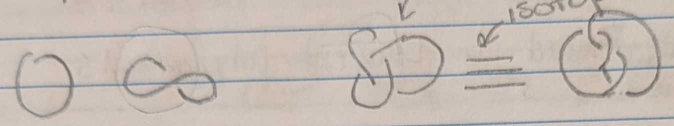
Notes on Khovanov Homology

A **Knot** K is an embedding $S^1 \hookrightarrow S^3$

Examples:

mirror is Rh tref.

isotopic



unknots   Trefoil   Trefoil

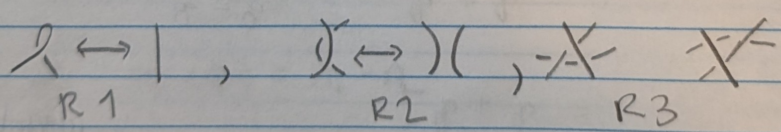
$\underbrace{\qquad\qquad\qquad}_{\text{mention isotopy (Reidemeister)}}$

* show we can orient (add arrows to pics)
* mention mirror image ($m(K)$)
* Fact: Rh trefoil $\not\equiv$ Lh tref.

A **Link** (n-component) is a smooth embedding $\underbrace{S^1 \times \cdots \times S^1}_{n} \hookrightarrow S^3$

* Again, can orient like knots

Reidemeister moves:



R1   R2   R3

* Note local (can rotate, mirror, etc)

A **link Invariant** F valued in category $\mathcal{C}$ is a machine that
1) takes in link diagram D
2) outputs $F(D) \in Ob(\mathcal{C})$

s.t. $D \sim D' \Rightarrow F(D) = F(D')$

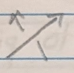
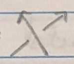
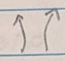
* Basically assigns a link something that it will assign an equiv. link as well

Jones Poly via skein relation

* Fun fact : ~~First~~ first poly invariant was Alexander poly in the 1920s. Jones poly was discovered by Vaughn Jones in the 1980s.

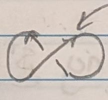
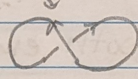
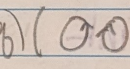
* Link invariant valued in Laurent polynomials

<u>Def.</u>
- Base case $J(O) = 1$
- skein relation : $q^{-2} J(L_+) - q^2 J(L_-) = (q^{-1} - q) J(L_0)$

$L_+ =$  $L_- =$  $L_0 =$ 

<u>Ex.</u>

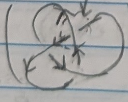
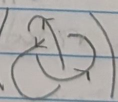
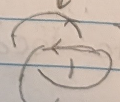
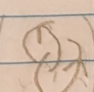
unknot

q^{-2}  $- q^2$  $= (q^{-1} - q)($  $)$

$q^{-2} \cdot 1 - q^2 \cdot 1 = (q^{-1} - q) J(O\,O)$

$\Rightarrow J(O\,O) = \dfrac{q^{-2} - q^2}{q - q^{-1}} = \dfrac{(q - q^{-1})(q + q^{-1})}{q - q^{-1}}$

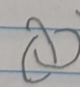
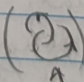
$= q + q^{-1}$

<u>Ex</u> $J($  $):$ $q^{-2}($  $) - q^2($  $) = (q^{-1} - q)($  $)$

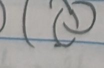
change

unknot

L_+ L_- L_0

$q^{-2}($  $) - q^2 = (q^{-1} - q)($  $)$

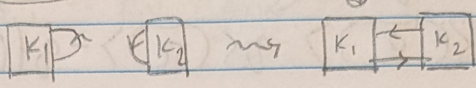
$= q + q^5$

$\Rightarrow J($  $) = \boxed{q^2 - q^6 - q^8}$

Facts: $J(L) = J(L^r)$

reverse orientation

$J(K_1 \# K_2) = J(K_1)J(K_2)$



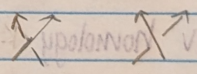
* Open Q: Does J detect unknot i.e. $J(n)=1 \Rightarrow$ unknot?

