

• Spaces: $\text{CGWH} \rightsquigarrow \text{Spectra}$ (Topology)

Spaces: $\text{Sm}/k \xrightarrow{\text{sheafy.}} \text{sheets}$ (Algebraic geometry).

$$\text{Sh}_{\text{Nis}}(\text{Sm}/k) =: \text{SpC}(k).$$

$$\begin{array}{ccc} X & \rightsquigarrow & h_X \leftarrow \text{Zariski sheaf} \\ \uparrow & & \downarrow \\ \text{Sm}/k & & \text{Nisnevich sheaf.} \end{array} \quad \text{Sm}/k \hookrightarrow \text{Sh}_{\text{Nis}}(\text{Sm}/k).$$

• $\text{SpC}(k) \quad (X, \alpha) \quad \alpha: \text{spec}k \rightarrow X, \quad x \in \text{SpC}(k).$

• Circles: $S^1 \quad S^1_s := (\Delta[0] \rightrightarrows \Delta[1] \rightarrow S^1_s) \cdot S^1_t$
 Top. SSet $A^1 \times \text{pt}$

Spaces: CGWH $\rightsquigarrow$ Spectra.

$$\mathrm{Spc}(k) \rightsquigarrow \mathrm{Spt}_s(k).$$

Stable Homotopy Categories.

$$\begin{array}{l} \text{s Spectrum, } \{E_n\}_{n \geq 0}, E_n \in \mathrm{Spc}(k), \\ S'_s \wedge E_n \xrightarrow{\delta_s} E_{n+1} \quad \leftarrow \text{structure map.} \end{array} \quad \left. \vphantom{\begin{array}{l} \text{s Spectrum, } \{E_n\}_{n \geq 0}, E_n \in \mathrm{Spc}(k), \\ S'_s \wedge E_n \xrightarrow{\delta_s} E_{n+1} \quad \leftarrow \text{structure map.} \end{array}} \right\} \begin{array}{l} \mathrm{Spt}_s(k) \\ \text{weak equivalences.} \end{array}$$

What we want to see,

$$E \longrightarrow E' \quad \text{in } \underline{\mathrm{Spt}_s(k)},$$

$$\text{if } \underbrace{\pi_n(E)}_{\downarrow \text{What is this thing?}} \xrightarrow{\cong} \pi_n(E').$$

for actual spt ,

$$\pi_n(x) \xrightarrow{\cong} \pi_n(y)$$

Stable Homotopy
Equivalence!

$X \in \text{Spc}(k) \rightsquigarrow \pi_n(X)$
 $\uparrow$
 sheaf with values in $\boxed{\text{SSet}}$

$\pi_n(X)$
 sheaf of homotopy groups,

$\rightsquigarrow$ we have notion of homotopy group

$U \longrightarrow \pi_n(X(U))_{\text{ssch}}$
 define homotopy sheaves for $\text{Spc}(k)$.

stable homotopy groups, $E \in \text{Spt}_S(k)$, $\pi_n(E) := \text{Colim}_m \pi_{n+m}(E)$.

$$E \longrightarrow E' \quad \text{in } \underline{\text{Spt}}_S(k), \quad (W)$$

$$\pi_n(E) \xrightarrow{\cong} \pi_n(E').$$

$$\text{SH}_S(k) = \text{Spt}_S(k) [W^{-1}].$$

$$\text{Spc}(k) \rightsquigarrow \text{SH}_S(k). \quad (\Sigma^1_S \text{ becomes } i_W)$$

$$\begin{array}{ccc}
 \text{Spt} & \longleftrightarrow & \text{Spt}_S(k) \\
 \downarrow & & \downarrow \\
 \text{SH} & \longrightarrow & \text{SH}_S(k)
 \end{array}$$

Affine line contractible

A^1 -local,

$E \longrightarrow E'$ is A^1 -stable,

if $\text{Hom}_{\text{SH}_S(k)}(E', F) \xrightarrow{\cong} \text{Hom}_{\text{SH}_S(k)}(E, F) \quad \forall F \text{ } A^1\text{-local}$

$\text{SH}_S(k) \longrightarrow \text{SH}_S^{A^1}(k)$. $\leftarrow$ here your Affine lines are contractible.

$$\text{Spc}(k) \longrightarrow \text{Spt}_S(k) \xrightarrow[\Sigma_2]{\text{S. sm}} \text{SH}_S(k) \xrightarrow{\text{localized}} \text{SH}_S^{A^1}(k) \rightsquigarrow \boxed{\text{SH}(k)}$$

$\nwarrow \Sigma_1^!$ invertible $\nearrow$ invert $\Sigma_1^!$

Motivic Spectra:

S'_t ← Tate cricle,

$E := \{E_n\}$, E_n - spectra,

$$S'_t \wedge E_n \xrightarrow{\delta'_t} E_{n+1}.$$

another way of stating it is, $E_{m,n} \in \text{Spc.}(k)$.

$$\delta_s: S'_s \wedge E_{m,n} \longrightarrow E_{m+n,n}.$$

$$\delta_t: S'_t \wedge E_{m,n} \longrightarrow E_{m,n+1}.$$

$$S'_s \wedge S'_t \wedge E_{m,n} \xrightarrow{S'_s \wedge \delta_t} S'_s \wedge E_{m,n+1}$$

such that

$$\tau \wedge \text{id} \downarrow$$

$\curvearrowright$



$\text{Spt}_{S,t}(k)$.
cat of (s,t) bi spectra

$$S'_t \wedge S'_s \wedge E_{m,n} \rightarrow S'_t \wedge E_{m+n,n} \longrightarrow \underline{E_{m+n,n+1}}$$

I want stable homotopy category for (s,t) bispectra. $\text{Spt}_{s,t}(k)$.

want $\pi_{p,q}(E) \xrightarrow{\cong} \pi_{p,q}(E')$, $E \rightarrow E'$ in $\text{Spt}_{s,t}(k)$.

