Theorem 3.13 (Classification theorem for vector bundles over compact spaces). *Suppose that B is compact. Let $\mathrm{Vect}_n(B)$ be the set of isomorphism classes of n-dimensional vector bundles over B, and write $[B, G_n]$ for the set of homotopy classes of maps $B \longrightarrow G_n$. Then the map*

$$\rho\colon [B, G_n] \longrightarrow \mathrm{Vect}_n(B) \qquad \textit{given by} \qquad f \longmapsto f^*\gamma_n$$

is a bijection.

3.3 The proof of the classification theorem

The rest of this chapter is dedicated to the proof of Theorem 3.13 and generalizing it beyond to compact spaces.

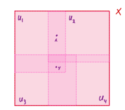
The proof of each step proceeds by first proving the necessary statements for trivializable bundles, and then by using the fact that there is a finite cover by opens over which the bundle is trivial. The main technical tool for doing this gluing is a *partition of unity*.

subject to; with respect to; using

Definition 3.18. A *partition of unity* for X subordinate to a finite open cover $\{U_i\}_{i=1}^n$ is n functions $\varphi_i\colon X \longrightarrow I$ such that for every $x \in X$,

For Ex.

$$\sum_{i=1}^{n} \varphi_i(x) = 1,$$



$\varphi_1(x) = 0.5$ $\quad$ $\varphi_1(y) = 0.5$
$\varphi_2(x) = 0.5$ $\quad$ $\varphi_2(y) = 0$
$\varphi_3(x) = 0$ $\quad$ $\varphi_3(y) = 0.25$
$\varphi_4(x) = 0$ $\quad$ $\varphi_4(y) = 0.25$

and such that the support of each φ_i is contained inside U_i.

For a finite cover these always exist. (For a more in-depth discussion of partitions of unity, see for example [Hat, Appendix to Chapter 1] or [AB06, Section 2.19].)

Step 1: ρ is well-defined

The function ρ is clearly well-defined as a function

$[B, G_n] \leftarrow$ homotopy classes of maps.

homomorphisms from $B \to G_n$

$$\mathrm{Hom}(B, G_n) \longrightarrow \mathrm{Vect}_n(B),$$

isomorphism classes of n-bundles.

from the set of maps $B \longrightarrow G_n$ to the isomorphism classes of n-bundles. The question of well-definedness therefore hangs on whether or not homotopic maps produce isomorphic vector bundles. This is implied by the following more general statement:

Lemma 3.19. *Let X be compact. Let $p\colon E \longrightarrow X \times I$ be a n-dimensional vector bundle. Let $f\colon X \longrightarrow I$ be any map, and write X_f for its graph inside $X \times I$. Then the isomorphism type of the restriction of E to X_f is independent of the choice of f. In particular, letting f be the constant map at 0 or 1 it follows that the restrictions of E to $X \times \{0\}$ and $X \times \{1\}$ are isomorphic.*

$$\begin{array}{ccc} E & & E_f \\ \downarrow & & \downarrow \\ X \times I & \supseteq & X_f \end{array}$$

Proof. For any $f\colon X \longrightarrow I$, write E_f for the restriction of E to X_f. We will show that E_f is isomorphic to E_0, the case where f is the constant map at 0.

$X_f = \{ (x, f(x)) \in X \times I \mid x \in X \}$

$$\begin{tikzcd} (X\times I)\times F \arrow[r] & E \arrow[d] \\ & X\times I \end{tikzcd}$$

To begin, consider the case of a trivial bundle $(X \times I) \times F$. The bundle E_f is isomorphic to the space

$$(X\times I\times F)_f = \{(x, t, y) \in X \times I \times F \mid t = f(x)\}. \subseteq X\times I\times F$$

$$\begin{tikzcd} (X\times I\times F)_f \arrow[r] & E_f \arrow[d] \\ & X_f \end{tikzcd}$$

There is an explicit isomorphism $E_f \longrightarrow E_0$ by

$$\overset{X\times I\times F}{(x, f(x), y)} \longmapsto \overset{X\times\{0\}\times F}{(x, 0, y)}.$$

