

Part 1 - Why derived categories?

functors like $M \otimes_R (-)$ and $\text{Hom}_R(M, -)$ are not

exact. I.e., they don't send SES's to SES's

$$\begin{array}{c}
 \begin{array}{ccccccc}
 0 \rightarrow N_1 & \rightarrow & N_2 & \rightarrow & N_3 & \rightarrow & 0 \\
 & \searrow & & \searrow & & \searrow & \\
 & M \otimes_R (-) & & M \otimes_R N_1 & \rightarrow & M \otimes_R N_2 & \rightarrow & M \otimes_R N_3 & \rightarrow & 0 \\
 & & & & & & & & & \text{right exact} \\
 & \searrow & & \searrow & & \searrow & & \searrow & & \\
 & \text{Hom}_R(-, M) & & 0 \rightarrow \text{Hom}_R(M, N_1) & \rightarrow & \text{Hom}_R(M, N_2) & \rightarrow & \text{Hom}_R(M, N_3) & & \text{left exact}
 \end{array}
 \end{array}$$

Sometimes they are exact. For example, if M is a free R -module $M \otimes_R (-)$, it is exact.

How do we measure the failure of exactness? Derived functors

$$M \otimes_R (-) \rightsquigarrow \text{Tor}_R^i(M, -)$$

Given LES is

$$\dots \rightarrow \text{Tor}_R^{-1}(M, N_2) \rightarrow \text{Tor}_R^{-1}(M, N_1) \rightarrow \text{Tor}_R^{-1}(M, N_2) \rightarrow \text{Tor}_R^{-1}(M, N_3) \rightarrow M \otimes_R N_1 \rightarrow M \otimes_R N_2 \rightarrow M \otimes_R N_3 \rightarrow 0$$

(coming from long exact sequence)

Problems

1) $P: \text{Sh}(X, \mathbb{Z}) \rightarrow \text{Ab}$ is left exact

So what's right derived functor $R^i P$

$$\text{Note, } R^i P(\mathbb{Z}_X) \cong H_{\text{sing}}^i(X, \mathbb{Z})$$

Singlet cohomology arises via an underlying chain complex of singlet cochains.

Can we do something similar for $R^i P$ or even $R^i F$?

2) How does one compute $L^i F(A)$

One finds a proj resolution $P^\bullet \rightarrow A$

$$\text{Then takes } L^i F(A) = H^i(F(P^\bullet))$$

Idea: Can we cook up a category such that $P^\bullet \simeq A$?

Part 2 - Triangulated Categories

Def

Let $\mathcal{C}$ be an additive cat w/ an automorphism $[1]: \mathcal{C} \rightarrow \mathcal{C}$, called shift,

along w/ a collection of diagrams

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

called distinguished triangles

We call $\mathcal{C}$ a triangulated category if

(Identity) $X \xrightarrow{id} X \rightarrow 0 \rightarrow A[1]$ is distinguished

(Rotation) $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ is distinguished iff $Y \rightarrow Z \rightarrow X[1] \rightarrow Y[1]$ is distinguished

(Completion) Any comm square can be completed to a comm. diagram of d.t.'s

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ X & \rightarrow & Y \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccccc} A & \rightarrow & B & \dashrightarrow & C & \dashrightarrow & A[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \rightarrow & Y & \dashrightarrow & Z & \dashrightarrow & X[1] \end{array}$$

(Octahedral) SKIP

Ex

1) A ab cat

$Ch(A)$ - cat of ch cplx

objs: ch cplx maps: ch maps

$K(A)$ - htpy cat of ch cplx

objs: ch cplx

$$\text{Hom}_{K(A)}(X^\bullet, Y^\bullet) = \text{Hom}_{Ch(A)}(X^\bullet, Y^\bullet) / \text{morphisms htpc to } 0$$

$$(A^\bullet[1])^n = A^{n+1} \quad \text{- shift functor}$$

The d.t.'s of $K(A)$ are given by

$$\begin{array}{ccccccc} X^\bullet & \longrightarrow & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & X[1] \\ \vdots & & \vdots & & \vdots & & \vdots \\ S \downarrow & & S \downarrow & & S \downarrow & & S \downarrow \\ & & \hookrightarrow & & \hookrightarrow & & \hookrightarrow \\ A^\bullet & \dashrightarrow & B^\bullet & \dashrightarrow & \text{Cone}(f) & \dashrightarrow & A[1] \end{array}$$

2) A map $f: A^* \rightarrow B^*$ is called a g-isom if $H^*(f)$ is an iso b.i.

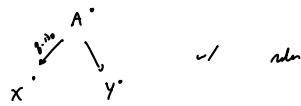
In $K(A)$ g.isom $\neq$ isom.

We can fix this by localizing at g.isoms

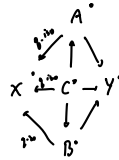
$D(A)$

Objs: ch. cplx in A

Mor: Equ. class of diagrams

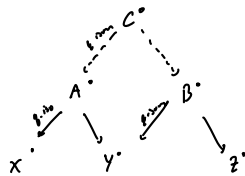


or also



Think localization
of rings
and modules

Composition:



The existence of C^* is a fact
about g.isoms and triangulated
categories.

It's a good exercise to prove that this occurs.

fact

$D(A)$ triangulated cat.

