Introduction to Spectra

- Recall homotopy groups for topological spaces.

$$\pi_n(X) := [(S^n, *), (X, *)]$$

- How do we compute them?
- How do we compute Homology groups?
  - pushouts don't respect homotopy.

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \\ D^2 \arrow[r] & S^2 \end{tikzcd} \cong \begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd} \sim \begin{tikzcd} S^1 \arrow[r, hook] \arrow[d] & \Delta \arrow[d] \\ \Delta \arrow[r] & \Diamond \cong S^2 \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "h"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} := \begin{tikzcd} A \arrow[r] \arrow[d] & M_f \arrow[d] \\ M_g \arrow[r] & D \end{tikzcd}$$

Exercise: replacing one of the corner is good enough.

$$\cdots \to H_i(A) \to H_i(B) \oplus H_i(C) \to H_i(D) \to H_{i-1}(A) \to \cdots$$

(Mayer-Vietoris sequence)

$$\begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, "h"] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd} := \begin{tikzcd} P_f \times_B P_g \arrow[r] \arrow[d] & P_g \arrow[d] \\ P_f \arrow[r] & B \end{tikzcd}$$

$P_f = \{(x, \gamma) \in A \times B^I, \gamma(0) = f(x)\}$

$$\to \pi_i(D) \to \pi_i(A) \oplus \pi_i(C) \to \pi_i(B) \to \pi_{i-1}(D) \to \cdots$$

Cofiber sequence: $\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, "h"] & B \arrow[d] \\ * \arrow[r] & C \end{tikzcd}$, $A \to B \to C$ h-cofiber sequence. $A \hookrightarrow X \to X/A$

Fiber sequence: $\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, "h"] & B \arrow[d] \\ * \arrow[r] & C \end{tikzcd}$ $A \to B \to C$ h-fiber sequence. $F \to E \to B$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, "h"] & B \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & C_f \arrow[r] & \Sigma A \end{tikzcd}$$

$A \to B \to C_f \to \Sigma A$ ← looks like a distinguished triangle.

$B \to C_f \to \Sigma A \to \Sigma B$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ A \arrow[r, "g"] & B \end{tikzcd} \sim \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & C_f \arrow[r] \arrow[d] & \Sigma A \arrow[d] \\ A \arrow[r] & B \arrow[r] & C_g \arrow[r] & \Sigma B \end{tikzcd}$$

$\Sigma \dashv \Omega$. they are adjunct. not inverses!

$A \xrightarrow{f} B \sim F_f \to A \to B$ | Σ / Ω are not automorphism.

What we want: cofiber and fiber sequence to coincide.

Blakers-Massey Excision:

$$\begin{tikzcd}
Q \arrow[rr, "m\text{-conn}"] \arrow[dr, "\cong"', "m+n-1\text{-connec}"] \arrow[dd, "n\text{-connec}"'] & & A \arrow[dd] \\
& R \arrow[ur] \arrow[dl] & \\
B \arrow[rr] & & P
\end{tikzcd}$$

← highly connected.

A map $A \xrightarrow{f} B$ is <u>m-connected</u>

if $\pi_i(A) \cong \pi_i(B)$ $i \le m-1$

epi for $i = m$.

homotopy pushout / pullback
coincide up to some level

we want to get rid of <u>connectivity</u>.

<u>observe</u>:

$A \longrightarrow B$ is <u>m-connected</u>.

$\Omega A \longrightarrow \Omega B$ is <u>m-1 connected</u>.

$$\begin{array}{ccccc}
[S^i, \Omega A] = & \pi_i(\Omega A) & \longrightarrow & \pi_i(\Omega B) & i \le m-2. \\
\wr\! \mid & \wr\! \mid & & \wr\! \mid & \\
[S^{i+1}, A] = & \pi_{i+1}(A) & \longrightarrow & \pi_{i+1}(B) & \text{iso for } i+1 \le m-1.
\end{array}$$

$A^2 \longrightarrow B^2$ <u>n-connected</u>. ← deloop it.

if $\exists A^1$: $\underline{\Omega A^1 = A^2} \simeq \Omega B^2 = B^1$

$A^1 \longrightarrow B^1$ <u>n+1-connected</u>.

delooping <u>increases</u> connectivity.

