The fiber bundle $\beta: \widetilde{E} \longrightarrow B$ is defined by taking the fiberwise 1-point compactification of E, so that the fiber of E above b & B is a copy of sn. This produces a new section so, the section at infinity given by beB -> point at infinity in its Alber. The Thom space of E is defined by Th(E):= E/so(B) het p:E→B be a vector bundle. Define a metric M:E→IR s.t. restriction of u to each fiber is a positive definite quadratic form. This how two imp. Fiber bundles sitting invide it: O Disk bundle: Let D(E) = feEE | Me) < 19 then $p^{-1}(b) \cong D^n$ $\forall b \in B$

mountes a map

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@ Sphere bundle: Let SLE) = {eEE | u(e)=13
                                      then p-1 (b) = sn-1 Y b(B)
               Then
                                Th(E) = D(E)/S(E)
                                space
             →eg. E=BxIR" since its a trivial bundle
               \Rightarrow D(E) \cong B \times D^{n} \quad \& \quad \&(E) \cong B \times S^{n-1} \quad \xrightarrow{B \times S^{n} \sqcup \{*\} \times S^{n}} 
\Rightarrow Th(E) = B \times D^{n} / B \times S^{n-1} \cong B \times S^{n} / B \times \{*\} \cup \{*\} \times S^{n} 
                                                           \cong (\beta+) \Lambda S^n
                                                                  * is disjoint from B
                                                           \cong \sum_{n} (\beta^{+})
                                 \Rightarrow \overset{\sim}{H}^{n+i} (Th(E)) \cong \overset{\sim}{H}^{n+i} (\Sigma^n(B_+))
                                                           \cong \mathcal{H}^{i}(\mathcal{B}_{+}) (: \mathcal{H}^{i}(\mathcal{X}) = \mathcal{H}^{i+1}(\mathcal{Z}\mathcal{X}))
                                                           = Hi(B) (from defn of
                                                                              homology of unpointed space)
  suspervion destroys up product > this is not a ring
 By kunneth thm,
     \widetilde{H}^*(S^n) \otimes \widetilde{H}^*(B_+) \cong \widetilde{H}^*(\Sigma^n B_+)
      \mathcal{H}^*(S^n) \otimes \mathcal{H}^*(B) \cong \mathcal{H}^*(\Sigma^n B)
         → Hi (Th(E)) = Hi-n (B) this iso, is given by multiplying
                                                   by generator in Hn(sn)
            ✓Prove lemma 4.22 and corollary 4.23
     Thom spaces work well with products of rector bundles.
• <u>hemma</u>: For vector bundles E \longrightarrow B & E' \longrightarrow B'
                            Th (EXE') & Th (E) ATh (E')
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D(EXE')
$$\cong$$
 D(E) \times D(E')

Restricting on $S(E \times E')$, we get

 $S(E \times E') \cong (S(E) \times D(E')) \cup (D(E) \times S(E'))$
 \Rightarrow Th(EXE') = $D(E \times E') / S(E \times E')$
 \cong D(E) \times D(E') $/ (S(E) \times D(E')) \cup (D(E) \times S(E'))$
 \cong $(D(E) / S(E)) \wedge (D(E') / S(E') = Th(E) \wedge Th(E')$

Pointed space with point [S(E)]

we think of
$$\mathcal{E}^k$$
 as a bundle over a point $\Rightarrow \text{Th}(\mathcal{E}^k) \cong D(\mathcal{E}^k)/S(\mathcal{E}^k) \cong D^k/S^{k-1} \cong S^k$

Introduce Thom Class defn 4.24.

I no project map $Th(E) \rightarrow B$, but θ sits nicely inside Th(E).

This includes

$$T^{H^n}(Th(E);A) \longrightarrow H^n(S^n)$$
 called restriction of u to the fiber above b.

Definition 4.24. A class $c \in \widetilde{H}^n(\operatorname{Th}(E); A)$ is called a *Thom class with coefficients in* A if for every $b \in B$ the restriction of c to the fiber above b is a generator for $\widetilde{H}^n(S^n; A)$.

