~~$d^0(v_1 \otimes v_2 \otimes v_3) = (m(v_1 \otimes v_2) \otimes v_3$~~

Left Handed Trefoil:

$q \backslash g_h$	0	-1	-2	-3
-1	$\mathbb{Z}$			
-3	$\mathbb{Z}$			
-5			$\mathbb{Z}$	
-7			$\mathbb{Z}_2$	
-9				$\mathbb{Z}$

Right Handed Trefoil

$q \backslash g_h$	0	1	2	3
1	$\mathbb{Z}$			
3	$\mathbb{Z}$			
5			$\mathbb{Z}$	
7				$\mathbb{Z}_2$
9				$\mathbb{Z}$

Khovanov Homology:

~~Let~~ Let V be the free graded $\mathbb{Z}$ module generated by v_+, v_-. Let degree of ~~v_+, v_- be~~ an element x, which is the tensor prod of v_+ and v_- be defined by

~~$p(v_+) = +1 \quad p(v_-) = -1$~~

$p(x) = \# v_+ - \# v_-$

Let L be a link/knot with n_+ +ve crossings and n_- negative crossings. To find the Khovanov homology

~~(1) Draw the cub~~

(1) Label the crossings (and edges, if necessary) and draw the cube of resolutions. Each vertex in the cube is a binary string of n bits (where n= no. of crossings).

(2) ~~Let~~ For each ~~string~~ ~~let r be the sum of its bits. Gather the vertic resolutions of 1~~

~~For~~ Each.

② For each string α, define,

$V_\alpha(L) = V^{\otimes k_\alpha}$ where k_α = # of distinct circles in the resolution corr. to α.

If $|\alpha|$ counts the sum of bits of ~~a string~~ α

define $V^{\bar{r}} := \bigoplus_{|\alpha| = r} V_\alpha(L)$

~~where $|\alpha|$ = sum of bits of α.~~

This free $\mathbb{Z}$ module has ~~Homological~~ (co)Homological grading g_h defined by,

$$g_h(v^i) = i - n_-$$

③ The differentials are constructed in the following way:

→ When moving from any resolution at $\bar{r}$-level to a resolution at $(\bar{r}+1)$-level, ~~there is a~~ consider the changes in 1-bit only. (The other bits remain unaffected). Denote these changes by arrows whose head has 1 and tail has 0 in the ~~bit~~ changing bit.

→ Geometrically, this change either merges 2 circles or splits a circle in 2. This gives rise to 2 maps

Merge map. $m : V \otimes V \longrightarrow V$

$$v_+ \otimes v_+ \longrightarrow v_+ \qquad v_+ \otimes v_- \longrightarrow v_-$$
$$v_- \otimes v_+ \longrightarrow v_- \qquad v_- \otimes v_- \longrightarrow 0$$

Split map $\Delta : V \longrightarrow V \otimes V$

$$v_+ \longrightarrow v_+ \otimes v_- + v_- \otimes v_+$$
$$v_- \longrightarrow v_- \otimes v_-.$$

Then each arrow is either a merge or split. Denote it by d_α^* for the lower ~~bit~~ string α.

→ Each d_α will be accompanied by a sign. If the position of the changed bit is denoted by $*$, then, the sign of d_α is determined as follows: Take the sum of all bits in α less than $*$. Call it $\xi_{j\alpha}$. Then the sign of d_α is $(-1)^{\xi_{j\alpha}}$. Then the differential $d^r : V^r \to V^{r+1}$ is $\sum_{|\alpha|=r} (-1)^{\xi_{j\alpha}} d_\alpha$.

(4) Sometimes it is necessary to indicate which circles are merging or splitting. When necessary, label each edge. Label ~~Denote~~ each circle in the resolution of α by the minimum edge it contains. The merge map m_{ij} merging circles i, j acts like identity on other circles but takes $V^i \otimes V^j \to V_{\min\{i,j\}}$. Similarly the split map Δ^{ij} splitting circles i, j acts as identity on other circles but splits $V_{\min\{i,j\}} \to V^i \otimes V^j$.

