Sheaves on Sites

$\mathcal{M}_S = [Sch/_S^{op}, sSet_*]$

Local model str :

1) Nisnevich up to htpy

2) $\mathbb{A}^1$ contractible

The cat of Nisnevich sh will play a crucial role.

I - Sites

Usual topology : specify open sets on a given set

Grothendieck topology :

1) Specify what collection of maps should be called "covers"

2) Do this for all schemes.

Def^n

A Groth. top $\mathcal{T}$ on $Sch/_k$ (or $Sm/_k$) is a set

$$\mathcal{T} = \{ \{ f_i : T_i \to T \}_{i \in I}$$

The elts are called coverings s.t.

1) $f : T' \to T$ isom, then $\{ f \} \in \mathcal{T}$ (sch covers itself)

2) $\{ f_i : T_i \to T \}$, $\{ g_{ij} : T_{ij} \longrightarrow T_i \} \in \mathcal{T}$

Then $\{ g_{ij} f_i : T_{ij} \to T \} \in \mathcal{T}$ (comp of cov is cov)

3) $\{ f_i : T_i \to T \} \in \mathcal{T}$ $g : V \longrightarrow T$

Then $\{ T_i \times_T V \to V \} \in \mathcal{T}$ (cov stable under base change)

Exs

1) Zariski $\quad Sch_{Zar}$

$\{ f_i : T_i \to T \}$ Zariski covering if

i) f_i open immersion

ii) $\bigcup f_i(T_i) = T$

3) Nisnevich $\quad Sch_{Nis}$

$\{ f_i : T_i \to T \}$ Nisnevich covering if

i) f_i étale

ii) $\forall\ \mathbb{F}/k,\ \bigcup f_i(T_i(\mathbb{F})) = T(\mathbb{F})$

2) Étale $\quad Sch_{ét}$

$\{ f_i : T_i \to T \}$ Étale covering if

i) f_i étale

ii) $\bigcup f_i(T_i) = T$

4) Smooth $\quad Sch_{sm}$

$\{ f_i : T_i \to T \}$ sm. covering if

i) f_i sm

ii) $\bigcup f_i(T_i) = T$

In order of coarseness,

Zariski $\longrightarrow$ Nisnevich $\longrightarrow$ Étale $\longrightarrow$ Smooth

Coarse $\qquad\qquad\qquad\qquad$ Fine

Ex

k fld

$i : \mathbb{A}^1 \setminus \{a\} \hookrightarrow \mathbb{A}^1$

$$f : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1, \quad x \mapsto x^n$$

Then $\{i, f\}$ étale covering of $\mathbb{A}^1$

But only Nisnevich if $x^n = a$ has a solution $/k$.

Def^n

Let $\mathcal{T}$ be a Groth top on Sch

A functor $F : Sch^{op} \to Sets$ is a $\underline{\mathcal{T}\text{-sheaf}}$ if

$\forall$ coverings $\{T_i \to T\}$

$$F(T) \to \prod F(T_i) \rightrightarrows \prod F(T_i \times_T T_j)$$

is an equalizer.

One should compare this w/ usual sheaf prop for Zariski top.

Then representable functors:

$$h_X : Sch^{op} \longrightarrow Set$$
$$T \longmapsto Hom(T, X)$$

If X is a sch, then h_X is a Zariski sheaf.

<u>fact</u>

1) F repr by a sch iff F covered by affine schs and F is a Zariski sheaf

2) h_X is a smooth (in fact fpqc) sheaf.

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II - Spaces revisited

$Spc(k) := sSh_{Nis}(Sm_{/k})$

– Nisnevich sh. w/ values in sSet.

Exs

1) $X \in Sm_{/k}$, then X is a sm shf $\Rightarrow$ Nisnevich sh.

