Introduction to Cobordisms:

What's the difference between a Klein bottle and a torus?

The Klein bottle is non orientable, while the torus is not.

In other words, $\exists W_T$ a 3-manifold s.t.

$$T = \partial W_T$$

but the same is not true for K.

We can ask this question in generality:

when is a closed n-manifold the boundary of an $(n+1)$-manifold?

Def: Let M and N be two n-manifolds. We say M and N are <u>cobordant</u> if

$\exists$ an $(n+1)$-manifold W s.t.

$$\partial W = M \sqcup N.$$

Here, W is called a cobordism between M and N.

Ex: (1) Any manifold $\overset{M}{\vee}$ is cobordant to itself, by way of $W := M \times I$.

(2) Various articles of clothing:

A pair of pants P, where $\partial P = S^1 \sqcup (S^1 \sqcup S^1)$

A sock S, where $\partial S = S^1 \sqcup \emptyset$

A shirt H, where $\partial H = S^1 \sqcup (S^1 \sqcup S^1 \sqcup S^1)$

(3) Cobordisms are not unique. $S^1 \times I$ is a cobordism between S^1 and S^1, but so is a torus with its ends cut off.

Next up: the cobordism group.

Def: The unoriented cobordism group $\mathfrak{N}_n$ is defined as follows.

As a set, $\mathfrak{N}_n$ consists of the equivalence classes of isomorphism classes of n-manifolds up to cobordism.

Addition is $\sqcup$. $([M] + [N] = [M \sqcup N])$

The empty manifold is an n-manifold, $\forall n$.

$\hookrightarrow [\emptyset]$ consists of all closed n-manifolds which are boundaries.

Lemma: $\mathfrak{N}_n$ is an abelian group.

Proof: We check well-definedness.

Suppose $[M] = [M']$ and $[N] = [N']$.

Then $\exists$ a cobordism W between M, M'

and $\exists$ " " W' " " N, N'.

Now $W \sqcup W'$ is a cobordism between $(M \sqcup N)$ and $(M' \sqcup N')$.

So $[M \sqcup N] = [M' \sqcup N']$ as desired.

Inverses: Observe that $[M] + [M] = [M \sqcup M]$

$$= \partial(M \times I)$$

$$= [\emptyset].$$

So every class has an inverse.

$\mathfrak{N}_n$ is abelian since $M \sqcup N \cong N \sqcup M$.

$\blacksquare$

Made with Goodnotes

We see that $\exp(\mathfrak{N}_n) = 2$. So if the group is finitely generated, this is enough to conclude that $\mathfrak{N}_n \cong (\mathbb{Z}/2)^k$ for some k.

This turns out to be true.

<u>Thm:</u> (Thom) The group $\mathfrak{N}_* = \bigoplus_{n=0}^{\infty} \mathfrak{N}_k$ is a graded $\mathbb{Z}/2$-algebra with the product given by $[M][N] = [M \times N]$

As a graded algebra,
$$\mathfrak{N}_* \cong \mathbb{Z}/2[x_i : i \geq 1, i \neq 2^j - 1]$$
with $|x_i| = i$.

No proof.

Stiefel-Whitney Numbers

Def: Let M be an n-manifold, and let $[M] \in H_n(M; \mathbb{Z}_2)$ be its $\mathbb{Z}_2$-fundamental class.
Let $r_1, \dots, r_n \in \mathbb{Z}_{\geq 0}$ be s.t.

$$r_1 + 2r_2 + \dots + nr_n = n.$$

Then the cohomology class

$$w_1(TM)^{r_1} \cdots w_n(TM)^{r_n}$$

is in $H^n(M)$ S-W class

The <u>$(r_1, \dots, r_n)$-Stiefel-Whitney number</u> is

$$(w_1(TM)^{r_1} w_2(TM)^{r_2} \cdots w_n(TM)^{r_n})[M] \in \mathbb{Z}_2.$$

This is denoted $w_1^{r_1} \cdots w_n^{r_n}[M]$.

