To show that $\pi_1(\mathrm{Th}(\gamma_k)) = 0$, it now suffices to show that $i_*\colon \pi_1(\mathrm{Gr}_{k-1}) \to \pi_1(\mathrm{Gr}_k)$ is surjective, so the image of $\pi_1(\mathrm{Gr}_{k-1})$ inside $\pi_1(\mathrm{Gr}_k)$ is all of $\mathbb{Z}/2$. But this is induced b the map $O(k-1) \to O(k)$ given by

$$\begin{aligned} O(k-1) &\longrightarrow O(k) \\ A &\longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This is surjective on connected components, which implies that the induced map $\pi_1 BO(k-1) \to \pi_1 BO(k)$ is surjective. Finally, we know that $\mathrm{Gr}_i \cong BO(i)$, so $\pi_1(\mathrm{Gr}_{k-1}) \to \pi_1(\mathrm{Gr}_k)$ is surjective. Hence,

$$\pi_1(\mathrm{Th}(\gamma_k)) = 0,$$

so the Hurewicz theorem applies and we may conclude that $\pi_i(\mathrm{Th}(\gamma_k)) = 0$ for $i < k$.

(2) Now we need to show that i_* is an isomorphism. On cohomology, i^* is an isomorphism $H^i(\mathrm{Gr}_{k+1}) \to H^*(\mathrm{Gr}_k)$ up to degree $k+1$. Hence, for cohomology of the Thom bundles, i^* is an isomorphism for $j < 2k+2$:

$$i^*\colon H^j(\mathrm{Th}(\gamma_{k+1})) \cong H^j(\mathrm{Th}(\gamma_k \oplus \varepsilon^1)).$$

Hence, Proposition 3.27 applies and therefore i_* is an isomorphism up to dimension $2k$. In particular i_* is an isomorphism on homotopy groups $\pi_{n+k+1}(-)$ for $n < k$. □

Theorem 3.28 (Thom). *For* $k > n+2$,

$$\mathfrak{N}_n \cong \pi_{n+k}(\mathrm{Th}(\gamma_k))$$

Notice that the right-hand-side of this isomorphism is well-defined by Lemma 3.25 for $k > n$.

3.4 L-equivalence and Transversality

To prove Theorem 3.28, we need a lot of results about smooth manifolds. Since the point of this class isn't to learn about smooth manifolds, we will cite a lot of these things without proof. Most of it comes out of Thom's original paper.

Remark 3.29. We will abuse notation and abbreviate $\mathrm{Gr}_k := \mathrm{Gr}_k(\mathbb{R}^N)$ for $N \geq 2k+5$. In the cases we care about in the lemmas below, we need a compact manifold; $\mathrm{Gr}_k(\mathbb{R}^N)$ is compact. Moreover, maps here are well-defined and independent of N when N is sufficiently large. Likewise, write $\gamma_k := \gamma_{kN}$.

Definition 3.30. Let $f\colon X^n \to M^p$ be a C^n map from an n-manifold to a p-manifold. Let $N^{p-q} \subseteq M$ be a submanifold of M of codimension q. For $y \in N$, $T_y M \supseteq T_y N$. Let $x \in f^{-1}(y)$. We say that f is **transverse to N at y** if

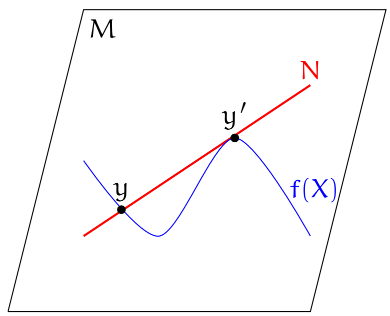
$$df_x \colon T_x X \to T_y M \underset{\text{quotient map}}{\to} {T_y M}/{T_y N}$$

is an epimorphism. → similar to surjective for $f: X \to Y$ morphism & $g_1, g_2: Y \to Z$ morphism $\left.\begin{matrix} g_1 \circ f = g_2 \circ f \\ \Rightarrow g_1 = g_2 \end{matrix}\right\} \to$ epimorphism

f is **transverse to N** if this holds for all x, y.

Notice that if $f^{-1}(y) = \emptyset$, transversality automatically holds.

Example 3.31. Let $X = \mathbb{R}$, $M = \mathbb{R}^2$, and $N = \mathbb{R}$.



At y, $T_x X \twoheadrightarrow {T_y M}/{T_y N}$ is transverse.

At y', $T_x X \xrightarrow{0} {T_{y'} M}/{T_{y'} N}$ is not transverse.