(5) After calculating the homology groups, calculate the quantum grading of each generator by the formula

$$q(x) = p(x) + r + n_+ - 2n_-.$$

Quantum gradings:

$g_h = 0$: $q(v_- \otimes v_-) = -2+0+2-2(0) = 0$

$q(v_- \otimes v_+ - v_+ \otimes v_-) = q(v_- \otimes v_+) = 0+0+2-2(0) = 2$

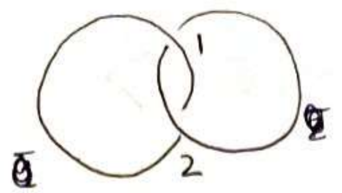
$g_h = 2$: $q(v_+ \otimes v_+) = 2+2+2-2(0) = 6$

$q(v_- \otimes v_+) = 0+2+2-2(0) = 4$

Chart:

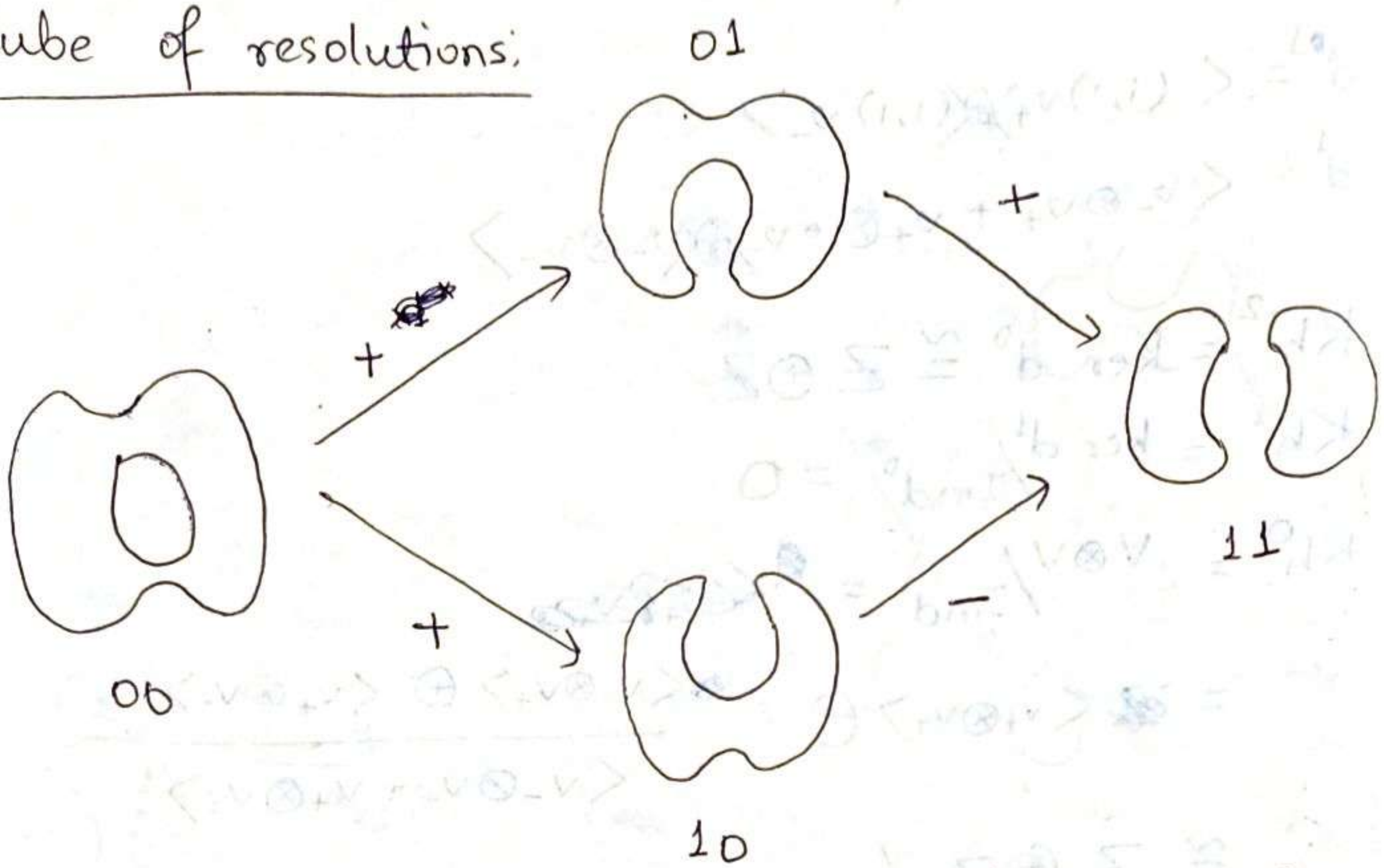
$q \backslash g_h$	0	1	2
0	$\mathbb{Z}$		
2	$\mathbb{Z}$		
4			$\mathbb{Z}$
6			$\mathbb{Z}$