We generalize this approach to a somewhat stronger statement. Let $f, f' \colon X \longrightarrow I$ be two functions, and suppose that

$$\{x \in X \mid f(x) \neq f'(x)\} \subseteq U \subseteq X$$

$$\begin{tikzcd} X \arrow[d, shift right, "f"'] \arrow[d, shift left, "f'"] \\ I \end{tikzcd}$$

for some open $U \subseteq X$ such that $U \times I$ in the trivialization cover of p. In other words, f and f' are the same except inside a patch over which p is trivial. Then we can define an isomorphism $g \colon E_f \longrightarrow E_{f'}$ by

$$\begin{tikzcd} E_f \arrow[rr, bend left, "g"] \arrow[d] & & E_{f'} \arrow[d] \\ X_f & & X_{f'} \end{tikzcd} \qquad \begin{tikzcd} E \arrow[d, "p"] \\ X\times I \end{tikzcd}$$

$$\underset{\in E_f}{g(e)} = \begin{cases} e \in E_{f'} & \text{if } p(e) \notin U \times I \\ \varphi^{-1}(x, f'(x), y) & \text{if } \varphi(e) = (x, f(x), y) \in U \times I \times F \end{cases}$$

where $\varphi \colon p^{-1}(U \times I) \longrightarrow U \times I \times F$ is the trivialization of p over $U \times I$. This is continuous because the points where f and f' are distinct are contained inside $U \times I$.

To glue these into a global isomorphism, he key observation is that the trivialization cover contains a subcover of sets of the form $\{U_\alpha \times I\}_{\alpha \in A}$, where the U_α cover X. Using the compactness of X we can then reduce to working within each of these sets separately, which is exactly the special case handled above.

$*$

To show that we can always trivialize over sets of the form $U \times I$ we first need the following observation: if E is trivializable over $U \times [a, b]$ and $U \times [b, c]$ then it is trivalizable over $[a, c]$. If the two trivialization isomorphisms agree on $U \times \{b\}$, we are done since we can just glue them together. Otherwise, given $\varphi_1 \colon E|_{U\times[a,b]} \longrightarrow U \times [a, b] \times F$ and $\varphi_2 \colon E|_{U\times[b,c]} \longrightarrow U \times [b, c] \times F$, there is an induced automorphism $h \colon U \times \{b\} \times F \longrightarrow U \times \{b\} \times F$ given by $\varphi_1\varphi_2^{-1}$. Extend this to an automorphism $h \colon U \times [b, c] \times F \longrightarrow U \times [b, c] \times F$ by ignoring the $[b, c]$-coordinate. This gives an alternate trivialization $h \circ \varphi \colon E|_{U\times[b,c]} \longrightarrow U \times [b, c] \times F$. This agrees with φ_1 on $U \times \{b\}$, and thus the earlier case applies.

Using this we show that the trivialization cover of p contains a subcover $\{U_\alpha \times I\}_{\alpha \in A}$ where $\{U_\alpha\}_{\alpha \in A}$ is a cover of X. Indeed, for any $(x, t) \in X \times I$

there exists an open subset $U_{xt} \times V_{xt}$ over which E is trivializable. Fixing x, since I is compact there exist $0 = t_0 < \cdots < t_n = 1$ such that $[t_{i-1}, t_i] \subseteq V_{xt'_i}$ for some $t'_i \in [0,1]$. Define $U_x \overset{\text{def}}{=} \bigcap_{i=1}^{n} U_{xt'_i}$, and note that E is trivializable over $U_x \times [t_{i-1}, t_i]$ for all i. Using the above observation, we conclude that U must be trivializable over $U_x \times I$, as desired.

Since X is compact, $\{U_\alpha\}_{\alpha \in A}$ contains a finite subcover $\{U_i\}_{i=1}^{n}$. Since this is finite, it has associated with it a subordinate partition of unity $\{\varphi_i\}_{i=1}^{n}$.

Define $f_i\colon X \longrightarrow I$ by

$$f_i(x) = f(x) \sum_{j=i+1}^{n} \varphi_j(x), \tag{3.20}$$

so that $f_0 = f$ and $f_n = 0$. Thus to prove the lemma it suffices to prove that $E_{f_{i-1}} \cong E_{f_i}$ for all $i \geq 1$; this is exactly the special case considered above, since f_{i-1} and f_i differ only inside the support of φ_i. □

Corollary 3.21. *If $f, g\colon X \longrightarrow Y$ are homotopic and $E \longrightarrow Y$ is a vector bundle over Y then f^*E and g^*E are isomorphic.*

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dd, bend right=60, "isomorphic"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \\ g^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

Proof. Let $h\colon X \times I \longrightarrow Y$ from f to g. Let $c_t\colon X \longrightarrow I$ be the constant function at t. By Lemma 3.19, the restrictions of h^*E to the graph of c_t is independent of t. By definition, f^*E (resp. g^*E) is isomorphic to the restriction of h^*E to X_{c_0} (resp. X_{c_1}). Since these restrictions are isomorphic, $f^*E \cong g^*E$. □

This completes Step 1 of the proof.

