Part 1 - Constructions of Simplicial Sets and Examples

We already saw some constructions of simp'l sets.

- Products, Coproducts, Mapping spaces, Suspensions

Let's see some more.

Lemma (Limits and Colimits)

sSet has all (small) limits and colimits

Pf

Set has all small (co)limits $\Rightarrow$ $sSet = Fun(\Delta, Set)$ has all small (co)limits

Moreover, they can be computed pointwise.

$\square$

Ex

1) Products and coproducts

$X, Y \in sSet$

$$X \times Y = \lim \left(\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & \Delta^0 \end{tikzcd} \right) \qquad X \sqcup Y = \operatorname{colim} \left(\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y \\ X & \end{tikzcd} \right)$$

2) Gluing and Quotients

$X, Y, Z \in sSet$, $Z \hookrightarrow X$, $Z \hookrightarrow Y$

$$X \underset{Z}{\sqcup} Y := \operatorname{colim} \left(\begin{tikzcd} Z \arrow[r, hook] \arrow[d, hook] & Y \\ X & \end{tikzcd} \right)$$, topologically, this is gluing X and Y along Z

$$X/Z = \operatorname{colim} \left(\begin{tikzcd} Z \arrow[r] \arrow[d, hook] & \Delta^0 \\ X & \end{tikzcd} \right)$$, topologically this is a quotient of X by Z

Ex Spheres

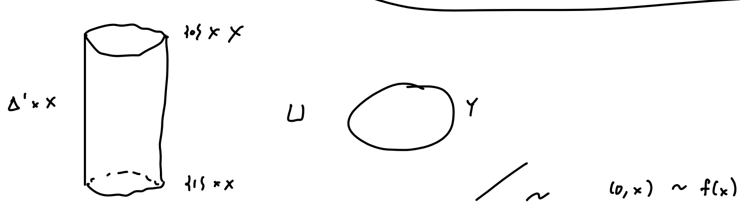
$\partial\Delta^n = sk_{n-1}\Delta^n$

$S^n = \Delta^n / \partial\Delta^n$

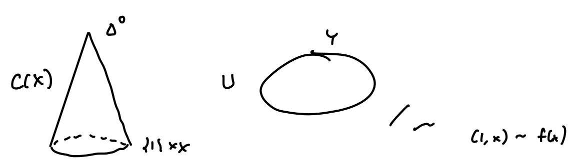
3) Mapping cylinder and mapping cone

$f : X \to Y$

$$M(f) := \operatorname{colim} \left(\begin{tikzcd} \Delta^0 \times X \arrow[r, "f"] \arrow[d, "d^0 \times id_X"'] & Y \\ \Delta^1 \times X & \end{tikzcd} \right)$$



$$C(f) := \operatorname{colim} \left(\begin{tikzcd} \Delta^0 \times X \arrow[r, "d^1 \times id_X"] \arrow[d] & \Delta^1 \times X \arrow[r] & M(f) \\ \Delta^0 & & \end{tikzcd} \right)$$



4) Skeleton

$X \in sSet$, $k \geq 0$

$$sk_k(X)_n = \{ \sigma \in X_n \mid \forall m \leq k, \ \underbrace{\Delta^n \longrightarrow \Delta^m \dashrightarrow X}_{\sigma} \}$$

So $sk_k(X)$ is obtained from X by throwing away all n-simplices w/ $n > k$.

More categorical: $\Delta_n \subseteq \Delta$ full subcat w/ objs $[m]$ for $m \leq n$

Truncated sSet: $sSet_n = \mathrm{Fun}(\Delta_n, sSet)$

The functor $\Delta_n \hookrightarrow \Delta$ induces a functor $sSet \xrightarrow{i_{n*}} sSet_n$

By Kan extension, admits a left adj $i_n^* : sSet_n \longrightarrow sSet$

$$i_{n!} X = \operatorname*{colim}_{[m] \in \Delta_n / X} \Delta^m$$

Then $sk_n X = i_{n!} i_n^* X$

Note, $i_n^* i_{n!} = id$

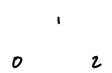
As a result,

$$X = \operatorname*{colim}_n sk_n X$$

Ex

Δ^2

$sk_0 \Delta^2$:



$sk_1 \Delta^2$:

Ex

$\partial \Delta^n = sk_{n-1} \Delta$

$\partial \Delta^n \setminus sk_{n-2} \Delta = \bigcup \Delta^{n-1}$

$\Lambda^n_k = \partial \Delta^n \setminus \Delta^{n-1}$

k^{th} $(n-1)$-face

Part 2 - Dold - Kan

Ab - cat of abelian grps

sAb = Fun(Δ^{op}, Ab) - cat of simplicial ab. grps

$A_\bullet \in$ sAb means that there are abelian grps A_n $\forall n$

as well as maps $A_n \to A_m$ for each map $[m] \to [n]$

$$d^i : [n-1] \longrightarrow [n]$$

$$\rightsquigarrow d_i : A_n \longrightarrow A_{n-1}$$

Define $\partial : A_n \to A_{n-1}$ by $\partial = \sum_{i=0}^{n} (-1)^i d_i$

Exercise

$$\partial^2 = 0$$

Hint

Copy the proof from intro to topology class.

