Part 1 - Why derived categories?

Functors like $M \otimes_R (-)$ and $\mathrm{Hom}_R(M, -)$ are not exact. I.e., they don't send SES's to SES's

$$0 \to N_1 \to N_2 \to N_3 \to 0 \quad \xrightarrow{M \otimes_R (-)} \quad M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0 \quad \text{right exact}$$

$$0 \to N_1 \to N_2 \to N_3 \to 0 \quad \xrightarrow{\mathrm{Hom}_R(-, M)} \quad 0 \to \mathrm{Hom}_R(M, N_1) \to \mathrm{Hom}_R(M, N_2) \to \mathrm{Hom}_R(M, N_3) \quad \text{left exact}$$

Sometimes they are exact. For example, if M is a free R-module $M \otimes_R (-)$, it is exact.

How do we measure the failure of exactness? Derived functors!

$M \otimes_R (-) \rightsquigarrow \mathrm{Tor}_R^i(M, -)$

Gives LES's

Connecting hom super important

$$\cdots \to \mathrm{Tor}_R^{-2}(M, N_3) \to \mathrm{Tor}_R^{-1}(M, N_1) \to \mathrm{Tor}_R^{-1}(M, N_2) \to \mathrm{Tor}_R^{-1}(M, N_3) \to M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0$$

Problems

1) $\Gamma : Sh(X, \mathbb{Z}) \to Ab$ is left exact

So admits right derived functors $R^i\Gamma$

Note, $R^i\Gamma(\underline{k}_X) \simeq H^i_{sing}(X; \mathbb{Z})$

Singular cohomology arises via an underlying ch cplx of singular cochains.

Can we do something similar for $R^i\Gamma$ or even R^iF?

2) How does one compute $L^iF(A)$

One finds a proj resolution $P^\bullet \to A$

Then takes $L^iF(A) = H^i(F(P^\bullet))$

Idea: Can we cook up a category such that $P^\bullet \simeq A$?

Part 2 - Triangulated Categories

Def^n

Let $\mathcal{C}$ be an additive cat w/ an automorphism $[1] : \mathcal{C} \to \mathcal{C}$, called <u>shift</u>,

along w/ a collection of diagrams

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

called <u>distinguished triangles</u>

We call $\mathcal{C}$ a <u>triangulated category</u> if

(Identity) $X \xrightarrow{id} X \to 0 \to A[1]$ is distinguished

(Rotation) $X \to Y \to Z \to X[1]$ is distinguished iff $Y \to Z \to X[1] \to Y[1]$ is distinguished

(Completion) Any comm square can be completed to a comm. diagram of d.t's

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r, dashed] \arrow[d] & C \arrow[r, dashed] \arrow[d, dashed] & A[1] \arrow[d, dashed] \\ X \arrow[r] & Y \arrow[r, dashed] & Z \arrow[r, dashed] & X[1] \end{tikzcd}$$

(Octahedral) SKIP

Ex

1) A ab cat

$Ch(A)$ - cat of ch cplxs
objs : ch cplxs mors : ch maps

$K(A)$ - htpy cat of ch cplxs
objs : ch cplxs

$$\mathrm{Hom}_{K(A)}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{Ch(A)}(X^\bullet, Y^\bullet) \big/ \text{morphisms htpc to } 0$$

$(A^\bullet[1])^n = A^{n+1}$ - shift functor

The d.t.'s of $K(A)$ are given by

$$\begin{tikzcd} X^\bullet \arrow[r] \arrow[d, dashed, "\wr"'] & Y^\bullet \arrow[r] \arrow[d, dashed, "\wr"'] & Z^\bullet \arrow[r] \arrow[d, dashed, "\wr"'] & X[1] \arrow[d, dashed, "\wr"'] \\ A^\bullet \arrow[r, dashed, "f"] & B^\bullet \arrow[r, dashed] & \mathrm{cone}^\bullet(f) \arrow[r, dashed] & A^\bullet[1] \end{tikzcd}$$

27) A mor. $f: A^\bullet \to B^\bullet$ is called a <u>q.isom</u> if $H^i(f)$ is an isom $\forall i$

In $K(\mathcal{A})$ q.isom $\not\Rightarrow$ isom.

We can fix this by localising at qisoms

$D(\mathcal{A})$

Objs: ch cplxs in $\mathcal{A}$

Mors: Equiv classes of diagrams

$$\begin{tikzcd} & A^\bullet \arrow[dl, "q.iso"'] \arrow[dr] & \\ X^\bullet & & Y^\bullet \end{tikzcd}$$

w/ rel^n

$$\begin{tikzcd} & A^\bullet \arrow[dl, "q.iso"'] \arrow[dr] & \\ X^\bullet & C^\bullet \arrow[l, "q.iso"'] \arrow[u] \arrow[d] \arrow[r] & Y^\bullet \\ & B^\bullet \arrow[ul, "q.iso"] \arrow[ur] & \end{tikzcd}$$

Think localisation of rings and modules

Composition:

$$\begin{tikzcd} & & C^\bullet \arrow[dl, dashed, "q.isom"'] \arrow[dr, dashed] & & \\ & A^\bullet \arrow[dl, "q.iso"'] \arrow[dr] & & B^\bullet \arrow[dl, "q.iso"'] \arrow[dr] & \\ X^\bullet & & Y^\bullet & & Z^\bullet \end{tikzcd}$$

The existence of $C^\bullet$ is a fact about q.isoms and triangulated categories.
It's a great exercise to prove that this occurs.