PeterMay (Operads) (E-spaces)

$$\left\{\begin{array}{l}
X := X_0,\ X_1,\ X_2,\ X_3,\ \ldots \\
\Omega X_1 = X_0,\quad X_1 \simeq \Omega X_2,\quad X_2 \simeq \Omega X_3. \\
X_0 \quad \Sigma X_0. \qquad\qquad X_n \xrightarrow{\simeq} \Omega X_{n+1}
\end{array}\right.$$

This is called <u>Ω-spectra</u>.

Note. $X_0 \simeq \Omega X_1$ $\quad X_0 \longrightarrow \Omega X_1$.

$\Rightarrow$ $\underline{\Sigma X_0 \longrightarrow X_1}$ $\quad X_i \simeq \Omega X_{i+1}$

$\Sigma X_i \longrightarrow X_{i+1}$.

Definition of a Spectrum:

Spectra X, is collection of spaces,

- $X_0, X_1, X_2, \dots$
- $\Sigma X_i \xrightarrow{\sigma_i} X_{i+1}$ ← structure map.
- a map between two spectra, $f: X \to Y$,

$$
\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f_0"'] & X_1 \arrow[r] \arrow[d, "f_1"'] & X_2 \arrow[r] \arrow[d, "f_2"'] & \cdots \\
Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots
\end{tikzcd}
$$

$$\Updownarrow$$

$$
\begin{tikzcd}
\Sigma X_i \arrow[r, "\sigma_i"] \arrow[d, "\Sigma f_i"'] \arrow[dr, phantom, "\circlearrowright"] & X_{i+1} \arrow[d, "f_{i+1}"] \\
\Sigma Y_i \arrow[r, "\sigma_i"'] & Y_{i+1}
\end{tikzcd}
$$

- homotopy pushout of spectrum

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "h.p."] & Y \arrow[d] \\
Z \arrow[r] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
X_n \arrow[r] \arrow[d] \arrow[dr, phantom, "h.p."] & Y_n \arrow[d] \\
Z_n \arrow[r] & W_n
\end{tikzcd}
$$

this square to commute. ←

$$
\begin{tikzcd}
\Sigma X_n \arrow[rr] \arrow[dr] \arrow[dd] & & \Sigma Y_n \arrow[dr] \arrow[dd] & \\
& X_{n+1} \arrow[rr] \arrow[dd] & & Y_{n+1} \arrow[dd] \\
\Sigma Z_n \arrow[rr] \arrow[dr] & & \Sigma W_n \arrow[dr, "h.p."] & \\
& Z_{n+1} \arrow[rr] & & W_{n+1}
\end{tikzcd}
$$

- $(X \vee Y)_n = X_n \vee Y_n$.

$(X \times Y)_n = X_n \times Y_n$.

$$
\begin{tikzcd}[column sep=small]
\Sigma (X \vee Y)_n = S^1 \wedge (X_n \vee Y_n) = (S^1 \wedge X_n) \arrow[d, "\sigma_n"] & \vee & (S^1 \vee Y_n) \arrow[d] \\
X_{n+1} & \vee & Y_{n+1}
\end{tikzcd}
$$

$X \wedge Y$. ← there is a way to define it.

- Eilenberg-Maclane Spectra.

$K(A,n)$,

$$\pi_i(\Omega K(A,n)) = \pi_{i+1}(K(A,n))$$

$$= \begin{cases} A & i+1=n, \text{ or } i=n-1 \\ 0 & \text{elsewhere} \end{cases}$$

$$= \pi_i(K(A,n-1)).$$

$$\Omega K(A,n) \simeq K(A,n-1).$$

$HA \leftarrow$ Eilenberg Maclane spectra.

$$\tilde{H}^n(X,A) \cong [X, K(A,n)].$$

- Thom spectra,
- Complex K theory spectra.
- $S^0, S^1, S^2, \cdots$

$\Sigma S^0 \cong S^1$, $\Sigma S^1 \cong S^2$, ...

suspension spectra.

