

Intro to Model Cats

Top

• Homotopy groups

• All the objects are fibrant.

geometric.

s Set

• All the objects are not fibrant.

Kan E_{∞}

1950's

Combinatorial

Quillen (1967):

Model Cat $(\mathcal{M}, \xrightarrow{\text{cofib}}, \xrightarrow{\text{fib}}, \xrightarrow{\text{weak equiv.}})$

Axioms: • $\mathcal{M}$ has all small limits and colimits.

• 2 out of 3 property for w.e. $\begin{array}{ccc} & \xrightarrow{\sim} & \\ \xrightarrow{\sim} & \searrow & \nearrow \\ & \xrightarrow{\sim} & \end{array}$

• Factorization

Motivation in top $A \xrightarrow{f} B$ $\left| \begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \searrow \text{mf} & & \nearrow \text{pf} \end{array} \right.$

$$A \xrightarrow{f} B$$

and

$$A \xrightarrow{f} B$$

(functorial)
Necess

• lifting

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ Y \downarrow & \nearrow & \downarrow s \\ C & \xrightarrow{\quad} & D \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ S \downarrow & \nearrow & \downarrow \\ C & \xrightarrow{\quad} & D \end{array}$$

• retract (f is called retract of g)

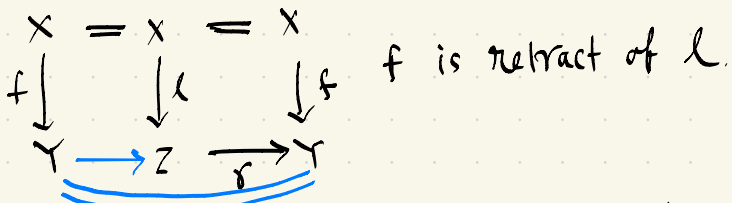
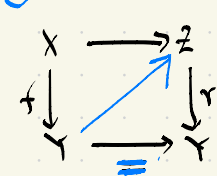
$$\begin{array}{ccccc} & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\ A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & A \\ \downarrow f & & \downarrow g & & \downarrow f \\ A' & \xrightarrow{\quad} & B' & \xrightarrow{\quad} & A' \\ & \xrightarrow{\quad} & & \xrightarrow{\quad} & \end{array}$$

if $g \in \mathcal{C}, \mathcal{F}, \mathcal{W}$
so does f .

Or: it comes with initial and terminal object.

A few Key Observations :

- $$\begin{array}{ccc}
 x & \xrightarrow{f} & y \\
 & \searrow & \nearrow \\
 & z &
 \end{array}$$
 $f = r \circ l$ if f has left lifting prop. w.r.t r , the f is retract of l .

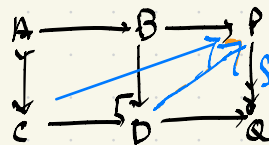


$\Rightarrow f_s$

- A map $f \in \underline{\mathcal{C}} \text{ or } (\underline{\mathcal{C}} \cap \mathcal{W}) \Leftrightarrow f$ has l.l.p. w.r.t maps in $\mathcal{I} \cap \mathcal{W}$ or $(\mathcal{I})$.
 A map $f \in \mathcal{F} \text{ or } (\mathcal{I} \cap \mathcal{W}) \Leftrightarrow f$ has r.l.p. w.r.t maps in $\underline{\mathcal{C}} \cap \mathcal{W}$ or $(\underline{\mathcal{C}})$.

- Any two of $\underline{\mathcal{C}}, \mathcal{F}, \mathcal{W}$ will determine the third. (signature feature)

- pushout of a cofibration (trivial) is a cofibration (trivial).
pullback of a fibration (trivial) is a fibration (trivial).



cofib.
 Sequential colimits.
 fibration
 Sequential limits.