Intuition:

d.b.'s replace SES's in abelian cat

Indeed, if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, then $A \rightarrow B \rightarrow C \xrightarrow{0} 0$ is a d.b.

fact

$H^*: D(A) \rightarrow A$

takes d.b.'s to LES's

Def

A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ of cat. cat is triangulated if

F takes d.b.'s to d.b.'s.

Ex

$$M \otimes_R (-) : R\text{-mod} \longrightarrow R\text{-mod}$$

$$\leadsto \text{Ch}(R\text{-mod}) \longrightarrow \text{Ch}(R\text{-mod})$$

$$\leadsto K(R\text{-mod}) \longrightarrow K(R\text{-mod})$$

For each $N^* \in R\text{-mod}$, we can find $P^* \xrightarrow[\text{q.isom}]{\simeq} N^*$ proj. mod.

Can then define

$$M \overset{L}{\otimes}_R N^* = M \overset{L}{\otimes}_R P^*$$

This turns out to not depend on P^* (up to isom.)

We then get a triangulated functor

$$M \overset{L}{\otimes}_R (-) : D(R\text{-mod}) \longrightarrow D(R\text{-mod})$$

$$\text{Moreover, } H^i(M \overset{L}{\otimes}_R N) \simeq \text{Tor}_R^i(M, N)$$

This is an example of derived functor.

Update

Derived cats create derived functors in very neat language.

Ex

$$\text{Sh}(X, \mathbb{Z})$$

$$R\Gamma : D(\text{Sh}(X, \mathbb{Z})) \longrightarrow D(\text{Ab})$$

$$H^i R\Gamma(\mathbb{Z}_X) \simeq H_{\text{sing}}^i(X; \mathbb{Z})$$

Exercise:

$$H^i R\Gamma(\mathbb{Z}_X) \simeq \text{Hom}(\mathbb{Z}_X, \mathbb{Z}_X[i])$$

So even Hom spaces create cohomology.

This makes cup products easy

$$\text{Hom}(\mathbb{Z}_X, \mathbb{Z}_X[i]) \times \text{Hom}(\mathbb{Z}_X, \mathbb{Z}_X[j]) \longrightarrow \text{Hom}(\mathbb{Z}_X, \mathbb{Z}_X[i+j])$$

$$(g, f) \longmapsto f[i] \circ g$$

Update

Derived cats give rise to cohomology theories on their Hom spaces!

Sheaf Theory	Cohomology Theory
$\mathcal{D}$ -modules " " $\mathcal{Q}\text{Coh}(X^{dR})$	de Rham cohomology
Sheaves on $X(\mathbb{C})$	Singh cohomology
$\mathcal{Q}\text{Coh}(X)$	Cohomology
$\mathcal{S}h(X_{\text{ét}})$ $X_{\text{ét}}$ is the étale site	Crisp cohomology

1) $\mathcal{D}\text{-mod}(X)$ cat of modules over sheaf of rings $\mathcal{D}(X)$ of differential operators

$$f: X \rightarrow pt$$

$$Rf_*: \mathcal{D}(\mathcal{D}\text{-mod}(X)) \longrightarrow \mathcal{D}(\mathcal{D}\text{-mod}(pt)) = \mathcal{D}(\mathbb{C}\text{-mod})$$

⌞ right derived global sections

Then

$$H^i(Rf_* \mathcal{O}_X) \simeq H_{\text{dR}}^i(X)$$

Alternatively,

$$H_{\text{dR}}^i(X) \simeq \text{Hom}(\mathcal{O}_X, \mathcal{O}_X[i])$$

2) Stalk sheaves

Basic idea: look at étale covers $U \rightarrow X$ (i.e. sm of mod étale)

"Hodge theory spaces" instead of open sets

$$\mathcal{S}h_{\text{ét}}(X) \text{ use functor } \text{Cor}_{\text{ét}}(X) \longrightarrow \text{Ab}$$

Can then define étale cohomology

$$H_{\text{ét}}^i(X) = H^i f_* (\mathbb{Z}_\ell)$$

Étale cohomology doesn't have a non-sheaf theoretic description.

3) $R\text{Goh}(X)$

2 definitions for derived cat of g. coh thus:

$$D(R\text{Goh}(X))$$

$$D_{\text{gcoh}}(X) \subseteq D(\mathcal{O}_X)$$

There is a natural functor

$$(*) \quad D(R\text{Goh}(X)) \longrightarrow D_{\text{gcoh}}(X)$$

These are often but not always equal.

If X Noeth. then (*) is an eqn of cats

Similar story for $\text{Coh}(X)$

$$D\text{Coh}(X) \text{ vs } D_{\text{coh}}(X) \subseteq D(\mathcal{O}_X)$$

Can create derived versions of our sheaf functors

$$\otimes, \text{Hom} \rightsquigarrow \overset{L}{\otimes}, R\text{Hom}$$

$$f_*, f^* \rightsquigarrow Rf_*, Lf^*$$

$$R \rightsquigarrow R^p$$

Then

$$H^i R^p(\mathcal{F}) = H^i(X, \mathcal{F}) = \check{H}^i(X, \mathcal{F})$$

Upshot

One can change sheaf theories in 2 main ways

1) Change the site

2) Change the coeffs in exotic ways

We've focused mostly on ①

A big goal of motives is to use ② to unify various versions of ①.