Step 2: Rephrasing as fiberwise-injective maps

In order to analyze ρ an alternate way of looking at vector bundles will be useful. We will need to be able to both construct an arbitrary bundle as a pullback of the universal bundle, and also show that, up to homotopy, this is unique. The goal is to construct a representation of vector bundles which is easier to compute with than the one we currently have. The key points here will be that fiberwise-injective maps $E \longrightarrow \mathbf{R}^\infty$ will correspond exactly to representations of E as a pullback of the universal bundle. Moreover, homotopic maps will correspond to homotopic representations. Given this representation, checking that ρ is surjective will correspond to a representation as a fiberwise-injective map existing, and checking that ρ is injective will correspond to checking that all fiberwise-injective maps are appropriately homotopic. In this step we develop the details of this representation.

Definition 3.22. Let $p\colon E \longrightarrow B$ be a vector bundle. A *fiberwise-injective map* $E \longrightarrow \mathbf{R}^\infty$ is a map which is a linear injection when restricted to $p^{-1}(b)$ for any $b \in B$.

Two fiberwise injections $g, g'\colon E \longrightarrow \mathbf{R}^\infty$ are *homotopic through fiberwise-injective maps* if there exists a homotopy $G\colon E \times I \longrightarrow \mathbf{R}^\infty$ such that for all $t \in I$, $G(\cdot, t)$ is fiberwise-injective.

Example 3.23. There is a fiberwise-injective map $\operatorname{proj}\colon \gamma_n \longrightarrow \mathbf{R}^\infty$ given by taking the composition

$\gamma_n := \{(\omega, v) \in G_n \times \mathbb{R}^\infty \mid \omega \in G_n + v \in \omega \subseteq \mathbb{R}^\infty\}$

$$\gamma_n \subseteq G_n \times \mathbf{R}^\infty \xrightarrow{\operatorname{proj}_2} \mathbf{R}^\infty.$$

Example 3.24. Suppose that B is compact and let $p\colon E \longrightarrow B$ be a rank-n fiber bundle. We can construct a fiberwise-injective map $E \longrightarrow \mathbf{R}^\infty$ as follows. Let $\{U_i\}_{i=1}^m$ be a finite subcover of the trivialization cover, and let $\{\varphi_i\}_{i=1}^m$ be a subordinate partition of unity. Over each U_i we can define a fiberwise-injective map

$$\widetilde{g}_i\colon p^{-1}(U_i) \xrightarrow{\tau_i} \overbrace{U_i \times \mathbf{R}^n}^{\text{trivialization cover}} \xrightarrow{pr_2} \mathbf{R}^n$$

$$\begin{array}{ccc} E & & U_i \times \mathbb{R}^n \to p^{-1}(U_i) \\ \downarrow p & & \downarrow \\ B & & U_i \end{array}$$

and then extend it to a map $g_i\colon E \longrightarrow \mathbf{R}^n$ (which will *not* be fiberwise-injective) by setting

$\varphi_i : U_i \to \mathbb{R}$ $\quad p : E \to X$

$$g_i(e) = \begin{cases} \underbrace{\varphi_i(p(e))}_{\text{scalar}}\underbrace{\widetilde{g}_i(e)}_{\text{vector}} & \text{if } e \in U_i \\ 0 & \text{otherwise.} \end{cases}$$

To assemble all of these to a fiberwise-injective map $g\colon E \longrightarrow \mathbf{R}^\infty$, we simply define

$$g(e) = (\overset{\in\mathbb{R}^n}{g_1(e)}, \overset{\in\mathbb{R}^n}{g_2(e)}, \ldots, \overset{\in\mathbb{R}^n}{g_m(e)}) \in (\mathbf{R}^n)^m \subseteq \mathbf{R}^\infty.$$

Thus we see that all bundles can be represented using fiberwise-injective maps. In fact, fiberwise injections represent more than just the bundle data: the set of fiberwise injections is in bijection with the representations of a bundle as a pullback of the universal bundle.