Lemma: Let M and N be n-manifolds.
For any $r_1, \dots, r_n$ with $r_1 + 2r_2 + \dots + nr_n = n$, we have

$$w_1^{r_1} \cdots w_n^{r_n}[M \sqcup N] = w_1^{r_1} \cdots w_n^{r_n}[M] + w_1^{r_1} \cdots w_n^{r_n}[N].$$

Proof:

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & M \arrow[d, "p_1"] \\ T(M \sqcup N) \arrow[r, "f"'] & M \sqcup N \end{tikzcd}
\qquad
\begin{tikzcd} p_2^*(M \sqcup N) \arrow[d, equal] & \\ TN \arrow[r] \arrow[d] & N \arrow[d, "p_2"] \\ T(M \sqcup N) \arrow[r, "f"'] & M \sqcup N \end{tikzcd}$$

We have: $w_n(TM) = w_n(p_1^* T(M \sqcup N))$

$$= p_1^*(w_n(T(M \sqcup N))) \in H^n(TM)$$

Likewise, $w_n(TN) = p_2^*(w_n(T(M \sqcup N))) \in H^n(TN)$

So $w_n(T(M \sqcup N)) \cong (w_n TM, w_n TN)$

Now we get $w_n(T(M \sqcup N))[M \sqcup N] = \langle w_n(T(M \sqcup N)), [M \sqcup N] \rangle$

$$= \langle (w_n TM, w_n TN), ([M], [N]) \rangle$$

$$= \langle w_n TM, [M] \rangle + \langle w_n TN, [N] \rangle$$

Ex: Compute S-W numbers for real projective spaces.

Recall: $H^*(\mathbb{R}P^n) \cong \mathbb{Z}_2[x]/x^{n+1}$

and $w_i(T\mathbb{R}P^n) = \binom{n+1}{i} x^i$.

S-W class

$$\binom{2k+1}{2k} = \frac{(2k+1)!}{(2k)!(1)!} = 2k+1 = n+1$$

First, suppose n is even.
Then
$$w_n(T\mathbb{R}P^n) = (n+1)x^n \neq 0$$

$\Rightarrow w_n[\mathbb{R}P^n] \neq 0$

Similarly, $w_1(T\mathbb{R}P^n) = (n+1)x \neq 0$

$\Rightarrow w_1[\mathbb{R}P^n] \neq 0$

In general, $w_1^{r_1} \cdots w_n^{r_n}[\mathbb{R}P^n] = \binom{n+1}{r_1} \cdots \binom{n+1}{r_n} \pmod 2$

These can vary, depending on $n+1$.

When $n = 2^k - 1$, all are nonzero.
When $n = 2^k$, all but w_1 and w_n are zero.

Now suppose $n = 2k-1$ is odd.

Note $(1+x)^{2k} = (1+2x+x^2)^k \equiv (1+x^2)^k$

Thus $\binom{2k}{2i} \equiv \binom{k}{i} \pmod 2$
and $\binom{2k}{2i+1} \equiv 0 \pmod 2$

$$(1+x)^{2k} = \sum_{i=0}^{2k} \binom{2k}{i} x^i$$
$$\shortparallel$$
$$(1+x^2)^k = \sum_{i=0}^{k} \binom{k}{i} x^{2i}$$

$$= \binom{2k}{0} x^0 + \binom{2k}{1} x + \binom{2k}{2} x^2 + \cdots$$
$$= \binom{k}{0} x^0 + \binom{k}{1} x^2 + \binom{k}{2} x^4$$

So $\forall i,\ w_{2i+1}(T\mathbb{RP}^n) = 0$.

Since any $w_1^{r_1} \cdots w_n^{r_n}$ must have at least one $r_i \neq 0$ for an odd i, we conclude that all odd S-W numbers are 0.

Thm: (Pontrjagin-Thom)

Let M be a smooth closed n-manifold.
Then,

$\exists$ a smooth, compact $(n+1)$-manifold B with $\partial B = M$

iff

all S-W numbers of M are 0.

Proof: ?..

Upshot: S-W numbers can identify exactly when a manifold is a boundary. By using the group structure on $\mathfrak{N}_n$, we conclude that S-W numbers can be used to detect cobordant manifolds.

Cor: M and N are cobordant iff their S-W numbers are equal.

Stability:

Want to talk about stability: properties of invariants of spaces which are preserved by the functor

$$X \longmapsto \Sigma X$$

taking a pointed space to its suspension.

Ex: Reduced homology and cohomology functors are stable by definition.

$h^i : \underline{Top_*^{op}} \to \underline{AbGp}$ comes with a natural suspension $\underline{\text{isomorphism}}$ $\sigma_i : h^i(X) \to h^{i+1}(\Sigma X)$.

Homotopy, in general, is not stable.

Ex: $\pi_3(S^2) \cong \mathbb{Z}$ but $\pi_4(\Sigma S^2) = \pi_4(S^3) \cong \mathbb{Z}_2$.

For homotopy groups, we get a stabilization homomorphism.