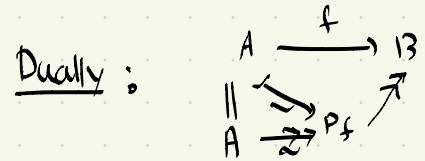
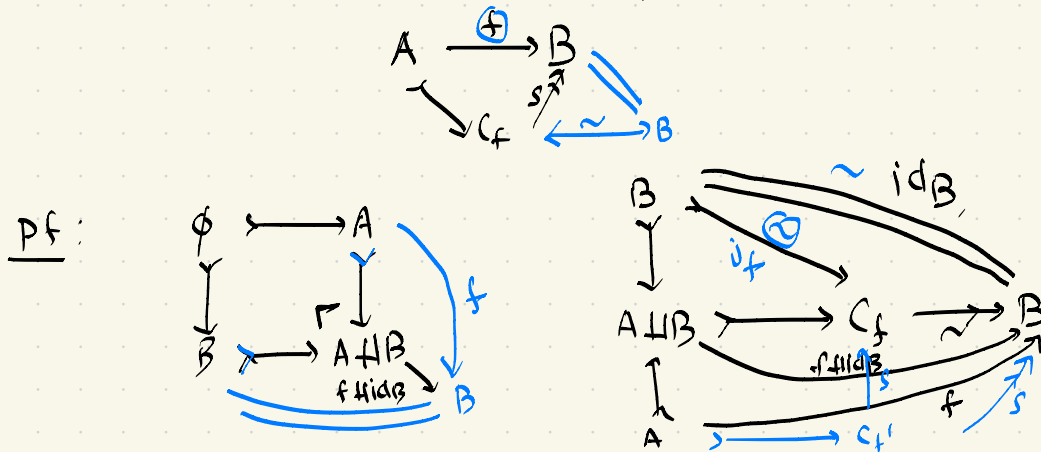
cofibrant object: $\phi \rightarrow X$

Fibrant object: $X \rightarrow 1$

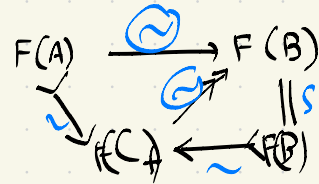
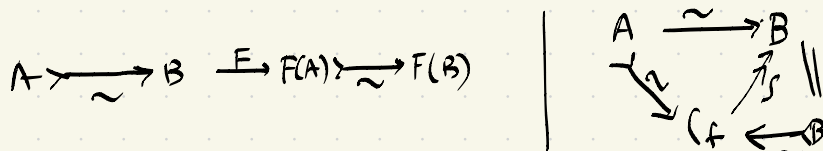
Q: When does a functor $F: M \rightarrow N$ (between Model cats) preserve weak equivalences?

Ken Brown's Lemma: (we see how all axioms take place)

$f: A \rightarrow B$ (A, B are cofibrant) then $\exists C_f$ and $\tilde{f}_f: B \xrightarrow{\sim} C$ s.t.



Cor: $F: M \rightarrow N$ preserve trivial cofib, then it sends between cofibrant objects to ~.

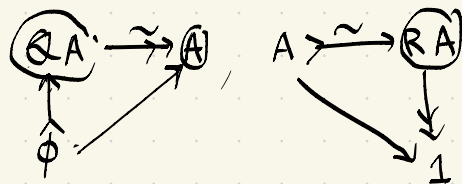


Homotopy Category

A cotibrant, B fibrant $\text{Hom}_{\text{on}}(A, B)$ left and right homotopy coincide.

$f, g : A \rightrightarrows B$, $f \sim g$ $\leftarrow$ (they have a notion of homotopy)

$$\text{Hom} \mathcal{M} := \mathcal{M}(R \& A, R \& B) / \sim$$



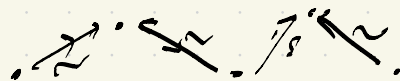
$$Q \rightrightarrows \text{id}_{\mathcal{M}}, \text{id}_{\mathcal{M}} \rightrightarrows R$$

$\text{hMcf} :=$ objects bifibrant objects.

$$[A, B] = \frac{\mathcal{M}(A, B)}{\sim}$$

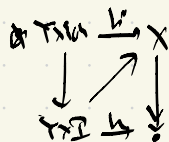
we can look them upto weak equivalences.

$$\mathcal{M}[W^{-1}]$$



$$\mathcal{M}, W, \boxed{\mathcal{M}[W^{-1}]} = \text{Hom} \mathcal{M}$$

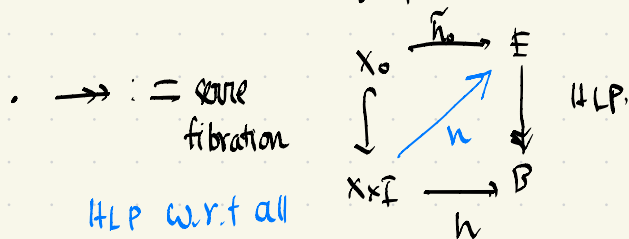
example.



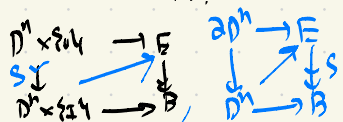
All objects are fibrant

Top / Top

• $\simeq :=$ weak homotopy equivalences

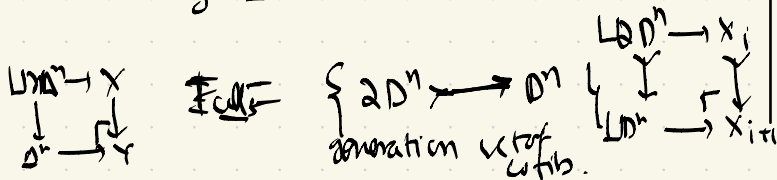


HLP w.r.t all co-complexes.