As a result, we have a ch. cplx

$$\cdots \to A_{n+1} \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \cdots \to A_1 \to A_0 \to 0$$

Therefore, we have a functor

$$C_\bullet(-) : sAb \longrightarrow Ch_{\geq 0}(Ab)$$

$$A_\bullet \longmapsto (A_\bullet, \partial)$$

called the Moore cplx functor.

This allows us to define (simplicial) homology,

$$H_\bullet(X) = H_\bullet(C_\bullet \mathbb{Z}[X])$$

Note : $H_\bullet(\mathrm{Sing}(X)) = H_\bullet^{sing}(X)$

—

We want for $C_\bullet$ to be an equiv of cats

Problem : degenerate simplices in $A_\bullet$

Solution : define the normalized ch. cplx

$$C_n^{norm}(A_\bullet) = \bigcap_{i=0}^{n-1} \ker(d_i : A_i \to A_{i-1})$$

This defines a funct

$$C_\bullet^{norm} : sAb \longrightarrow Ch_{\geq 0}(Ab)$$

Thm [Dold–Kan]

$$C_\bullet^{norm} : sAb \longrightarrow Ch_{\geq 0}(Ab)$$

is an equiv of cats.

Moreover, $C_\bullet^{norm}$ sends htpys to ch. htpys.

The inverse functor:

Let $C_\bullet \in Ch_{\geq 0}(Ab)$

$$\Gamma(C_\bullet)_n = \bigoplus_{[n] \twoheadrightarrow [k]} C_k$$

Making $\Gamma(C)$ into a simpl. ab. gp is tricky

Let's see an example.

Ex

$C \quad \cdots \to 0 \to 0 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \to 0$

$\Gamma(C)_0 = C_0$

$\Gamma(C)_1 = C_0 \oplus C_1$

$\Gamma(C)_2 = C_0 \oplus C_1^{\oplus 2} \oplus C_2$

$$\left(\begin{array}{c|c} id & \partial \end{array} \right) \qquad \left(\begin{array}{c|c|c|c} id & \partial & \partial & 0 \\ \hline 0 & id & id & \partial \end{array} \right)$$

Most of the work in proving Dold-Kan

is setting the combinatorics to construct Γ

in general and prove it is actually a functor.

Application

$$H^n(X, A) = [C_\bullet^{norm}(\mathbb{Z}[X]), A[n]]$$

$$= [\mathbb{Z}[X], \Gamma(A[n])]$$

$$= [X, U(\Gamma(A[n]))] \qquad \begin{tikzcd} sAb \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & sSet \arrow[l, shift left, "{\mathbb{Z}[-]}"] \end{tikzcd}$$

Note, $U(\Gamma(A[n]))$ is the Eilenberg-Maclane space for A!

We can then describe cohomology via E-M spaces.

Part 3 - Pointed Sets

$sSet_* = Fun(\Delta^{op}, Set_*)$ ~~pointed simp'l sets~~

Constructions

$X, Y \in sSet_*$

1) $X \vee Y = X \amalg_* Y$ wedge

2) $X \wedge Y = (X \times Y)/(X \vee Y)$ smash

3) Suspension $S^1 \wedge X$

Can define $S^n = S^1 \wedge S^{n-1}$

$$S^n = \Delta^n/\partial\Delta^n$$

Simp'l sets $\longleftrightarrow$ simp'l homology

Pointed simp'l sets $\longleftrightarrow$ reduced simp'l homology

add 1 pt

$$X_+ \longmapsfrom X$$

$$Ab \underset{u}{\overset{\tilde{\mathbb{Z}}[\cdot]}{\rightleftarrows}} Set_* \underset{u}{\rightleftarrows} Set$$

Note $u(A)$ is pt'd by 0.

$$\tilde{\mathbb{Z}}[X] = \mathbb{Z}[X]/\mathbb{Z}[\cdot]$$

Many relationships,