Hopf Link:



$n_+ = 0$
$n_- = 2$

Cube of resolutions:



$$\underset{\substack{r=0 \\ g_h=-2}}{V^{\otimes 2}} \xrightarrow{d^0} \underset{\substack{r=1 \\ g_h=-1}}{V \oplus V} \xrightarrow{d^1} \underset{\substack{r=2 \\ g_h=0}}{V^{\otimes 2}}$$

$d^0(v_1 \otimes v_2) = (m(v_1, v_2),\ m(v_1, v_2))$

$d^1(v_1, v_2) = \Delta v_1 - \Delta v_2$

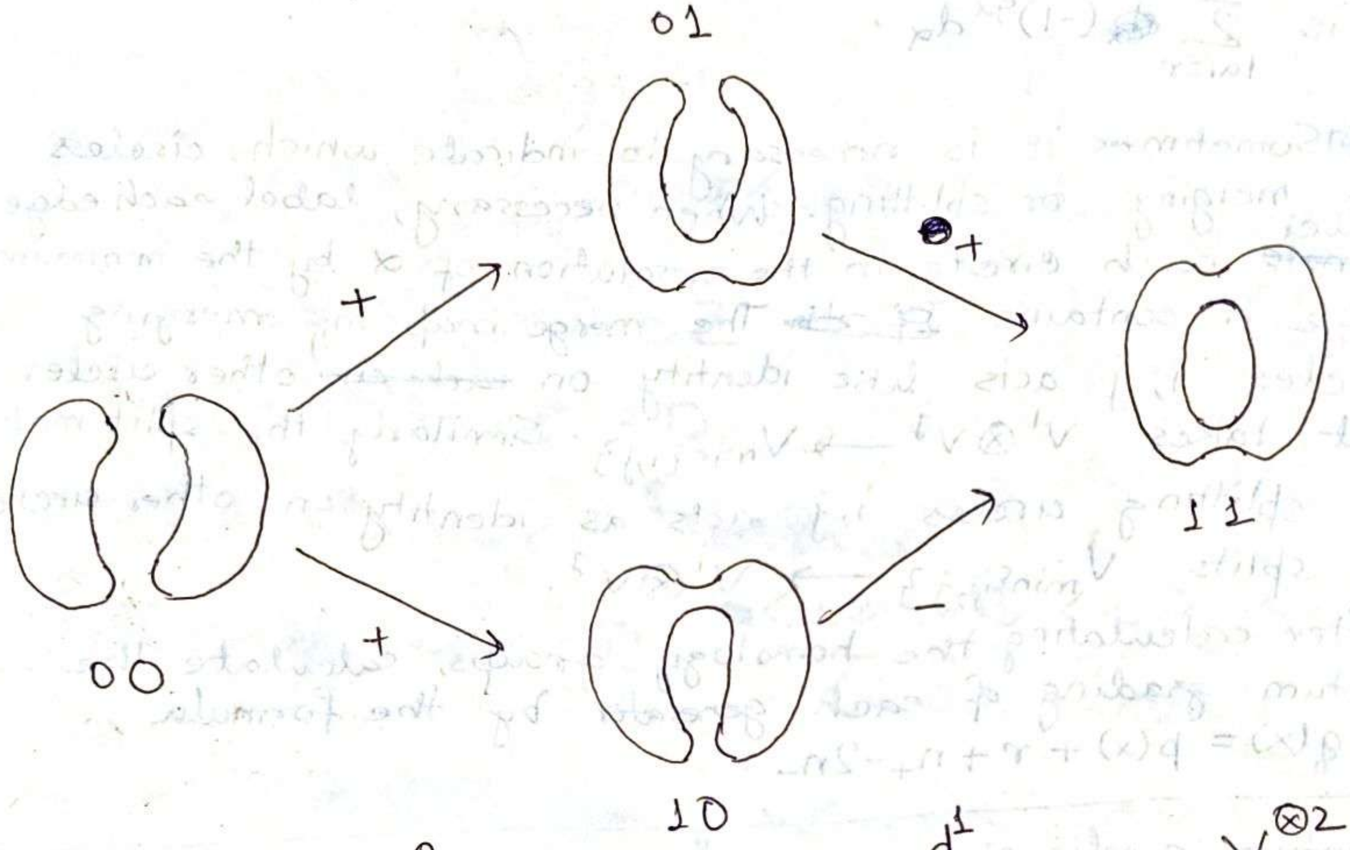
$V \otimes V = \langle v_+ \otimes v_+ \oplus v_- \otimes v_+ \oplus v_+ \otimes v_- \oplus v_- \otimes v_- \rangle$