For any map of pointed spaces $f: X \to Y$, smashing with a circle induces

$$\Sigma f : \Sigma X \to \Sigma Y$$

which induces a function

$$[X, Y] \to [\Sigma X, \Sigma Y].$$

For homotopy groups, this is an isomorphism.

Thm: (Freudenthal Suspension Theorem)

For an n-connected pointed CW-complex X, the stabilization homomorphism

$$\pi_k(X) \longrightarrow \pi_{k+1}(\Sigma X)$$

is an isomorphism if $k \leq 2n$.

Our goal will be to prove an analogous result for Thom spaces.

Observation: suspending a Thom space of a bundle E corresponds to adding a trivial bundle.

Recall: ① $Th(\varepsilon^k) \cong S^k \wedge B_+$?

② $Th(E \times E') \cong Th(E) \wedge Th(E')$

③ $Th(E \oplus \varepsilon^k) \cong Th(E) \wedge S^k$

Lemma: For $k > n$, the group $\pi_{n+k}(Th(\gamma_k))$ is independent of k.

We won't prove this, but we know it's true for cohomology groups.

?

Thom isomorphism $\Rightarrow \widetilde{H}^{n+k}(Th(\gamma_k)) \cong H^n(G_k)$.

Since $n < k$, this is the group ?

$$\mathbb{Z}_2\{w_1^{i_1} \cdots w_n^{i_n} : i_1 + 2i_2 + \cdots + ni_n = n\}$$

Clearly, this is independent of k.

This is our first encounter with <u>stable homotopy groups</u>.

Ex: Consider the sequence of spaces

$$Th(\gamma_0),\ Th(\gamma_1),\ Th(\gamma_2),\ \ldots$$

The classifying map of the bundle $\gamma_n \oplus \varepsilon^1$ induces a map

$$\Sigma Th(\gamma_n) \cong Th(\gamma_n \oplus \varepsilon^1) \xrightarrow{\sigma} Th(\gamma_{n+1}).$$

The adjoint of this is an inclusion

$$\sigma' : Th(\gamma_n) \hookrightarrow \Omega Th(\gamma_{n+1}).$$

If these were equivalences, we would have an Ω-spectrum (and thus a represented cohomology theory).

We can force this to become an Ω-spectrum in the following way.

Define

$$MO_n = \operatorname{colim}\left(Th(\gamma_n) \to \Omega Th(\gamma_{n+1}) \to \Omega^2 Th(\gamma_{n+2}) \to \cdots\right)$$

where the maps are given by the adjoints.

We claim that this is an Ω-spectrum, and that

$$\pi_{n+k} Th(\gamma_k) \cong \operatorname{colim}_k \left(\pi_{n+k}(MO_k)\right)$$

for k large enough.

$$MO_n = \operatorname{colim}\left(Th\,\gamma_n \to \Omega Th\,\gamma_{n+1} \to \Omega^2 Th\,\gamma_{n+2} \to \cdots\right)$$

$$= \operatorname{colim}\left(\Omega^2 Th\,\gamma_{n+2} \to \Omega^3 Th\,\gamma_{n+3} \to \cdots\right)$$

$$= \operatorname{colim}\left(\Omega\left(\Omega Th\,\gamma_{n+2} \to \Omega^2 Th\,\gamma_{n+3} \to \cdots\right)\right)$$

Filtered: $Th\,\gamma_n \hookrightarrow \Omega Th\,\gamma_{n+1}$

$$\Rightarrow \quad = \Omega \operatorname{colim}\left(\Omega Th\,\gamma_{n+2} \to \Omega^2 Th\,\gamma_{n+3} \to \cdots\right)$$

So MO_n is representable

Rmk: With our new cohomology theory, we can ask: what useful characteristic classes exist with values in MO?

The existence of char. classes in a general cohomology theory relies on one additional property: the fact that the cohomology of $\mathbb{RP}^\infty$ is a polynomial ring with one generator.

Then char. classes are defined and are similar to ones we have seen.

What is the cohom. theory represented by MO? We can compute its value on a point by

$$MO^n(S^0) \cong [S^0, \operatorname{colim} \Omega^k Th(\gamma_{n+k})]$$
$$\cong \operatorname{colim} [S^0, \Omega^k Th(\gamma_{n+k})]$$
$$\cong \operatorname{colim} [S^k, Th(\gamma_{n+k})].$$

Thm: (Thom) When $k > n+2$,

$$\mathfrak{N}_n \cong \pi_{n+k} Th(\gamma_k).$$