3 notions of equivalence in spectra:

- **homotopy equivalence**: $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ $\quad g \circ f \sim id_X$, $f \circ g \simeq id_Y$. $\quad X \wedge I_+ \xrightarrow{H} Y$.

- **level equivalence**: $X \xrightarrow{f} Y$ l.e. if $X_n \xrightarrow{f_n} Y_n$ weak homotopy equivalence $\forall n$.

- **Stable equivalence**: $\pi_k(X) = \operatorname{colim}\left(\pi_k(X_0) \longrightarrow \pi_{k+1}(X_1) \longrightarrow \cdots \pi_{k+n}(X_n) \to \cdots\right)$

  close to me

  $X \xrightarrow{f} Y$ s.e. if $\pi_k(X) \cong \pi_k(Y)$ $\forall k$.

Freudenthal suspension theorem:

$S^1 \xrightarrow{\Sigma} S^2 \xrightarrow{\Sigma} S^3 \to \cdots$ $\quad\Big|\quad$ X

· Σ Ω these are inverses in spectra (upto stable equivalence).

$$\pi_{k+1}(\Sigma X) \cong \pi_k(X) \cong \pi_{k-1}(\Omega X).$$

$$\pi_k(X) = \operatorname{colim}_n \left(\cdots \to \pi_{k+n}(X_n) \to \pi_{k+n+1}(\Sigma X_n) \to \pi_{k+n+1}(X_{n+1}) \to \pi_{k+n+2}(\Sigma X_{n+1}) \to \cdots \right)$$

$= \operatorname{colim}$ (Yellow portion).

$$\pi_{k+1}(\Sigma X) = \operatorname{colim}_n \left(\cdots \to \pi_{k+1+n}(\Sigma X_n) \to \pi_{k+1+n+1}(\Sigma\Sigma X_n) \to \pi_{k+1+n+1}(\Sigma X_{n+1}) \to \cdots \right)$$

$= \operatorname{colim}$ (Yellow portion) cor. • the following are equivalent.

$$X \xrightarrow{f} Y \quad \text{S.E}$$

$$\Updownarrow$$

$$\Sigma X \xrightarrow{\Sigma f} \Sigma Y \quad \text{S.E}$$

$$\Updownarrow$$

$$\Omega X \xrightarrow{\Omega f} \Omega Y \quad \text{S.E.}$$

$$\begin{tikzcd}
\pi_{n+1}(\Sigma X) \arrow[r, phantom, "\cong"] \arrow[d, "\Sigma f"] & \pi_n(X) \arrow[r, phantom, "\cong"] \arrow[d, "f"] & \pi_{n-1}(\Omega X) \arrow[d, "\Omega f"] \\
\pi_{n+1}(\Sigma Y) \arrow[r, phantom, "\cong"] & \pi_n(Y) \arrow[r, phantom, "\cong"] & \pi_{n-1}(\Omega Y).
\end{tikzcd}$$

• $X \longrightarrow \Omega\Sigma X$ and $\Sigma\Omega X \longrightarrow X$ are also stable equivalence

$$\pi_k(X) \cong \pi_{k+1}(\Sigma X) \cong \pi_k(\Omega \Sigma X).$$

• homotopy cofiber and fiber sequences coincide.

$X \xrightarrow{f} Y \xrightarrow{g} Z$ cofiber sequence

$\Updownarrow$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] \arrow[dl, phantom, "\ulcorner h"] & * \arrow[d] \arrow[dl, phantom, "h \ulcorner"] \\
* \arrow[r] & Z \arrow[r] & \Sigma X
\end{tikzcd}$$

$Z \cong C\cancel{X} \cup_X Y$, $\quad \Sigma X \cong CY \cup_Y Z$

$$\begin{tikzcd}
X \arrow[r, "\simeq"] \arrow[d] & Y \times_Z PZ \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & * \arrow[d] \\
Y \arrow[r, "="] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "h \ulcorner"] & Z \arrow[d] \\
& * \arrow[r] & \Sigma Y \times_Z PZ \simeq CY \cup_Y Z
\end{tikzcd}$$

$\Sigma X \simeq \Sigma Y \times_Z PZ$.