$V \oplus V = \langle (v_+, 0) \oplus (0, v_+) \oplus (v_-, 0) \oplus (0, v_-) \rangle$

$$\therefore\ d^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad d^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Hopf Link #2:

$n_+ = 2$
$n_- = 0$



$$\underset{\substack{r=0 \\ g_h = 0}}{V^{\otimes 2}} \xrightarrow{d^0} \underset{\substack{r=1 \\ g_h = 1}}{V \oplus V} \xrightarrow{d^1} \underset{\substack{r=2 \\ g_h = 2}}{V^{\otimes 2}}$$

$$V \otimes V = \mathbb{Z}(v_+ \otimes v_+) \oplus \mathbb{Z}(v_- \otimes v_+) \oplus \mathbb{Z}(v_+ \otimes v_-) \oplus \mathbb{Z}(v_- \otimes v_-)$$

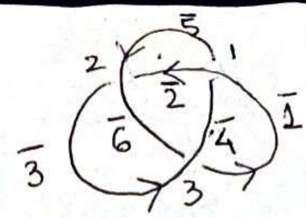
$$V \oplus V = \mathbb{Z}((1,0)v_+) \oplus \mathbb{Z}((0,1)v_+) \oplus \mathbb{Z}((1,0)v_-) \oplus \mathbb{Z}((0,1)v_-)$$

$$d^0(v_1 \otimes v_2) = (m(v_1, v_2), m(v_1, v_2))$$

$$d^1((v_1, v_2)) = \Delta v_1 - \Delta v_2$$

$d^0 =$

Left Handed Trefoil:



$n_+ = 0$
$n_- = 3$

$\ker d^0 = \langle v_- \otimes v_- \rangle \oplus \langle v_- \otimes v_+ - v_+ \otimes v_- \rangle$

$\operatorname{Im} d^0 = \langle (1,1) v_+ \rangle \oplus \langle (1,1) v_- \rangle$

$\ker d^1 = \langle (1,1) v_+ \rangle \oplus \langle (1,1) v_- \rangle$

$\operatorname{Im} d^1 = \langle v_- \otimes v_+ + v_+ \otimes v_- \rangle \oplus \langle v_- \otimes v_- \rangle$