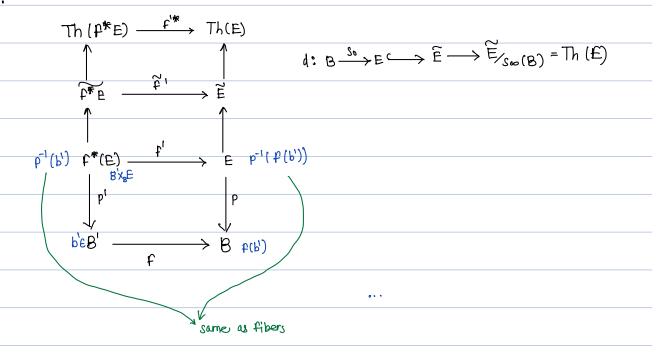
The usual cases of interest are when $A = \mathbb{Z}/2$ and when $A = \mathbb{Z}$. When $A = \mathbb{Z}$ this is often simply called a *Thom class*.

A bundle with a choice of Thom class with coefficients in A is called an A-oriented vector bundle.

Thom classes are natural w.r.t. pullbacks.

Proposition 4.25. Let $p: E \longrightarrow B$ be an oriented vector bundle, let $c \in \widetilde{H}^n(\operatorname{Th}(E))$ be the chosen Thom class, and let $f: B' \longrightarrow B$ be any map. There is an induced map $\operatorname{Th}(f): \operatorname{Th}(f^*(E)) \longrightarrow \operatorname{Th}(E)$ which takes c to a Thom class for f^*E .

pp:



Theorem 4.26 (Thom Isomorphism Theorem). Let c be a Thom class with coefficients in A for the n-dimensional bundle $p: E \longrightarrow B$. The homomorphism

$$\Phi: H^i(B; \mathbb{Z}/2) \longrightarrow \widetilde{H}^{i+n}(\operatorname{Th}(E); \mathbb{Z}/2) \qquad b \longmapsto p^*(b) \smile c$$

 $is\ an\ isomorphism\ for\ all\ i.$

• "Uc" ;

$$\Delta$$
: E+ \longrightarrow E+ Λ E+ induces Th(E) \longrightarrow E+ Λ Th(E)

by kunneth theorem,

$$H^*(E) \otimes H^*(Th(E)) \xrightarrow{\cup} H^*(Th(E))$$

 $(P^*: H^*(B) \longrightarrow H^*(E)$ is an iso ?)

precomposing with $P^* \otimes 1$

 H^* (B) \otimes H^* (Th(E)) \longrightarrow Th*(Th(E))

 ϕ = restriction of this map where 2nd coordinate is set to be c.

- · IF E is trivial, we saw that Thorn class exist
- Suppose B=UUV (U,V are open)

& Thom iso. holds for Elu, Elu & Elunv

het Thom clases : Cu, cv, Cunv associated to above.

since Thom iso. holds for Eluny, for i=-1:

We therefore have a Mayer-Vietoris sequence,

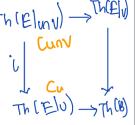
which begins:

$$\widetilde{H}^n(\operatorname{Th}(E)) \hookrightarrow \widetilde{H}^n(\operatorname{Th}(E|_U)) \oplus \widetilde{H}^n(\operatorname{Th}(E|_V)) \stackrel{u-v}{\longrightarrow} \widetilde{H}^n(\operatorname{Th}(E|_{U\cap V})) \longrightarrow \cdots$$

The inclusion $\operatorname{Th}(E|_{U\cap V}) \longrightarrow \operatorname{Th}(E|_U)$ induces a homomorphism

$$\widetilde{H}^n(\operatorname{Th}(E|_U)) \longrightarrow \widetilde{H}^n(\operatorname{Th}(E|_{U\cap V});$$

since the theorem holds for $E|_{U\cap V}$ and Thom classes are preserved under pullbacks (and are usual because of the specified orientation), the image of c_U is $c_{U\cap V}$. Thus the image of $c_U \oplus c_V$ in the middle term is 0; since the sequence is exact there is a unique element $c \in \widetilde{H}^n(\operatorname{Th}(E))$ which hits $c_U \oplus c_V$.



The Mayer–Vietoris sequence above receives a homomorphism from the Mayer–Vietoris sequence

$$\cdots \longrightarrow \widetilde{H}^{i}(E|_{U\cap V}) \longrightarrow \widetilde{H}^{i}(E) \longrightarrow \widetilde{H}^{i}(E|_{U}) \oplus \widetilde{H}^{i}(E|_{V}) \longrightarrow \widetilde{H}^{i}(E|_{U\cap V}) \longrightarrow \cdots$$

induced by cupping with the appropriate Thom class. By the induction hypothesis this is an isomorphism on all terms other than the terms $\widetilde{H}^{i}(E)$; by the five lemma these must also be isomorphisms, and the proof of this case is complete.