Proposition 3.25. *Let* proj *be the fiberwise-injective map* $\gamma_n \longrightarrow \mathbf{R}^\infty$ *defined in Example 3.23. Let* $p\colon E \longrightarrow B$ *be a vector bundle. There is a bijection*

$$\left\{ \begin{array}{c} \textit{pullback squares} \\ \begin{array}{ccc} E & \xrightarrow{f'} & \gamma_n \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \xrightarrow{f} & G_n \end{array} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{fiberwise-injective maps} \\ E \longrightarrow \mathbf{R}^\infty \end{array} \right\}.$$

sending the square on the left to $\mathrm{proj} \circ f'$.

Proof. The map $\mathrm{proj} \circ f'$ is fiberwise-injective, since the restriction of f' to any fiber is an isomorphism, and the composition of an isomorphism and an injection is an injection. On the other hand, suppose we are given a fiberwise-injective map $g\colon E \longrightarrow \mathbf{R}^\infty$. Define $f\colon B \longrightarrow G_n$ and $g'\colon E \longrightarrow \gamma_n$ by

$$f(b) = g(p^{-1}(\underbrace{p(e)}_{b})) \qquad \text{and} \qquad g'(e) = (f(p(e)), g(e)).$$

$= g(p^{-1}(b))$

The map g factors as $\mathrm{proj} \circ g'$. Thus we obtain a commutative square

$$\begin{array}{ccc} E & \xrightarrow{g'} & \gamma_n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_n} \\ B & \xrightarrow{f} & G_n. \end{array}$$

$$\begin{array}{ccccc} \gamma_n & \xleftarrow{g'} & E & \xrightarrow{g} & \mathbb{R}^\infty \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle p} & & \\ G_n & \xleftarrow{f} & B & & \end{array}$$

$E \xrightarrow{g'} \gamma_n \xrightarrow{\mathrm{proj}} \mathbb{R}^\infty$

To check that it's a pullback square, it suffices to construct an isomorphism between E and the fiber product (see Example 3.3)

$$B \times_{G_n} \gamma_n = \{(b, (\omega, x)) \in B \times \gamma_n \subseteq B \times (G_n \times \mathbf{R}^\infty) \mid f(b) = \omega\}$$

compatible with the projection maps; this is the map

$$\overset{\in E}{e} \longmapsto (\underbrace{p(e)}_{b}, \overbrace{(f \circ \underbrace{p(e)}_{b}, g(e))}^{\in G_n \times \mathbb{R}^\infty})\,.$$

This is a fiberwise isomorphism because g is fiberwise-injective and the fibers are finite-dimensional vector spaces, and is therefore a bundle isomorphism, as desired.

This gives functions in both directions which are mutually inverse, so it is a bijection, as desired. $\square$

Moreover, if two fiberwise-injective maps are homotopic through fiberwise-injective maps, the bottom maps in the corresponding pullback squares are also homotopic:

Lemma 3.26. *Let* $G\colon E \times I \longrightarrow \mathbf{R}^\infty$ *be a homotopy through fiberwise-injective maps. Then the maps* $B \longrightarrow G_n$ *corresponding to* $G(\cdot, 0)$ *and* $G(\cdot, 1)$ *are homotopic.*

Proof. Consider the map $F\colon B \times I \longrightarrow G_n$ given by $F(b,t) = G(p^{-1}(b) \times \{t\})$. By definition, $G(p^{-1}(b) \times \{t\})$ is an n-dimensional subspace of $\mathbf{R}^\infty$, and thus gives a point in G_n. This is continuous because G is, and thus gives a homotopy as desired. $\square$

Thus to show that there is exactly one homotopy class of maps $B \longrightarrow G_n$ corresponding to a vector bundle, it suffices to show that all fiberwise-injective maps are homotopic through fiberwise-injective maps.

Lemma 3.27. *Let $p \colon E \longrightarrow B$ be a vector bundle. Any two fiberwise-injective maps $E \longrightarrow \mathbf{R}^\infty$ are homotopic through fiberwise-injective maps.*

Proof. Let $g_0, g_1 \colon E \longrightarrow \overset{\mathbf{R}^\infty}{\cancel{B}}$ be the two fiberwise-injective maps. Whenever $g_0(e) \neq 0$ it must also be the case that $g_1(e) \neq 0$, since g_i can only map the 0 in each fiber to 0 (since the restriction to each fiber is a *linear* injection).

It is tempting to define G by setting $G(e,t) = g_0(e)t + g_1(e)(1-t)$. However, in the case when $g_0(e), g_1(e) \neq 0$ but $g_1(e) = \lambda g_0(e)$ for some negative scalar λ, this will have a problem: when $t = -\lambda/(1-\lambda)$ this will be 0, and $G(-,t)$ will not be injective on the fiber containing e. Luckily, this is the *only* thing that can go wrong, and thus this formula shows that g_0 and g_1 are homotopic through fiberwise injections if it is never the case that $g_0(e) = \lambda g_1(e)$ for any e with $g_0(e) \neq 0$.