$\therefore \quad Kh^{-2} = \ker d^0 \cong \mathbb{Z} \oplus \mathbb{Z}$

$Kh^{-1} = \ker d^1 / \operatorname{Im} d^0 = 0$

$Kh^0 = V \otimes V / \operatorname{Im} d^1 =$

$= \langle v_+ \otimes v_+ \rangle \oplus \dfrac{\langle v_- \otimes v_+ \rangle \oplus \langle v_+ \otimes v_- \rangle}{\langle v_- \otimes v_+ + v_+ \otimes v_- \rangle}$

$\cong \mathbb{Z} \oplus \mathbb{Z}$. (gen. by $v_+ \otimes v_+$ and $v_- \otimes v_+$)

Quantum gradings of gen.:- $\quad q(x) = p(x) + r + n_+ - 2n_-$

$p(x) = \# v_+ - \# v_-$

-2
$q(v_- \otimes v_-) = -2 + 0 + 0 - 2(2) = -6$

$q(v_- \otimes v_+ - v_+ \otimes v_-) = q(v_- \otimes v_+) = 0 + 0 + 0 - 2(2) = -4$

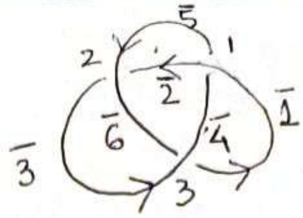
0
$q(v_+ \otimes v_+) = 2 + 2 + 0 - 2(2) = 0$

$q(v_- \otimes v_+) = 0 + 2 + 0 - 2(2) = -2$

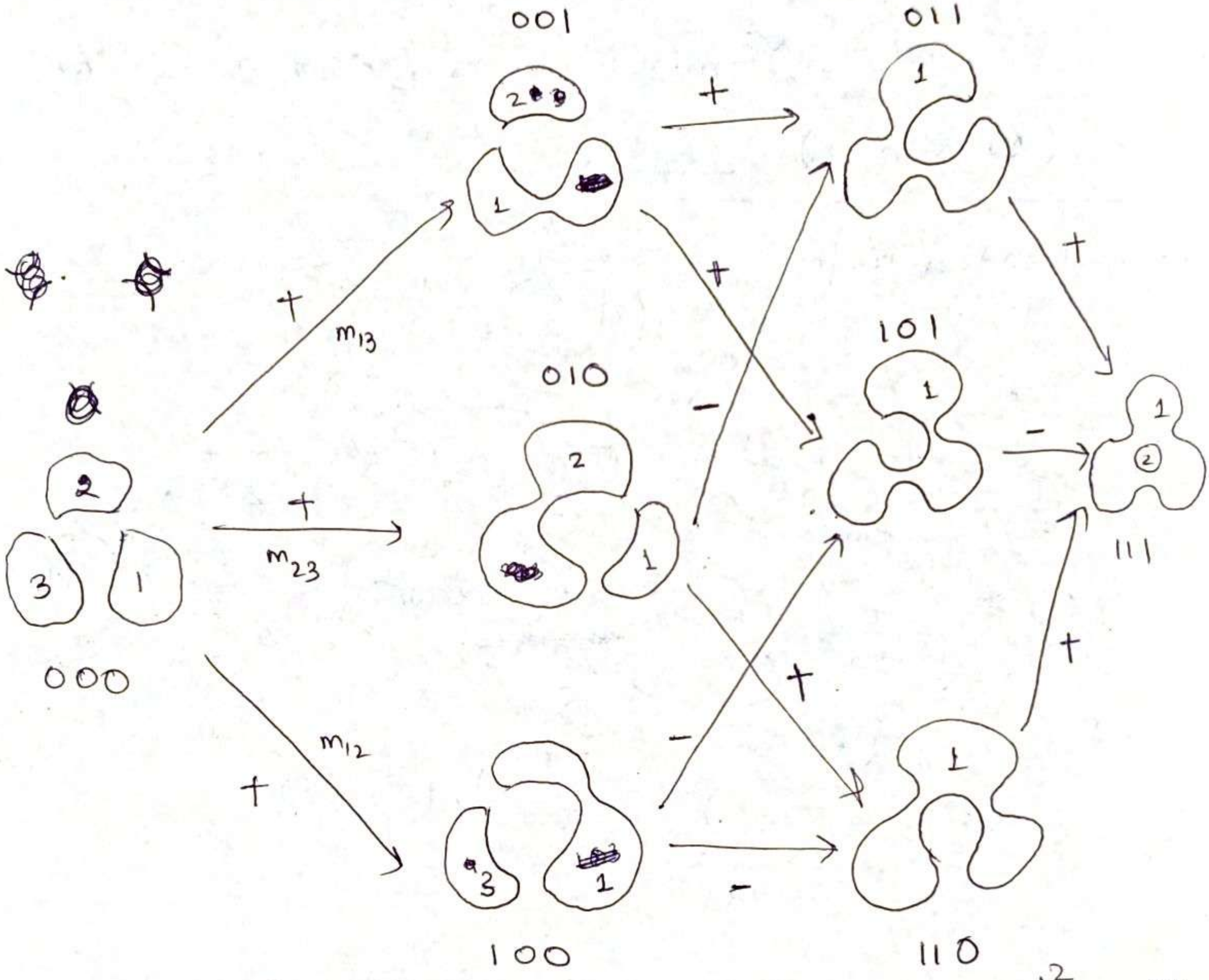
$\therefore$ Chart:

$q \backslash gh$	-2	-1	0
0			$\mathbb{Z}$
-2			$\mathbb{Z}$
-4	$\mathbb{Z}$		
-6	$\mathbb{Z}$		

Left Handed Trefoil:



$n_+ = 0$
$n_- = 3$



$$V^{\otimes 3} \xrightarrow{d^0} V^{\otimes 2} \oplus V^{\otimes 2} \oplus V^{\otimes 2} \xrightarrow{d^1} V \oplus V \oplus V \xrightarrow{d^2} V^{\otimes 2}$$

$r=0$	$r=1$	$r=2$	$r=3$
$g_h=-3$	$g_h=-2$	$g_h=-1$	$g_h=0$

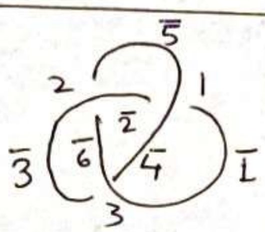
$\cancel{d^0(v_1 \otimes v_2 \otimes v_3) = (m(v_1 \otimes v_2) \otimes v_3, \ldots}$

$$d^0(v_1 \otimes v_2 \otimes v_3) = \big(m(v_1 \otimes v_3) \otimes v_2,\ v_1 \otimes m(v_2 \otimes v_3),\ m(v_1 \otimes v_2) \otimes v_3\big)$$