Definition 3.32. A homotopy $X \times [0, 1] \to Y$ is an **isotopy** if for all $t \in [0, 1]$, the map $X \times \{t\} \to Y$ is smooth.

Definition 3.33. Let N be a submanifold of a manifold M of codimension q. A **tubular neighborhood** of N in M is an embedding of a q-disk bundle on N into M such that N is the zero section.

Theorem 3.34. *Assume that*

- X *is a smooth n-manifold;*
- M *is a p-manifold;*
- $N \subseteq M$ *is a paracompact submanifold of M of codimension q;*
- T *is a tubular neighborhood of N in M;*
- $f\colon X \to M$ *is a C^n map;*
- $y \in T_y M$ *and* $x \in f^{-1}(Y)$.

Then we may conclude the following.

(a) *If* $f: X \to M$ *is transverse to* N*, then* $f^{-1}(N)$ *is a* C^n *submanifold of* X *of codimension* q*.*

(b) *There is a homeomorphism* A *of* T *arbitrarily close to the identity and equal to the identity on* ∂T*, such that* $A \circ f$ *is transverse to* N*.*

(c) *If* $f: X \to M$ *is transverse to* N*, then* N *is compact. Then for any* A *(as in (b)) sufficiently close to the identity,* $A \circ f$ *is transverse to* N *and* $f^{-1}(N)$*,* $(A \circ f)^{-1}(N)$ *are isotopic in* X*.*

Remark 3.35. Theorem 3.34(b) says that we may always wiggle the tubular neighborhood a little bit so that, after with composing with f, it is transverse to N. Theorem 3.34(c) implies that $f^{-1}(N)$ and $(A \circ f)^{-1}(N)$ are isomorphic. Similar results hold for manifolds M with boundary.

Now let's relate this to cobordism.

Theorem 3.36. *Let* $f, g: X \to M$ *be* C^m *maps where* $m \geq n$*, both transverse to a submanifold* N *of codimension* q*. If* f *and* g *are homotopic, then* $f^{-1}(N)$ *is cobordant to* $g^{-1}(N)$*.*

Proof. We may assume that this homotopy is smooth. So consider $F: X \times I \to M$. By Theorem 3.34(b) there is some A such that $A \circ F$ is transverse to N. By Theorem 3.34(c), $f^{-1}(N)$ is isotopic to $(A \circ F|_{X \times \{0\}})^{-1}(N)$. In particular, this implies that $f^{-1}(N)$ is *cobordant* to $(A \circ F|_{X \times \{0\}})^{-1}(N)$. Likewise, $g^{-1}(N)$ is cobordant to $(A \circ F|_{X \times \{1\}})^{-1}(N)$.

Now by Theorem 3.34(a), $(A \circ F)^{-1}(N)$ is a submanifold of $X \times I$ with boundary

$$(A \circ F|_{X \times \{0,1\}})^{-1}(N) = (A \circ F|_{X \times \{0\}})^{-1}(N) \cup (A \circ F|_{X \times \{0\}})^{-1}(N)$$

Composing with the cobordisms in the first paragraph, we obtain a cobordism between $f^{-1}(N)$ and $g^{-1}(N)$. $\square$

This motivates the following definition.

Definition 3.37. Let W_0, W_1 be two k-submanifolds of an n-manifold X. Then we say that W_0 and W_1 are **L-equivalent in X** if there exists a manifold Y with boundary $W_0 \sqcup W_1$ and an embedding $f: Y \to X \times I$ such that

$$f^{-1}(X \times \{0\}) = W_0 \text{ and } f^{-1}(X \times \{1\}) = W_1.$$

We write $L_k(X)$ for the set of L-equivalence classes of k-submanifolds.

Example 3.38. If $W_0 = S^1 \sqcup S^1$ and $W_1 = S^1$ inside the plane X, but $W_0 \cap W_1 \neq \emptyset$, then there's no embedded cobordism between them. But there is an embedded pair of pants linking them in $X \times I$.

Lemma 3.39. *If $n > 2k+2$, then $L_k(S^n)$ is an abelian group. The map $\phi\colon L_k(S^n) \to \mathfrak{N}_k$ taking the L-equivalence class of W to the cobordism class of W is an isomorphism.*

$\mathfrak{N}_k$

equivalence classes of isomorphism classes up to cobordism.

Proof. For $n > 2k+2$, any two embedded k-submanifolds can be homotoped (and indeed, isotoped) to be disjoint. Thus, disjoint union is a well-defined operation on $L_k(S^n)$.

We say that $[\emptyset]$ is the identity in $L_k(S^n)$.

$L_k(S^n)$ has inverses given by the horseshoe L-equivalence: Therefore, $2[W] = 0$, so $[W] = -[W]$. Hence $L_k(S^n)$ is a group.