Moreover, the relation "homotopic through fiberwise-injective maps" is an equivalence relation, so it suffices to construct a chain of maps which are each homotopic to each other through fiberwise-injective maps.

Consider the injection $L_0 \colon \mathbf{R}^\infty \longrightarrow \mathbf{R}^\infty$ defined by

$$(a_1, a_2, a_3, \dots) \longmapsto (a_1, 0, a_2, 0, a_3, \dots).$$

Both g_0 and $L_0 \circ g_0$ are fiberwise-injective, and there does not exist an e such that $g_0(e) = \lambda L_0(g_0(e))$ for a negative λ. Thus the above formula shows that g_0 and $L_0 \circ g_0$ are homotopic through fiberwise-injective maps. Analogously define $L_1 \colon \mathbf{R}^\infty \longrightarrow \mathbf{R}^\infty$ to send $(a_1, a_2, \dots)$ to $(0, a_1, 0, a_2, 0, \dots)$, so that g_1 and $L_1 \circ g_1$ are homotopic through fiberwise-injective maps. The maps $L_0 \circ g_0$ and $L_1 \circ g_1$ are homotopic through fiberwise-injective maps, since they never share any nonzero coordinates Thus g_0 and g_1 are homotopic through fiberwise-injective maps, as desired. □

Step 3: Checking bijectivity

With the above results, we can now prove that ρ is a bijection.

Proof that ρ is a bijection. First, consider surjectivity. Given a rank-n vector bundle $p \colon E \longrightarrow B$, it can be represented as a pullback of the universal bundle if and only if (by Proposition 3.25) the set of fiberwise injections $E \longrightarrow \mathbf{R}^\infty$ is nonempty. By Example 3.24, these always exist, so ρ is surjective.

Now consider injectivity. Suppose that $\rho([f]) = \rho([f'])$, so that there exists an isomorphism $\alpha\colon f^*(\gamma_n) \longrightarrow (f')^*(\gamma_n)$. In particular, this means that there exists a diagram

$$\begin{tikzcd}
\gamma_n \arrow[d] & f^*\gamma_n \arrow[l] \arrow[rr, "\alpha"] \arrow[dr] \arrow[rr, bend left, leftrightarrow, "\cong"] & & (f')^*\gamma_n \arrow[r] \arrow[dl] & \gamma_n \arrow[d] \\
G_n & & B \arrow[ll, "f"] \arrow[rr, "f'"] & & G_n
\end{tikzcd}$$

In this diagram, both the left square and the right square are pullback squares, and thus correspond to fiberwise injections $f^*\gamma_n \longrightarrow \mathbf{R}^\infty$ and $(f')^*\gamma_n \longrightarrow \mathbf{R}^\infty$. Since all fiberwise-injective maps are homotopic through fiberwise-injective maps, (by Lemma 3.27) applying Lemma 3.26 shows that f and f' are homotopic. Thus $[f] = [f']$, as desired. $\square$

Set Here

3.4 Beyond compactness

Theorem 3.13 is beautiful, but somewhat unsatisfying. Firstly, although compact spaces arise often, we often want to work with more general spaces. Moreover, the space G_n is itself not compact, and so it appears that we are classifying all vector bundles on compact spaces using a structure on a noncompact space (which is aesthetically unsatisfying). It turns out that the above proof actually works in a much wider class of spaces, which will in particular include all CW-complexes (and thus also G_n).

In order to do this, let us inspect the proof above to see where compactness was used:

(a) In the proof of Lemma 3.19 it is used to ensure that the partition of unity $\{\varphi_i\}_{i=1}^n$ exists, in order to define the maps f_i (see (3.20)).

(b) In the same proof it is also used because the final isomorphism is a composition of m isomorphisms $E_{f_{i-1}} \longrightarrow E_{f_i}$.

(c) In Example 3.24 it is used to ensure that the partition of unity $\{\varphi_i\}_{i=1}^m$ exists, in order to extend the maps $\widetilde{g}_i$ continuously to all of E.

(d) The finiteness of m is also used in order to have $(\mathbf{R}^n)^m \subseteq \mathbf{R}^\infty$. This portion of the proof will still work if m is *countably* infinite, although not if it is uncountably infinite, since almost all of the coordinates in the function we construct will be 0.