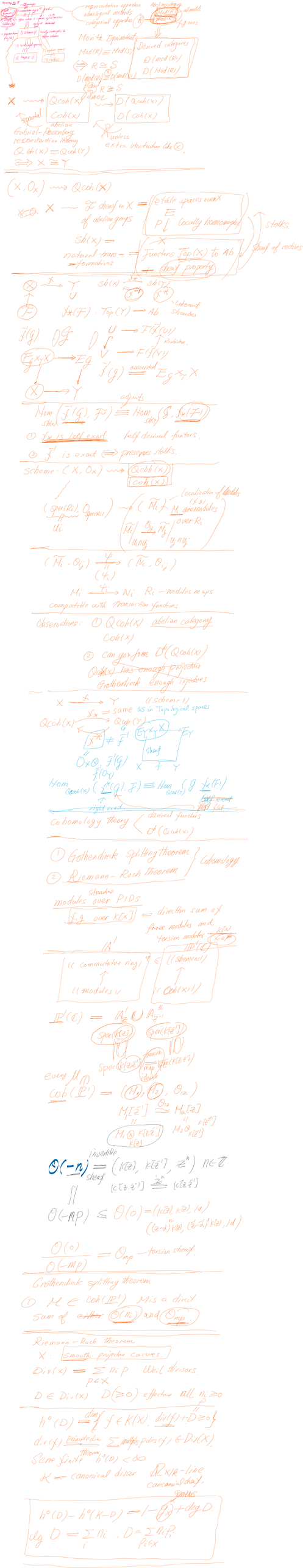

Elementary AG II

Friday, July 18, 2025 12:53 PM

((Non commutative ring)) $\supset$ ((commutative ring))^{op} $R \xleftarrow{f} S$

((finite g. generated domains)) $\subseteq$ ((commutative ring))^{op}

((integral schemes/k)) $\subseteq$ ((affine scheme)) $(\mathrm{Spec}(R), \mathcal{O}_R) \xrightarrow{(f, f^\#)} (\mathrm{Spec}(S), \mathcal{O}_S)$

topological space   structure sheaf

((projective schemes)) $\mathrm{Proj}(R)$ $\subseteq$ ((Schemes))   locally isomorphic to affine schemes

((local ringed spaces)) $\supseteq$ ((topos))

((algebraic spaces)) $\supseteq$ ((stacks))

representation approaches
homological methods
categorical approaches $R \to$ Abelian category: $\mathrm{Mod}(R)$ all modules, $\mathrm{mod}(R)$ f.g. ones

$\Downarrow$

Derived categories
$D(\mathrm{mod}(R))$
$D(\mathrm{Mod}(R))$

Morita Equivalent:

$\mathrm{Mod}(R) \equiv \mathrm{Mod}(S)$

$\Longleftrightarrow R \cong S$

$D(\mathrm{mod}(R)) \overset{\otimes}{\cong} D(\mathrm{mod}(S))$

$\overset{\text{Paul Balmer}}{\Longleftrightarrow} R \cong S$

$X \rightsquigarrow$ $\mathrm{Qcoh}(X)$, $\mathrm{Coh}(X)$ abelian $\rightsquigarrow$ $D(\mathrm{Qcoh}(X))$, $D(\mathrm{Coh}(X))$

separated

Gabriel-Rosenberg reconstruction theory

$\mathrm{Qcoh}(X) \equiv \mathrm{Qcoh}(Y)$

$\Longleftrightarrow X \cong Y$

✗ unless extra structures like $\otimes$.

$(X, \mathcal{O}_X) \rightsquigarrow \mathrm{Qcoh}(X)$

~~$X, \mathcal{O}$~~ X — $\mathcal{F}$ sheaf on X of abelian groups $\equiv$ étale spaces over X: $E \xrightarrow{p} X$ locally homeomorphic

stalks

sheaf of sections

$Sh(X) \equiv$ natural transformations $\equiv$ Functors $\mathrm{Top}(X)$ to Ab + sheaf property

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$X \xrightarrow{f} Y$, $U \subseteq Y$, $\mathcal{F}$ on X

$$\begin{tikzcd} Sh(X) \arrow[r, bend left, "f_*"] & Sh(Y) \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

f^* — coherent sheaves

$f_*(\mathcal{F}) : \mathrm{Top}(Y) \to Ab$

$U \longrightarrow \mathcal{F}(f^{-1}(U))$

$V \longrightarrow \mathcal{F}(f^{-1}(V))$ Restriction.

$f^{-1}(\mathcal{G})$ $\mathcal{O}\mathcal{G}$

$$\begin{tikzcd} E_{\mathcal{G}} \times_Y X \arrow[r] \arrow[d] & E_{\mathcal{G}} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

$f^{-1}(\mathcal{G}) \overset{\text{associated}}{\equiv} E_{\mathcal{G}} \times_Y X$

adjoints

$$\mathrm{Hom}_{Sh(X)}(f^{-1}(\mathcal{G}), \mathcal{F}) \equiv \mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_*(\mathcal{F}))$$

① f_* is left exact   left derived functors.

② f^{-1} is exact $\Longleftrightarrow$ preserves stalks.