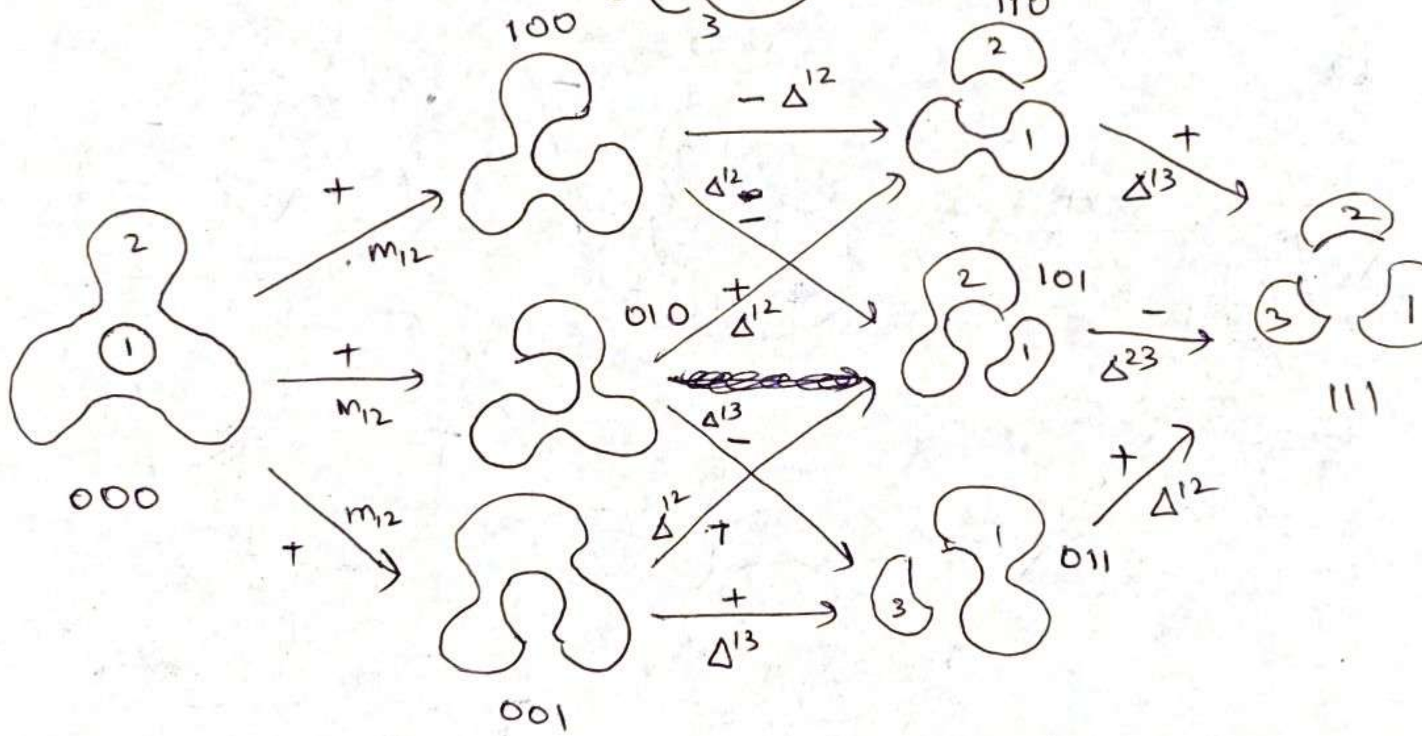
$$d^1(v_1 \otimes v_2,\ v_3 \otimes v_4,\ v_5 \otimes v_6) = \big(m(v_1 \otimes v_2) - m(v_3 \otimes v_4),\ m(v_1 \otimes v_2) - m(v_5 \otimes v_6),\ m(v_3 \otimes v_4) - m(v_5 \otimes v_6)\big)$$

$$d^2((v_1, v_2, v_3)) = \Delta v_1 - \Delta v_2 + \Delta v_3$$

Right Handed Trefoil:



$n_+ = 3$
$n_- = 0$



$$V^{\otimes 2} \longrightarrow V \oplus V \oplus V \longrightarrow V^{\otimes 2} \oplus V^{\otimes 2} \oplus V^{\otimes 2} \longrightarrow V^{\otimes 3}$$

$r = 0$, $g_h = 0$ $\qquad$ $r = 1$, $g_h = 1$ $\qquad$ $r = 2$, $g_h = 2$ $\qquad$ $r = 3$, $g_h = 3$

$$d^0(v_1 \otimes v_2) = \big(m(v_1 \otimes v_2),\ m(v_1 \otimes v_2),\ m(v_1 \otimes v_2)\big)$$

$$d^1((v_1, v_2, v_3)) = (\Delta v_2 - \Delta v_1,\ \Delta v_3 - \Delta v_1,\ \Delta v_3 - \Delta v_2)$$

$$d^2((v_1 \otimes v_2,\ v_3 \otimes v_4,\ v_5 \otimes v_6)) = (\Delta v_1) \otimes v_2 - v_3 \otimes (\Delta v_4) + (\Delta v_5) \otimes v_6.$$