More on Simplicial Homotopy

$$(\mathcal{C}, \rightarrowtail, \twoheadrightarrow, \xrightarrow{\sim})$$
$$\mathcal{C} = sSet$$

- $X \xrightarrow{\simeq} Y \in sSet \overset{|\,|}{\rightsquigarrow} |X| \xrightarrow{\simeq} |Y|$ weak homotopy equivalence,

- cofibrations:

  for Top spaces, $\leftarrow$ cofibrantly generated,

  I cell/maps $:= \{\partial\Delta^n \rightarrowtail \Delta^n\}$

  . not all injections are cofibrations.

· $\underline{\text{sSets}}$. $K \hookrightarrow L$ inclusion in simplicial sets.

Lemma: $K \subseteq \underbrace{K(0) \subseteq K(1) \subseteq \cdots \subseteq \operatorname{colim}}_{\exists} K(i) = L$

$$\begin{tikzcd}
\exists \coprod_J \partial\Delta[i] \arrow[r] \arrow[d] & K(i-1) \arrow[d] \\
\coprod_J \Delta[i] \arrow[r] & K(i) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

$\underbrace{\left[\partial\Delta[i] \xrightarrow[\text{define}]{} \Delta[i]\right]}_{\text{focus on this}} \rightsquigarrow K \hookrightarrow L$ is a cofibration.

look at lifting w.r.t these to get all your trivial fibrations.

$$\begin{tikzcd}
\partial\Delta[i] \arrow[r] \arrow[d, tail] & X \arrow[d, two heads, "\sim", "S"'] \\
\Delta[i] \arrow[r] \arrow[ur] & Y
\end{tikzcd}$$

$$Q \overset{\sim}{\Rightarrow} \mathrm{id}$$

$$\begin{tikzcd}
\emptyset \arrow[r, tail] \arrow[d, equal] & QA \arrow[r, two heads, "\sim"] \arrow[d] & A \arrow[d, "f"] \\
\emptyset \arrow[r, tail] & QB \arrow[r, "\sim"] & B
\end{tikzcd}$$

<u>factorization</u>: $X \xrightarrow{f} Y$ sSet,

$$\begin{tikzcd}
X \arrow[r, tail] \arrow[rr, bend right, "f"'] & Z \arrow[r, two heads] & Y
\end{tikzcd}$$

D_1 for which $\left\{ \begin{tikzcd}
\partial\Delta[n] \arrow[r] \arrow[d, tail] & X \arrow[d] \\
\Delta[n] \arrow[r] & Y
\end{tikzcd} \right\}$

$$\begin{tikzcd}
X \arrow[d] \\
2
\end{tikzcd}$$

$$\begin{tikzcd}
\coprod_{d \in D_1} \partial\Delta[n_d] \arrow[r] \arrow[d, hook] & X \arrow[d, tail] \arrow[dr, bend left, "f"] & \\
\coprod_{d \in D_1} \Delta[n_d] \arrow[r] \arrow[rr, bend right] & X_1 \arrow[r] & Y
\end{tikzcd}$$

iterate this to get X_2.

$$D_2 = \left\{ \begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, tail] & X_1 \arrow[d, tail] \\ \Delta[n] \arrow[r] & Y \end{tikzcd} \right\}$$

$$\begin{tikzcd} \coprod\limits_{d\in D_2} \partial\Delta[n_d] \arrow[r] \arrow[d, tail] & X_1 \arrow[d, tail] \arrow[ddr, bend left] & \\ \coprod\limits_{d\in D_2} \Delta[n_d] \arrow[r] \arrow[drr, bend right] & X_2 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr] & \\ & & Y \end{tikzcd}$$

$$X \hookrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 - - -$$

$$Z = \operatorname{colim} X_i = \bigcup X_i \qquad X \hookrightarrow Z$$

$\boxed{Z \overset{\sim}{\twoheadrightarrow} Y}$ ← to show this.

$$
\begin{tikzcd}
\partial\Delta[n] \arrow[r] \arrow[d] & Z \arrow[d, "s"] \\
\Delta[n] \arrow[r] \arrow[ur, "f"] & Y
\end{tikzcd}
\qquad Z = \operatorname{colim} X(i)
$$

$$
\begin{tikzcd}
\partial\Delta[n] \arrow[r, "f"] \arrow[d, hook] & X(m) \arrow[d, hook] \arrow[dr] & \\
\Delta[n] \arrow[r] \arrow[rr, bend left] \arrow[dr] & X(m+1) \arrow[r] \arrow[d, hook] & Z \arrow[dl] \\
& Y &
\end{tikzcd}
$$

Small object argument:

“ filtered colimits commute with finite colimit.

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

$$\mathrm{Hom}(A, \operatorname{colim}(X_i)) \cong \lim \mathrm{Hom}(A \longrightarrow X_i)$$

← A satisfies small object argumen

small objects.