$W \cup W$
$= W \cup (W^c \cap W)$
$= (W \cup W^c) \cap (W \cup W)$

Now to show that the map $\phi\colon L_k(S^n) \to \mathfrak{N}_k$ is an isomorphism, it suffices to check that this is a bijection since these have the same group structure.

To check surjectivity, assume $[W] \in \mathfrak{N}_k$. Then there is an embedding

$$W \hookrightarrow \mathbb{R}^{2k+2} \hookrightarrow S^n$$

(recall that we are assuming that $n \geq 2k+2$ Remark 3.29). So $[W]$ is a class in $L_k(S^n)$.

To check injectivity, consider an embedded submanifold $W \hookrightarrow S^n$ such that $[W] = 0$ in $\mathfrak{N}_k$. Write $W = \partial B$ for a $(k+1)$-manifold B. Embed B into S^n via

$$f\colon B \hookrightarrow \mathbb{R}^{2(k+1)} \hookrightarrow S^n.$$

Use Urysohn's Lemma to pick a function $\phi\colon B \to I$. Then $\phi^{-1}(0) = W$, and $\phi^{-1}(1) = \emptyset$. Then $(f, \phi)\colon B \to S^n \times I$ witnesses an L-equivalence between W and $\emptyset$. Hence, $[W] = 0$ in $L_k(S^n)$. □

Construction 3.40. For X an n-manifold, we define a map $J\colon L_{n-k}(X) \to [X, \mathrm{Th}(\gamma_k)]$ by first choosing an embedding $X \hookrightarrow \mathbb{R}^N$. Then for each $w \in W$, we have a normal bundle at w inside X:

L-equivalence class

$$N_w W := (T_w W)^{\perp} \cap T_w X.$$

Then $N_w W$ is a k-plane in $\mathbb{R}^N$, so an element of $\mathrm{Gr}_k = \mathrm{Gr}_k(\mathbb{R}^N)$ (see Remark 3.29). This gives a map $f\colon W \to \mathrm{Gr}_k$.

Now let N be a tubular neighborhood of W in X; think of it as a pullback of the disk bundle of γ_k: $N = f^*(D(\gamma_k))$. Then f induces a map $\tilde{f}\colon \mathrm{Th}(f^*(\gamma_k)) \to \mathrm{Th}(\gamma_k)$. So define

$$f'\colon X \to \mathrm{Th}(\gamma_k)$$

Remark 3.29. We will abuse notation and abbreviate $\mathrm{Gr}_k := \mathrm{Gr}_k(\mathbb{R}^N)$ for $N \geq 2k+5$. In the cases we care about in the lemmas below, we need a compact manifold; $\mathrm{Gr}_k(\mathbb{R}^N)$ is compact. Moreover, maps here are well-defined and independent of N when N is sufficiently large. Likewise, write $\gamma_k := \gamma_{kN}$.

by

$$f'(x) = \begin{cases} * & \text{if } x \notin N, \\ \widetilde{f}(x) & \text{if } x \in N. \end{cases}$$

The image of W under J is this map $f'\colon X \to \mathrm{Th}(\gamma_k)$.

Now, Gr_k embeds into $\mathrm{Th}(\gamma_k)$ as the zero section. So

$$(f')^{-1}(\mathrm{Gr}_k) = W.$$

Why do we find this construction useful? Let $X = S^{n+k}$. Then we have a map

$$L_n(S^{n+k}) \to \pi_{n+k}\,\mathrm{Th}(\gamma_k).$$

To prove Theorem 3.28, we want to show that this is an isomorphism of groups. Then we may apply Lemma 3.39 to conclude that

$$\mathfrak{N}_k \cong L_n(S^{n+k}) \cong \pi_{n+k}\,\mathrm{Th}(\gamma_k)$$

Remark 3.41. When might we expect $[X, Y]$ to be a group?

If we have $f, g\colon X \to Y$ with X **cogrouplike**, meaning that it has a nice map $p\colon X \to X \vee X$, then we might define the product of f and g by

$$X \xrightarrow{p} X \vee X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\text{fold}} Y.$$

For example, the pinch map $S^n \to S^n \vee S^n$ satisfies this property.

Alternatively, if we had a retraction map $r\colon Y \times Y \to Y \vee Y$, then we might define the product of f and g by

$$X \xrightarrow{\text{diag}} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{r} Y \vee Y \xrightarrow{\text{fold}} Y.$$

Classically, the conditions for this second approach to work were answered in **cohomotopy theory**, which studies homotopy classes of maps into spheres instead of out of spheres. This theory is now pretty much defunct.