$\Updownarrow$

$X \simeq Y \times_Z PZ$

$\Longleftrightarrow$ $X \to Y \to Z$ fiber sequence.

- homotopy pushout/pullbacks coincide.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"'] & D \arrow[ul, phantom, "\ulcorner" near start] \end{tikzcd} \quad \text{iff} \quad \begin{tikzcd} C_f \arrow[d, "\text{is}"] \\ C_g \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] \arrow[d] & D \arrow[ul, phantom, "\ulcorner" near start] \arrow[d] \\ * \arrow[r] & C_f \simeq C_g \arrow[ul, phantom, "\ulcorner" near start] \end{tikzcd} \iff \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C_f \arrow[d, "\text{is}"] \\ C \arrow[r] & D \arrow[ul, phantom, "\ulcorner" near start] \arrow[r] & C_g \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" near start] & B \arrow[d, dash] \\ C \arrow[r] & D \end{tikzcd} \quad \text{iff} \quad \begin{tikzcd} F_f \arrow[d, "\text{is}"] \\ F_g \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" near start] & B \arrow[d] \\ C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" near start] & D \arrow[d] \\ * \arrow[r] & C_f \simeq C_g \end{tikzcd}$$

h. coh.

$\Rightarrow$ fiber sequence.

- Spectra is additive.

$X_n \vee Y_n \hookrightarrow X_n \times Y_n$, ; $X \vee Y \simeq X \times Y$. [product and coproduct coincide]

$$
\begin{tikzcd}
\pi_k(X \vee Y) \arrow[r, phantom, "\cong"] & \pi_k(X) \oplus \pi_k(Y) \arrow[d, phantom, "\wr\|"] \\
\pi_k(X \times Y) \arrow[r, phantom, "\cong"] & \pi_k(X) \times \pi_k(Y)
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & X \vee Y \arrow[d] \\
* \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{\mathrm{Top}} \arrow[r] \arrow[d] & X_{\mathrm{Top}} \vee Y_{\mathrm{Top}} \arrow[d] \\
* \arrow[r] & CX \vee Y \simeq Y_{\mathrm{Top}}.
\end{tikzcd}
$$

$$Y \overset{s}{\hookrightarrow} X \vee Y.$$

$$
\begin{tikzcd}
\cdots \arrow[r] & \pi_k(X) \arrow[r] & \pi_k(X \vee Y) \arrow[r] & \pi_k(Y) \arrow[l, bend left, "s"] \arrow[r] & \cdots
\end{tikzcd}
$$

$$\pi_k(X \vee Y) \cong \pi_k(X) \oplus \pi_k(Y).$$

Model structure on Spectra:

top.

- '$\twoheadrightarrow$' : serre fibration
- '$\xrightarrow{\sim}$' : weak homotopy equivalence
- '$\hookrightarrow$' : $I = \{ S^{n-1} \longrightarrow D^n \}_{n \geq 0}$.

  and cofibrant generation.

spectra

- '$\twoheadrightarrow$' : $X_n \longrightarrow Y_n$ serre fibration.

$$
\begin{tikzcd}
X_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_n"] & \Omega X_{n+1} \arrow[d] \\
Y_n \arrow[r] & \Omega Y_{n+1}
\end{tikzcd}
$$

- '$\sim$' : stable homotopy equivalence
- '$\hookrightarrow$' $I = (F_d S^{n-1}_+ \longrightarrow F_d D^n_+)_{d \geq 0, n \geq 0}$.

  basepoint.

$$
\begin{array}{ccccc}
S^{n-1} & \Sigma S^{n-1} & \cdots & \Sigma^i S^{n-1} & \cdots \\
0 & 1 & & i &
\end{array}
$$

$$
\begin{array}{cccccc}
* & * & \cdots & S^{n-1} & \Sigma S^{n-1} & \cdots \\
0 & & & d & d+1 &
\end{array}
$$