<u>Fact</u>

$D(\mathcal{A})$ triangulated cat.

Intuition:

d.t.'s replace SES's in abelian cats

Indeed, if $0 \to A \to B \to C \to 0$, then $A \to B \to C \xrightarrow{0} A[1]$ is a d.t.

<u>Fact</u>

$H^0: D(\mathcal{A}) \to \mathcal{A}$

takes d.t.'s to LES's

Def^n

A functor $F: \mathcal{C} \to \mathcal{D}$ of tri cats is <u>triangulated</u> if

F takes d.t.'s to d.t.'s.

Ex

$M \otimes_R (-) : R\text{-mod} \longrightarrow R\text{-mod}$

$\rightsquigarrow \quad Ch(R\text{-mod}) \longrightarrow Ch(R\text{-mod})$

$\rightsquigarrow \quad K(R\text{-mod}) \longrightarrow K(R\text{-mod})$

For each $N^\bullet \in R\text{-mod}$, we can find $P^\bullet \underset{q.\text{isom}}{\simeq} N^\bullet$ proj. such

Can then define

$$M \overset{L}{\underset{R}{\otimes}} N^\bullet = M \overset{L}{\underset{R}{\otimes}} P^\bullet$$

This turns out to not depend on $P^\bullet$ (up to q.isom)

We then get a triangulated functor

$$M \overset{L}{\underset{R}{\otimes}} (-) : D(R\text{-mod}) \longrightarrow D(R\text{-mod})$$

Moreover, $H^i(M \overset{L}{\underset{R}{\otimes}} N) \simeq \mathrm{Tor}_R^i(M,N)$

This is an example of derived functor.

Upshot

Derived cats encode derived functors in very neat language.

Ex

$Sh(X, \mathbb{Z})$

$R\Gamma : D(Sh(X,\mathbb{Z})) \longrightarrow D(Ab)$

$H^i R\Gamma(\underline{\mathbb{Z}}_X) \simeq H^i_{sing}(X;\mathbb{Z})$

Exercise:

$$H^i R\Gamma(\underline{\mathbb{Z}}_X) \simeq \mathrm{Hom}(\underline{\mathbb{Z}}_X, \underline{\mathbb{Z}}_X[i])$$

So even Hom spaces encode cohomology.

This makes cup products easy

$$\mathrm{Hom}(\mathbb{Z}_X, \mathbb{Z}_X[i]) \times \mathrm{Hom}(\mathbb{Z}_X, \mathbb{Z}_X[j]) \longrightarrow \mathrm{Hom}(\mathbb{Z}_X, \mathbb{Z}_X[i+j])$$

$$(g, f) \longmapsto f[i] \circ g$$

Upshot

Derived cats give rise to cohomology theories on their Hom spaces!

Sheaf Theory	Cohomology Theory
$\mathcal{D}$-modules = $QCoh(X^{dR})$	de Rham cohomology
Sheaves on $X(\mathbb{C})$	Singular cohomology
$QCoh(X)$	Coherent cohomology
$Sh(X_{Inf})$ — X_{Inf} is the infinitesimal site	Crystalline cohomology

—

1) $\mathcal{D}\text{-mod}(X)$ cat of modules over sheaf of rings $\mathcal{D}(X)$ of differential operators

$f : X \to pt$

$Rf_* : D(\mathcal{D}\text{-mod}(X)) \longrightarrow D(\mathcal{D}\text{-mod}(pt)) = D(\mathbb{C}\text{-mod})$

right derived global sections

Then

$$H^i(Rf_* \mathcal{O}_X) \simeq H^i_{dR}(X)$$

Alternatively,

$$H^i_{dR}(X) \simeq Hom(\mathcal{O}_X, \mathcal{O}_X[i])$$

2) Étalé sheaves

Basic idea: look at étale covers $U \to X$ (i.e. sm of rel dim 0)

"think covering spaces" instead of open sets

$Sh_{ét}(X)$ are functors $\underbrace{Cov_{ét}(X)}_{\text{the étale site}} \longrightarrow Ab$

Can then define étale cohomology

$$H^{\bullet}_{ét}(X) = H^{i} f_{*}(\underline{\mathbb{Z}}_{X})$$

Étale cohomology does not have a non-sheaf theoretic description.

3) $QCoh(X)$

2 definitions for derived cat of q. coh shvs:

$$D(QCoh(X)) \qquad\qquad D_{qcoh}(X) \subseteq D(\mathcal{O}_X)$$

There is a nat'l functor

$$(*) \quad D(QCoh(X)) \longrightarrow D_{qcoh}(X)$$

These are often but not always equiv.

If X Noeth. then $(*)$ is an equiv of cats

Similar story for $Coh(X)$

$$D\,Coh(X) \quad \text{vs} \quad D_{coh}(X) \subseteq D(\mathcal{O}_X)$$

Can create derived versions of our sheaf functors

$$\otimes, \ Hom \rightsquigarrow \overset{L}{\otimes}, \ RHom$$

$$f_*, \ f^* \rightsquigarrow Rf_*, \ Lf^*$$

$$\Gamma \longrightarrow R\Gamma$$

Then

$$H^{i} R\Gamma(\mathcal{F}) \simeq H^{i}(X, \mathcal{F}) \simeq \check{H}^{i}(X, \mathcal{F})$$

<u>Upshot</u>

One can change sheaf theories in 2 main ways

1) Change the site

2) Change the coeffs in exotic ways

We've focused mostly on ①

A big goal of motives is to use ② to unify various versions of ①.