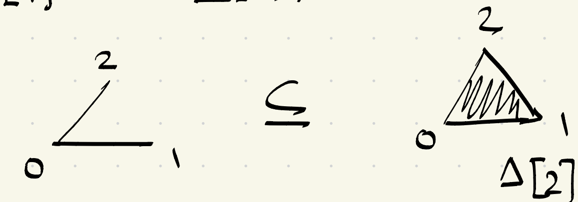
$$A \longrightarrow \operatorname{colim} X_i$$

$$\begin{tikzcd}
 & & & A \arrow[d] \arrow[dr] & \\
X_0 \arrow[r] & X_1 \arrow[r, dashed] & \cdots \arrow[r, dashed] & X_m \arrow[r] & \operatorname{colim} X_i
\end{tikzcd}$$

Fibrations:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\ \Delta^n \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

$$\Lambda^k[n] \longrightarrow \Delta[n].$$



$\Lambda^0[2] \subseteq \Delta[2]$

$$\begin{tikzcd} {|\Lambda^k_n|} \arrow[r] \arrow[d, "s"] & {|X|} \arrow[d] \\ {|\Delta^n|} \arrow[r] \arrow[ur] & {|Y|} \end{tikzcd} \sim \begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook, "s"'] & X \arrow[d, two heads] \\ \Delta^n \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

$X \twoheadrightarrow Y$

<u>Kan fibrations</u>

$\{ D^n \times \{0\} \xrightarrow{\sim} D^n \times I \}$

<u>J-cells.</u>

- $A \rightarrowtail B \in sSet$, $\quad X \twoheadrightarrow Y \in sSet$,

$\underline{S}(-,X)$ $\quad \underline{S}(-,Y)$ $\quad \underline{S}(A,-)$ $\quad \underline{S}(B,-)$

$$
\begin{tikzcd}
\underline{S}(B,X) \arrow[r, two heads, "\text{claim}"'] \arrow[rr, bend left] \arrow[dr, bend right] & \underline{S}(A,X) \times_{\underline{S}(A,Y)} \underline{S}(B,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{S}(B,Y) \arrow[d] \\
& \underline{S}(A,X) \arrow[r] & \underline{S}(A,Y)
\end{tikzcd}
$$

moreover, $\quad A \overset{\sim}{\rightarrowtail} B \quad X \overset{\sim}{\twoheadrightarrow} Y$

if one of them is trivial, universal map is also trivial.

we will prove this, for $A = \phi$, $Y = *$,

$$
\begin{tikzcd}
\underline{S}(B,X) \arrow[r] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \cdot \arrow[d] \\
& \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

$\underline{S}_*$

$X \dashv \underline{S}$

$$\begin{tikzcd} K \times B \arrow[r] \arrow[d, hook, "s"'] & X \arrow[d] \\ L \times B \arrow[r] \arrow[ur] & \bullet \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} K \arrow[r] \arrow[d, hook, "s"'] & \underline{S}(B, X) \arrow[d] \\ L \arrow[r] \arrow[ur] & \bullet \end{tikzcd}$$

$\leftarrow$ to show fibration.

$X \twoheadrightarrow \cdot \qquad X \xrightarrow{\sim} \hspace{-0.5em}\twoheadrightarrow *$

$$\boxed{|K \times B| = |K| \times |B|}$$

$$\begin{tikzcd} {|K \times B|} \arrow[d, "s"'] & {|K| \times |B|} \arrow[d, "s"] \\ {|L \times B|} \arrow[r, "="] & {|L| \times |B|} \end{tikzcd}$$

$|K| \xrightarrow{\sim} |L|$

$$\begin{tikzcd} K \times B \arrow[r] \arrow[d, hook] & X \arrow[d, "s"] \\ L \times B \arrow[r] \arrow[ur] & \bullet \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} K \arrow[r] \arrow[d, hook] & \underline{S}(B, X) \arrow[d, "s"] \\ L \arrow[r] \arrow[ur] & \bullet \end{tikzcd}$$

Whitehead lemma (Simplicial Sets):

$A \xrightarrow{\simeq} B$ bifibrant object $\Rightarrow A \xrightarrow{\cong} B$

$\phi \hookrightarrow X.$

if $A \xrightarrow{\sim} B$ fibrant then we have homotopy equivalence $A \longrightarrow B$.

all simplicial sets are cofibrant

$$\begin{tikzcd} A \arrow[rr, "\sim"] \arrow[dr, hook, "\sim"] & & B \\ & C \arrow[ul, bend left, "\text{homoty inv}"] \arrow[ur, two heads, "\sim"] & \arrow[l, bend left, "\text{inv}"] \end{tikzcd}$$

enough to state the theorem for $A \overset{\sim}{\twoheadrightarrow} B$, $A \overset{\sim}{\rightarrowtail} B$.

