- Spaces: CGWH $\rightsquigarrow$ Spectra. (Topology)

Spaces: $Sm/k \longrightarrow$ sSets (Algebraic geometry).

$\quad\quad\Big\}$ sheafify.

$$sSh_{Nis}(Sm/k) =: Spc(k).$$

$$\begin{array}{ccc} X & \longrightarrow & h_X \\ \cap & & \\ Sm/k & & \end{array}$$

$h_X \leftarrow$ Zariski sheaf $\Downarrow$ Nisnevich sheaf.

$$Sm/k \hookrightarrow sSh_{Nis}(Sm/k).$$

- $Spc_{\cdot}(k)$ $\quad (X, x)$ $\quad x : \mathrm{spec}\,k \longrightarrow X$, $\quad X \in Spc(k)$.

- Circles: $\quad S^1$ $\quad S^1_s := \left(\Delta[0] \rightrightarrows \Delta[1] \longrightarrow S^1_s\right)$. $\quad S^1_t$

$\quad\quad$ Top. $\quad$ sSet $\quad\quad\quad\quad \mathbb{A}^1 \setminus \{0\}$.

Spaces: CGWH $\rightsquigarrow$ Spectra.

$$Spc(k) \rightsquigarrow Spt_s(k).$$

<u>Stable Homotopy Categories.</u>

s Spectrum, $\{E_n\}_{n \geq 0}$. $E_n \in Spc(k)$, $\Bigg\}$ $Spt_s(k)$

$\leftarrow$ structure maps.

$$S^1_s \wedge E_n \xrightarrow{\sigma_s} E_{n+1}$$

$\{$ weak equivalences.

What we want to see,

$E \longrightarrow E'$ in <u>$Spt_s(k)$</u>,

if $\underbrace{\pi_n(E)}_{}$ $\xrightarrow[\cong]{}$ $\pi_n(E')$.

What is this thing?

for actual spt,

$$\pi_n(X) \xrightarrow{\cong} \pi_n(Y)$$

Stable homotopy Equivalence!

$X \in Spc(k) \rightsquigarrow \pi_n(X)$. $U \longrightarrow \underbrace{\pi_n(X(U))}_{sset}$.

sheaf of homotopy groups.

sheaf with values in $\boxed{sSet} \rightsquigarrow$ we have notion of homotopy group

define homotopy sheaves for $Spc(k)$.

Stable homotopy groups, $E \in Spt_s(k)$, $\pi_n(E) := \operatorname{colim}_m \pi_{n+m}(E)$.

$E \longrightarrow E'$ in $\underline{Spt_s(k)}$, (W)

$\pi_n(E) \xrightarrow{\simeq} \pi_n(E')$.

$SH_s(k) = Spt_s(k)[W^{-1}]$.

$Spc(k) \rightsquigarrow SH_s(k)$. ($\Sigma_s^1$ becomes iso)

$$\begin{tikzcd} Spt \arrow[r, hook] \arrow[d] & Spt_s(k) \arrow[d] \\ SH \arrow[r] & SH_s(k) \end{tikzcd}$$

Affine line contractible

$\mathbb{A}^1$- local,

$E \longrightarrow E'$ is $\mathbb{A}^1$- stable,

if $\mathrm{Hom}_{SH_s(k)}(E', F) \xrightarrow{\cong} \mathrm{Hom}_{SH_s(k)}(E, F)$. $\forall$ F $\mathbb{A}^1$-local.

$SH_s(k) \longrightarrow SH_s^{\mathbb{A}^1}(k)$. ← here your Affine lines are contractible.

$$Spc(k) \longrightarrow Spt_s(k) \xrightarrow[\substack{S.Sm \\ \Sigma_n}]{} SH_s(k) \xrightarrow[\text{localized}]{} SH_s^{\mathbb{A}^1}(k) \rightsquigarrow \boxed{SH(k)}$$

Σ_s^1 invertible

invert Σ_t^1

Motivic Spectra:

S^1_t ← tate cricle,

$E := \{E_n\}$, E_n - s spectra,

$$S^1_t \wedge E_n \xrightarrow{\sigma^1_t} E_{n+1}.$$

another way of stating it is, $E_{m,n} \in \mathrm{Spc}.(k)$.

$$\sigma_s : S^1_s \wedge E_{m,n} \longrightarrow E_{m+1,n}.$$

$$\sigma_t : S^1_t \wedge E_{m,n} \longrightarrow E_{m,n+1}.$$

such that

$$\begin{tikzcd}
S^1_s \wedge S^1_t \wedge E_{m,n} \arrow[rr, "S^1_s \wedge \sigma_t"] \arrow[d, "\tau \wedge id"'] & \arrow[d, phantom, "\circlearrowright"] & S^1_s \wedge E_{m,n+1} \arrow[d] \\
S^1_t \wedge S^1_s \wedge E_{m,n} \arrow[r] & S^1_t \wedge E_{m+1,n} \arrow[r] & E_{m+1,n+1}
\end{tikzcd}$$

$\mathrm{Spt}_{s,t}(k)$.
cat of (s,t) bi spectra

I want stable homotopy category for (s,t) bispectra. $Spt_{s,t}(k)$.