$\boxed{SH_s^{A'}}$ $\xrightarrow{\text{sheaf}}$ $\text{stable in homotopy groups.}$

$\cup \longmapsto \text{Colim}_m \text{Hom}_{\boxed{SH_s^{A'}(k)}} (S_s^{p-q} \wedge S_t^{q+m} \wedge \Sigma_s^\infty U_+, E_m)$

$SH(k) = \text{Spt}_{s,t}(k).$

$\text{Spc}(k) \longrightarrow \text{Spt}_s(k) \xrightarrow[\Sigma_s]{S \cdot \Sigma_m} SH_s(k) \xrightarrow{\text{localized}} SH_s^{A'}(k) \longrightarrow \boxed{SH(k)}$
 $\Sigma_s^{\infty}, \Sigma_t^{\infty}$
 $\text{Spc}_*(k) \xrightarrow{\Sigma_s^{\infty}} SH_s^{A'}(k) \xrightarrow{\Sigma_t^{\infty}} SH(k)$
 $\parallel S$
 $\Sigma_{s,t}^{\infty}$
 $\Sigma_s^{\infty}, \Sigma_t^{\infty}$
 is invertible
 $A' \text{ contractible.}$

Propn: $X \in \mathcal{S}m/k$, $E \in \mathcal{S}p_{S,t}(k)$.

$$\begin{aligned} &\{ \\ &\varinjlim_{S,t} X_+ \end{aligned}$$

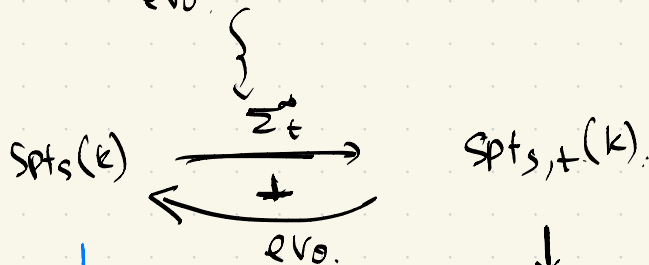
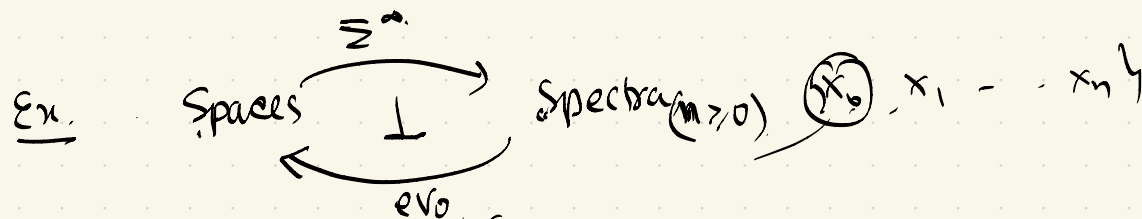
then,
$$\underbrace{\text{Hom}_{SH(k)}(\varinjlim_{S,t} X_+, E)}_{\substack{\uparrow \\ SH_S^{A!}(k)}} = \varinjlim_n \underbrace{\text{Hom}_{SH_S^{A!}(k)}(S_t^{-1} \varinjlim_S X_+, E_n)}_{\text{in } SH_S^{A!}(k)}$$

Def: $E = \{E_{m,n}\}$ some (S,t) -bigrada.
 E_n is an S -spectrum. $\{E_{0,n}, E_{1,n}, \dots\} =: E_n$

$$\text{Hom}_{SH(k)}(\Sigma_{s,t}^{\infty} X_+, E) = \text{Hom}_{\text{Spt}_{S,t}(k)}(\Sigma_{s,t}^{\infty} X_+, E^f) / \simeq.$$

$\Sigma_{s,t}^{\infty}(\mathbb{Z}/s)_+ \xrightarrow{\text{cofibrant}} \Sigma_{s,t}^{\infty} X_+ \xrightarrow{\text{fibrant}} E^f$

$$\textcircled{*} \boxed{HOM(A, B) = M(QA, RB) / \simeq} = \text{Hom}_{\text{Spt}_S}(\Sigma_S^{\infty} X_+, (E^f)_0) / \simeq.$$



↓
we want to look at.

$(E_0^f) \leftarrow$ choose this.

$$E_0^f := \operatorname{colim}_n ((E_0)^f \longrightarrow (\Omega_+(E_1)^f) \longrightarrow \dots)$$

E_0

$$\begin{aligned} \underbrace{\operatorname{Hom}_{\operatorname{Spt}_S(k)}\left(\sum_s^\infty X_+, E_0^f\right)}_{\sim} &= \operatorname{Hom}_{\operatorname{Spt}_S(k)}\left(\sum_s^\infty X_+, \operatorname{colim}\left((E_0)^f \longrightarrow \dots\right)\right) / \sim \\ &= \operatorname{colim} \operatorname{Hom}_{\operatorname{Spt}_S(k)}\left(\sum_s^\infty X_+, \Omega_+^n(E_n^f)\right) / \sim \\ &= \operatorname{colim} \operatorname{Hom}_{\operatorname{Spt}_S(k)}\left(S_+^{\wedge} \wedge \sum_s^\infty X_+, E_n^f\right) / \sim \\ &= \operatorname{colim} \operatorname{Hom}_{\operatorname{Spt}_S^{\wedge}(k)}\left(\sum_s^\infty X_+, E_n\right) \quad \square \end{aligned}$$