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & A \arrow[d, "sf"] \\ B \arrow[ur, "s"] \arrow[r, "\overline{\overline{\ }}"', "id_B"'] & B \end{tikzcd}$$

$f \circ s = id_B$, we wan to show $s \circ f \cong id_A$.

Notion of Homotopy here.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

$f \simeq g$ if $\exists H$:

$$\begin{tikzcd} X \amalg X \arrow[r, "{(i_0, i_1)}"] \arrow[dr, "{(f, g)}"'] & X \times \Delta[1] \arrow[d, "H"] \\ & Y \end{tikzcd}$$

$$\begin{tikzcd} A \amalg A \arrow[r, "sf \amalg id_A"] \arrow[d, hook] & A \arrow[d, "s"', "f"] \\ A \times \Delta[1] \arrow[ur, "H"] \arrow[r, "f \circ pr_A"'] & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times \Delta[1] \arrow[l, "pr_A"'] \arrow[dl, bend left] \\ B & \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \arrow[ddr, bend left, "id"] & \\ A \arrow[r] \arrow[drr, bend right, "sf"'] & A \amalg A \arrow[dr] & \\ & & A \end{tikzcd}$$

• $A \overset{\sim}{\rightarrowtail} B$ (this is the part where we use the fact A, B fibrant)

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, tail, "f"', "s"] & A \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, "p"] & * \end{tikzcd}$$

$p \circ f = id_A$, $\quad$ | $f \circ p \overset{\text{to show}}{\simeq} id_B$

$$\begin{tikzcd} B \amalg B \arrow[r, "f \circ p \amalg id_B"] \arrow[d, hook] & B \arrow[d, two heads] \\ B \times \Delta[1] \arrow[r] \arrow[ur, dashed, "? H"] & * \end{tikzcd}$$

$\sim$

$$\begin{tikzcd} B \amalg B \arrow[d, tail] \arrow[drr] & & \\ B \amalg_A A \times \Delta[1] \amalg_A B \arrow[rr] \arrow[d, tail, "s"] & & B \arrow[d, two heads] \\ B \times \Delta[1] \arrow[rr] \arrow[urr, "H"] & & * \end{tikzcd}$$

$$\underset{\text{cofibrant}}{X} \longrightarrow \underset{\text{fibrant}}{Y}$$ then homotopy makes sense.

Top. $\quad Y$ is always fibrant, $\pi_q = [S^q, X]$.

cofibrant objects.

sSets. $\quad Y$ may not be fibrant

for fibrant Y, $\boxed{\pi_q(Y) = \pi_0(S^q, Y)}$

$\boxed{\pi_q X \cong \pi_q |X|}$

in sSets.
$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d] \\
* \arrow[r] & B
\end{tikzcd}
\quad \longrightarrow \pi_n(A) \longrightarrow \pi_n(F) \longrightarrow \pi_n(B) \longrightarrow \cdots
$$

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d] \\
G \arrow[r] & B
\end{tikzcd}
$$

$\rightsquigarrow$ Mayer Vietoris sequence of homotopy groups.

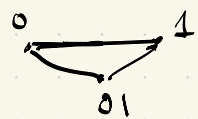
Given any simplicial set $\rightsquigarrow$ Fibrant simplicial sets.

will solve the Fibrant issues. $\hookrightarrow$ $\boxed{\pi_q(X) := \pi_0\, \underline{S}_\cdot(S^q, FX)}$

<u>Kan</u>: Ex^∞.

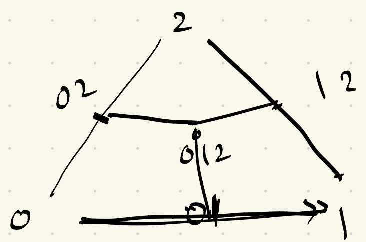
$sd\, \Delta[n]$.

- Δ^1 $\quad sd\, \Delta^1$



$$\begin{tikzcd} \Delta^0 \arrow[r, "d^0"] & \Delta^1 & \Delta^0 \arrow[l, "d^1"'] \end{tikzcd}$$

- Δ^2 $\quad sd\,\Delta^2$



$\leftarrow$ Barycentric Subdivision

$$\Delta[n] \longrightarrow X.$$

$$\lim_{sSet}\{ sd\, \Delta[n] \longrightarrow X \} =: Ex(X)_n.$$

$$X \longrightarrow Ex(X) \longrightarrow Ex^2 Ex(X) \longrightarrow \cdots$$

$\operatorname{colim} Ex^n(X) = Ex^\infty(X)$. is fibrant.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \text{cofib} & & \text{fib.} \end{array}$$ ← talking about homotopy makes sense.

in top all your objects are fibrant.

<u>s sets.</u> your objects cofibra $\phi \hookrightarrow X$.

they don't need to be fibrant.