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scheme $(X, \mathcal{O}_X) \rightsquigarrow$ $\mathrm{Qcoh}(X)$, $\mathrm{coh}(X)$

$(\mathrm{Spec}(R_i), \mathcal{O}_{\mathrm{Spec}(R_i)})$, U_i $\rightarrow$ $(\tilde{M}_i)$ localization of modules (x.g) M_i are modules over R_i

$\tilde{M}_i|_{U_i \cap U_j} \xrightarrow{\theta_{ij}} \tilde{M}_j|_{U_j \cap U_i}$

$(\tilde{M}_i, \theta_{ij}) \xrightarrow{\varphi} (\tilde{N}_i, \theta_{ij})$, $\varphi = (\varphi_i)$

$M_i \xrightarrow{\varphi_i} N_i$ R_i-modules maps

compatible with transaction functions

Observations: ① $\mathrm{Qcoh}(X)$ abelian category, $\mathrm{Coh}(X)$

② can you form $D^b(\mathrm{Qcoh}(X))$

~~$\mathrm{Qcoh}(X)$ has enough projective~~

Grothendieck enough injectives

$X \xrightarrow{f} Y$ ((scheme))

$f_* =$ same as in Topological spaces

$$\begin{tikzcd} \mathrm{Qcoh}(X) \arrow[r, bend left, "f_*"] & \mathrm{Qcoh}(Y) \arrow[l, bend left] \end{tikzcd}$$

$\boxed{f^*} \neq f^{-1}$

$= \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{G})$

$$\begin{tikzcd} E_Y \times_Y X \arrow[r] \arrow[d, "\text{sheaf}"'] & E_Y \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$$\mathrm{Hom}_{\mathrm{Qcoh}(X)}(f^*(\mathcal{G}), \mathcal{F}) \equiv \mathrm{Hom}_{\mathrm{Qcoh}(Y)}(\mathcal{G}, f_*(\mathcal{F}))$$

right exact   ~~left exact~~   flat

Cohomology theory: derived functors; $D^b(\mathrm{Qcoh}(X))$

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① Grothendieck splitting theorem
② Riemann-Roch theorem
} Cohomology.

Structure modules over PIDs

f.g over $k[x]$ = direction sum of free modules and torsion modules $\frac{k[x]}{(x-a)^m}$

$\mathbb{A}^1$: ((commutative ring))^{op} $\leftarrow$ ((modules)) $\subseteq$ $\mathbb{P}^1(\mathbb{C})$: ((schemes)) $\leftarrow$ ((Coh(X)))

$\mathbb{P}^1(\mathbb{C}) = \mathbb{A}^1_z \amalg \mathbb{A}^1_{z^{-1}}$

$\mathrm{Spec}(k[z]) \supseteq \mathrm{Spec}(k[z, z^{-1}]) \overset{\text{transition map identity}}{\equiv} \mathrm{Spec}(k[z, z^{-1}]) \subseteq \mathrm{Spec}(k[z^{-1}])$

every $\mathcal{M} \in$

$\mathrm{Coh}(\mathbb{P}^1) = (M_1, M_2, \theta_{12})$

$M_1[z^{-1}] \overset{\theta_{12}}{\cong} M_2[z]$

$M_1[z^{-1}] = M_1 \otimes_{k[z]} k[z, z^{-1}]$, $M_2[z] = M_2 \otimes_{k[z^{-1}]} k[z^{\pm 1}]$

$\mathcal{O}(-n) = (k[z], k[z^{-1}], z^n)$ $n \in \mathbb{Z}$ invertible sheaf

$k[z, z^{-1}] \overset{z^n}{\cong} k[z, z^{-1}]$

$\mathcal{O}(-mp) \subseteq \mathcal{O}(0) = (k[z], k[z], Id)$

$\mathcal{O}(-mp) = ((z-\alpha)^n k[z], (z^{-1}-\alpha^{-1})^n k[z], Id)$

$\frac{\mathcal{O}(0)}{\mathcal{O}(-mp)} = \mathcal{O}_{mp}$ — torsion sheaf.

Grothendieck splitting theorem

① $\mathcal{M} \in \mathrm{Coh}(\mathbb{P}^1)$ $\mathcal{M}$ is a direct sum of ~~either~~ $\mathcal{O}(n_i)$ and $\mathcal{O}_{mp}$.

Riemann-Roch theorem

X: smooth projective curves.

$\mathrm{Div}(X) = \sum_{p \in X} n_i p$ Weil divisors.

$D \in \mathrm{Div}(X)$ $D(\geq 0)$ effective all $n_i \geq 0$ ($n_i = n_p$)

$h^0(D) = \dim\{f \in K(X),\ \mathrm{div}(f) + D \geq 0\}$

$\mathrm{div}(f) \xrightarrow{\text{principle divisor}} \sum$ multiples poles$(f) \in \mathrm{Div}(X)$

Serre finite theorem $h^0(D) < \infty$

$K \leftarrow$ canonical divisor $\Omega_{X/k}$ - line canonical sheaf.

$$\boxed{\begin{aligned} h^0(D) - h^0(K-D) &= 1 - g_X + \deg D \\ \deg D = \sum_i n_i, \quad D &= \sum_{P_i \in X} n_i P_i \end{aligned}}$$

genus