So $X \in Spc(k)$

2) $S \in sSet$, then constant shf $\underline{S} \in Spc(k)$

Compare w/ stacks when constant gpoid is allowed

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$Spc_\bullet(k)$ - cat of ptd spaces
i.e., spaces $X \in Spc(k)$ w/ a map $x: \operatorname{Spec} k \to X$

We have functors

$$Spc_\bullet(k) \underset{\sqcup \bullet}{\overset{\text{forget}}{\rightleftarrows}} Spc(k)$$

We can make $Spc_\bullet(k)$ a closed symm mon cat under p/w $\wedge$,

$$(X \wedge Y)^{pre}(T) = X(T) \wedge Y(T)$$

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Two circles:

1) Simplicial circle

$$S^1_s = \operatorname{coeq}([0] \rightrightarrows [1]) \quad \rightsquigarrow \quad S^n_s = \overset{n}{\wedge} S^1_s$$

2) Tate circle

$$S^1_t = \mathbb{G}_m \quad \rightsquigarrow \quad S^n_t = \overset{n}{\wedge} \mathbb{G}_m$$

III - Towards stable htpy cat

What is the right cat of spectra?

Need: combo of S^1_s and S^1_t

Today: we will study just S^1_s - sus.

Def^n

An s-spectrum E is a seq of ptd spaces $\{E_n\}_{n \geq 0}$

w/ str maps

$$S^1_s \wedge E_n \to E_{n+1}$$

A map of S-spectra $E \to E'$ consists of maps $E_n \to E_n'$ s.t.

$$\begin{tikzcd} S_s^1 \wedge E_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & S_s^1 \wedge E_n' \arrow[d] \\ E_n \arrow[r] & E_{n+1}' \end{tikzcd}$$

Write $Spt_s(k)$ for cat of s-spectra.

Ex

$\Sigma_s^\infty X$: n^{th} term is $S_s^n \wedge X$.

Let $(X, x) \in Spc_\bullet(k)$. Define a presheaf

$$\pi_n(X, x) : Sm_{/k}^{op} \longrightarrow Set$$

$$T \longmapsto \pi_n(X(T), x|T)$$

This defines a Nisnevich-sheaf $\pi_n(X, x)$

If $n \geq 2$, $\pi_n(X, x)$ sh. of ab grps

In fact π_n defines a functor

$$\pi_n : Spc_\bullet(k) \longrightarrow Sh_{Nis}(Sm_{/k})$$

Defn

A map $E \to E'$ b/w s-spectra is an <u>s-stable weak equiv</u> if

$$\pi_n(E) \longrightarrow \pi_n(E')$$

isom of sheaves $\forall n \in \mathbb{Z}$.

Defn

$SH_s(k) = Spt_s(k)$ localized at s-stable weak equivs.

Remark

Recall Spt from Kershaw's talk

There is a constant sheaf functor:

$$Spt \longrightarrow Spt_s(k)$$

This induces a functor

$$SH \longrightarrow SH_s(k)$$

Next goal: make $\mathbb{A}^1$-contractible

$F \in Spt_S(k)$ is <u>$\mathbb{A}^1$-local</u> if

$\forall\, U \in Sm_{/k}$, $n \in \mathbb{Z}$, $U \times \mathbb{A}^1 \to U$ induces a bijection

$$\mathrm{Hom}_{SH_S}(\Sigma_s^\infty U_+, \Sigma_s^n F) \longrightarrow \mathrm{Hom}(\Sigma_s^\infty (U \times \mathbb{A}^1)_+, \Sigma_s^n F)$$

A map $E \to E'$ of s-spectra is $\mathbb{A}^1$-stable w.e. if

$$\mathrm{Hom}(E', F) \xrightarrow{\sim} \mathrm{Hom}(E, F)$$

$\forall$ F $\mathbb{A}^1$-local.

<u>Lemma</u>

$X \times \mathbb{A}^1 \to X$ is $\mathbb{A}^1$-stable weak equiv

$\Rightarrow$ $\Sigma_s^\infty(\mathbb{A}^1)$ contractible.

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Can localize to get $SH_s^{\mathbb{A}^1}(k)$