Jones poly from Kauffman Bracket

Kauffman Bracket : ① $\langle \varnothing \rangle = 1$
(unnormalized) ② $\langle O L \rangle = (q+q^{-1})\langle L \rangle$ * Look familiar?
③ $\langle \times \rangle = \langle)(\rangle - q \langle \asymp \rangle$

* note unoriented 0-resolution 1-resolution

Let n = # of crossings
n_+ = # of positive crossings
n_- = # of negative

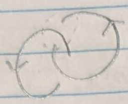


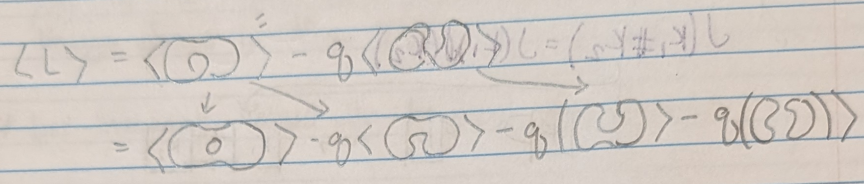
writhe $wr(D) = n_+ - n_-$

Then can write $J(L) = (-q)^{-n_-} \cdot q^{wr(L)} \langle L \rangle$

Alternatively could define oriented ③ as follows

positive: $\langle \times \rangle = q \langle)(\rangle - q^2 \langle \asymp \rangle$
negative: $\langle \times \rangle = q^{-2} \langle \asymp \rangle - q^{-1} \langle)(\rangle$

Example:  $n=2,\ n_+=2,\ n_-=0$



$= q^4+q^2+1+q^{-2}$

$\Rightarrow \hat{J} = (-1)^0 q^2(q^4+q^2+1+q^{-2}) = q^6+q^4+q^2+1$

* Might be thinking not what I said earlier, but normalized $\left(\frac{\hat{J}(H)}{q+q^{-1}}\right) = \boxed{q^5+q}$

Jones poly via Khovanov bracket

Khovanov homology, $Kh(\cdot)$, is a $(\mathbb{Z}\oplus\mathbb{Z})$-graded (co)homology theory whose graded Euler char. recovers the Jones poly.

*Extra grading is usually called quantum or internal grading

finite rank $\forall i,j$ — * collection of modules equipped w/ 2 differential maps

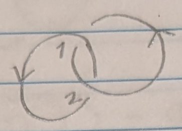
Def. Let $C = \left(\bigoplus_{i,j\in\mathbb{Z}} C^{i,j}, d\right)$ be a bigraded chain complex of $\mathbb{Z}$-modules.

The graded Euler characteristic of C is

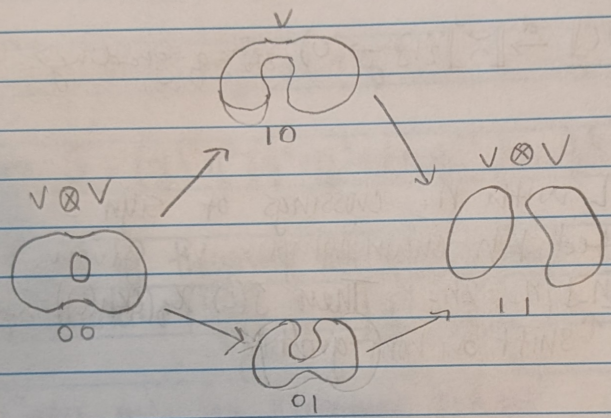
$$\chi_q(C) = \sum_{i,j} (-1)^i q^j \operatorname{rank} H^{i,j}(C)$$

* recall χ of CW-complex can be computed using any CW decomp $\Rightarrow$ don't need to know differential to compute $\chi_q(C)$

Example: Back to Hopf link



Cube of complete resolutions:



Each complete resolution corresponds to binary string $u \in \{0,1\}^2$, perform quantum grading shift $|u|$ on associated $\mathbb{Z}$-module, $|u|$ is number of 1's appearing in bitstring u.

Kh bracket: $[\![H]\!] = (\underline{V \otimes V} \to V \oplus V\{1\} \to V \otimes V\{2\})$

Kh Chain complex:

$$CKh(H) = (\underline{V \otimes V} \to V \oplus V\{1\} \to V \otimes V\{2\})[n_-]\{n_+ - 2n_-\}$$

$$= (\underline{V \otimes V} \to V \oplus V\{1\} \to V \otimes V\{2\})[0]\{2\}$$

$$= (\underline{V \otimes V}\{2\} \to V \oplus V\{3\} \to V \otimes V\{4\})$$

				χ
$gr_q = 6$	0	0	1	1
$gr_q = 4$	1	2	2	1
$gr_q = 2$	2	2	1	1
$gr_q = 0$	1	0	0	1
	$gr_h = 0$	$gr_h = 1$	$gr_h = 2$	

$C^0 = V \otimes V\{2\}$

$V = \mathbb{Z}v_+ \oplus \mathbb{Z}v_-$

	deg	Shift 2
$v_+ \otimes v_+$:	2	4
$v_- \otimes v_+$:	0	2
$v_+ \otimes v_-$:	0	2
$v_- \otimes v_-$:	-2	0

$\Rightarrow \chi_q(Kh(H)) = q^6 + q^4 + q^2 + 1 = \hat{J}(H)$