<u>want</u> $\underline{\pi_{p,q}(E)} \xrightarrow{\cong} \pi_{p,q}(E')$, $E \to E'$ in $Spt_{s,t}(k)$.

$\boxed{SH_s^{\mathbb{A}^1}}$ <u>sheaf</u> in stable homotopy groups.

$$U \longmapsto \operatorname*{colim}_m \operatorname{Hom}_{\boxed{SH_s^{\mathbb{A}^1}(k)}}\left(S_s^{p-q} \wedge S_t^{q+m} \wedge \Sigma_s^\infty U_+, E_m \right)$$

$$SH(k) = Spt_{s,t}(k).$$

$$Spc_\cdot(k) \longrightarrow Spt_s(k) \xrightarrow[\substack{s.Sm \\ \varepsilon_n}]{} SH_s(k) \xrightarrow[\text{localized}]{} SH_s^{\mathbb{A}^1}(k) \longrightarrow \boxed{SH(k)}$$

Σ_s^1, Σ_t^1, is invertible

$\mathbb{A}^1$-contractible.

$$\begin{tikzcd}
& SH_s^{\mathbb{A}^1}(k) \arrow[dr, bend left, "\Sigma_t^\infty"] \arrow[d, phantom, "\wr\wr"] & \\
Spc_\cdot(k) \arrow[ur, bend left, "\Sigma_s^\infty"] \arrow[rr, bend right, "\Sigma_{s,t}^\infty"'] & {} & SH(k)
\end{tikzcd}$$

Prop^n: $X \in Sm/k, \quad E \in Spt_{s,t}(k)$.

$$X \rightsquigarrow \Sigma^\infty_{s,t} X_+$$

then, $$\underbrace{\mathrm{Hom}_{SH(k)}(\Sigma^\infty_{s,t} X_+, E)}_{\text{in } SH(k)} = \operatorname{colim}_n \mathrm{Hom}_{SH_s^{\mathbb{A}^1}(k)}\underbrace{(S_t^n \wedge \Sigma_s^\infty X_+, E_n)}_{\text{in } SH_s^{\mathbb{A}^1}(k)}$$

$$\uparrow$$
$$SH_s^{\mathbb{A}^1}(k).$$

Pf: $E = \{E_{m,n}\}$ some (s,t)-bispectra.

E_n is an s-spectrum. $\{E_{0n}, E_{1n}, \dots\} =: \mathbb{E}_n$

$$\underset{\text{cofibrant}}{\Sigma^\infty_t(\Sigma^\infty_s X_+)} \qquad \text{fibrant}$$

$$\mathrm{Hom}_{SH(k)}(\Sigma^\infty_{s,t}X_+, E) = \mathrm{Hom}_{Spt_{s,t}(k)}(\Sigma^\infty_{s,t}X_+, E^f)/\simeq$$

$$= \mathrm{Hom}_{Spt_s}\underline{(\Sigma^\infty_s X_+, (E^f)_0)}/\simeq$$

(*) $\boxed{\mathrm{Ho}\,\mathcal{M}(A,B) = \mathcal{M}(QA, RB)/\simeq}$

__Ex.__

$$\begin{tikzcd} \text{Spaces} \arrow[r, bend left, "\Sigma^\infty"] \arrow[r, phantom, "\perp"] & \text{Spectra}_{(n\geq 0)} \arrow[l, bend left, "ev_0"] \end{tikzcd} \qquad \{X_0, X_1, \dots, X_n\}$$

$$\begin{tikzcd} Spt_s(k) \arrow[r, "\Sigma^\infty_t"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[d] & Spt_{s,t}(k) \arrow[l, bend left, "ev_0"] \arrow[d] \\ {} & {} \end{tikzcd}$$

We want to look at.

(E^f_0) ← check this.

$$E^f_0 := \operatorname{Colim}_n \left((E_0)^f \longrightarrow (\Omega_t (E_1)^f) \longrightarrow \cdots \right)$$

E_0

$$\underline{\operatorname{Hom}_{Spt_S(k)}(\Sigma^\infty_S X_+, E^f_0)}_{\sim} = \operatorname{Hom}_{Spt_S(k)}\left(\Sigma^\infty_S X_+, \operatorname{Colim}\left((E_0)^f \to \cdots\right)\right)/\sim$$

compact.

$$= \operatorname{Colim} \operatorname{Hom}_{Spt_S(k)}\left(\Sigma^\infty_S X_+, \Omega^n_t (E_n)^f\right)/\simeq.$$

$$= \operatorname{Colim} \operatorname{Hom}_{Spt_S(k)}\left(S^n_t \wedge \Sigma^\infty_S X_+, E^f_n\right)/\simeq.$$

$$(*) \quad = \operatorname{Colim} \operatorname{Hom}_{SH^{\mathbb{A}^1}_S(k)}\left(\Sigma^\infty_S X_+, E_n\right). \blacksquare$$