Lemma 3.42. *J is independent of the choice of $X \hookrightarrow \mathbb{R}^N$.*

Proof. If $i_0, i_1\colon X \hookrightarrow \mathbb{R}^N$, we may assume for large enough N that $i_0(X) \cap i_1(X) = \emptyset$. Moreover, we may assume that there is an embedding $X \times I \to \mathbb{R}^N$ that is i_0 on $X \times \{0\}$ and i_1 on $X \times \{1\}$. Finally, we may assume that $X \times I$ is embedded orthogonally to its boundary.

Let W be some k-submanifold of X. The embedding above restricts to an embedding of $W \times I \to \mathbb{R}^N$. A tubular neighborhood N of $W \times I$ under this embedding is orthogonal to the boundary of $X \times I$ by our assumption; thus $N \cap X \times \{0\}$ is a tubular neighborhood of $i_0(W)$ and $N \cap X \times \{1\}$ is a tubular neighborhood of $i_1(W)$. We can then apply the construction of J to this N and the embedding of $W \times I$ to produce a map $X \times I \to \mathrm{Th}(\gamma_k)$. This restricts to the maps constructed for W under i_0 and i_1, respectively. Thus, the two maps are homotopic. ∎

Lemma 3.43. *L-equivalent submanifolds give homotopic maps under J.*

Proof sketch. Let W_0, W_1 be L-equivalent. So there is some submanifold $B \subseteq S^{n+k} \times I$ with $B \cap (S^{n+k} \times \{i\}) = W_i$. Let T be a tubular neighborhood of B. Then $T \cap (S^{n+k} \times \{i\})$ is a tubular neighborhood of W_i. We get a map

$$S^{n+k} \times I \to \mathrm{Th}(\gamma_k)$$

that is a homotopy. $\square$

Now claim that J is a group homomorphism.

Given $W, W' \subseteq S^{n+k}$, and tubular neighborhoods N, N' of W and W' inside S^{n+k}, $N \sqcup N'$ is a tubular neighborhood of $W \sqcup W'$. Applying J to W and W', we get two maps $f\colon W \to \mathrm{Gr}_k$ and $f'\colon W' \to \mathrm{Gr}_k$. The group operation on $L_n(S^{n+k})$ is given by disjoint union, so if we collapse everything outside of a tubular neighborhood of $W \sqcup W'$ inside S^{n+k}, we can realize the disjoint union of f with f' as

$$S^{n+k} \to S^{n+k} \vee S^{n+k} \xrightarrow{f \vee f'} \mathrm{Th}(\gamma_k) \vee \mathrm{Th}(\gamma_k) \xrightarrow{\nabla} \mathrm{Th}(\gamma_k).$$

This is exactly the same as the group operation on $[S^{n+k}, \mathrm{Th}(\gamma_k)]$. Hence, J is a group homomorphism.

This next lemma shows that J is injective.

Lemma 3.44. *Let $f, f'\colon X \to \mathrm{Th}(\gamma_k)$. If f is homotopic to f' and both are transverse to Gr_k, then $f^{-1}(\mathrm{Gr}_k)$ is L-equivalent to $(f')^{-1}(\mathrm{Gr}_k)$.*

Proof sketch. Let $F\colon X \times I \to \mathrm{Th}(\gamma_k)$ be a homotopy. Then $F^{-1}(\mathrm{Gr}_k)$ is a submanifold, and gives the desired L-equivalence. $\square$

This lemma actually gives us something more: an inverse to J sending $f\colon S^{n+k} \to \mathrm{Th}(\gamma_k)$ to $f^{-1}(\mathrm{Gr}_k)$. Hence, we have shown the following.

Lemma 3.45. $L_n(S^{n+k}) \cong \pi_{n+k}(\mathrm{Th}(\gamma_k))$.

Modulo checking some details, this in fact shows Theorem 3.28.

3.5 Characteristic Numbers and Boundaries

Corollary 3.46 (Corollary to Theorem 3.28)**.** *If M is an n-manifold all of whose characteristic numbers are zero, then M is the boundary of an $(n+1)$-manifold.*

Proof. Suppose that we have an n-manifold M and an embedding $M \hookrightarrow S^{n+k}$ (recall $k \geq n+2$, so such an embedding exists). Under the isomorphism $L_n(S^{n+k}) \cong \pi_{n+k}(\mathrm{Th}(\gamma_k))$, we have a map $f\colon S^{n+k} \to \mathrm{Th}(\gamma_k)$ with $M = f^{-1}(\mathrm{Gr}_k)$. Thus, f restricts to a map $\tilde{f}\colon M \to \mathrm{Gr}_k$.