• cofibrations :=

$$J = \{ D^n \times S^1 \xrightarrow{\sim} D^n \times \{1\} \}$$

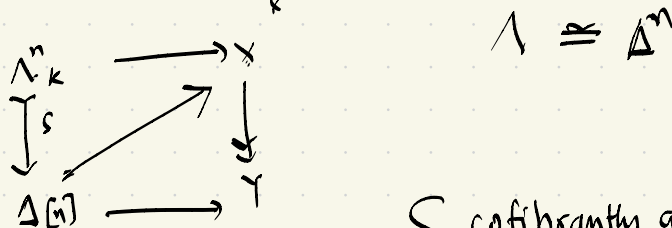


All objects are cofibrant

SSet / SSet

• $\simeq := X \xrightarrow{\sim} Y$ if $|X| \xrightarrow{\sim} |Y|$

• $\rightarrow :=$ Kan fibrations



{ cofibrantly generated model cats }

• $\rightarrow :=$ $K \subset L$

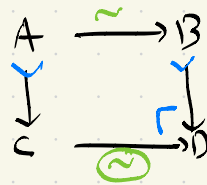
$$I = K \subseteq K(0) \subseteq K(1) \subseteq \dots \subseteq \text{colim}(K(i)) = L$$

$$\begin{array}{ccc} \partial \Delta[n] & \rightarrow & K(0) \\ \downarrow & & \downarrow \\ \Delta[n] & \rightarrow & K(1) \end{array}$$

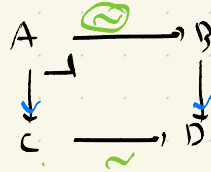
{ $\partial \Delta[n] \rightarrow \Delta[n]$ }
generators for cofibration

Proper Model Cat :

$\mathcal{M}$ is called left proper if



right proper.



$\mathcal{M}$ is both left and right proper then it's called proper.

e.g. Set, skel, Top, Top*, Spectra.

Quillen Equivalence:

We want a functor between model cats preserving the structure.

A left Quillen functor, $F: M \rightarrow N$

- F has right adjoint, G
- F preserves cofibrations and trivial cofibrations

is called Quillen Equivalence if:

$$\forall A \in \mathcal{C}_M, X \in \mathcal{F}_N$$

equivalently define right Quillen Functor.

G preserves fibrations and trivial fibrations.

$$FA \rightarrow X \quad \hookrightarrow \quad A \rightarrow GX$$

$$\begin{array}{ccccc} A & & X & \xrightarrow{\quad} & GA \\ \downarrow & & \downarrow s & \nearrow & \downarrow \\ B & & Y & \xrightarrow{\quad} & GB \end{array} \quad \hookrightarrow \quad \begin{array}{ccccc} FX & \xrightarrow{\quad} & A \\ \downarrow s & \nearrow & \downarrow \\ FY & \xrightarrow{\quad} & B \end{array}$$

$$M(A, GX) \cong N(FA, X) \text{ restrict to } \underbrace{WM(A, GX) \cong WN(FA, X)}$$

$$\begin{cases} M[W_M^{-1}] \cong M[W_N^{-1}] \\ \downarrow \quad \uparrow \\ hM_{cf} \cong hN_{cf} \end{cases}$$

Ex. 9.

- $S\&T \xrightarrow{||} Top$ Quillen equivalence
 - $S\&T \xrightarrow{C|+} cSet.$
 - $S\&T. \xrightarrow{\cong} Ab.$
- } Quillen Functor.

Remark: if both F and G preserve weak equivalence, F is a Quillen Equivalence

$\Leftrightarrow \forall$ fibrant objects $A \in M, X \in N,$

$$A \xrightarrow{\sim} GFA$$

$$FGX \xrightarrow{\sim} X$$

Start with F is Quillen equivalence.

$$\underline{A \xrightarrow{\sim} G \otimes X} \quad \underline{X = FA}$$

$\{ F$

$$FA \xrightarrow{\sim} FGX$$

$$\searrow \sim \quad \downarrow \textcircled{S}$$

analogous co is G .

$$FA \xrightarrow{\sim} X$$

$\{ G$

$$GFA \xrightarrow{\sim} GX$$

$$\textcircled{D} \nearrow A \nearrow \sim$$

F, G Quillen functor that preserve weak equivalences.

F is Quillen Equiv $\Rightarrow \forall$ ~~fibrant~~ A, X , $A \xrightarrow{\sim} GFA$, $FGA \xrightarrow{\sim} A$

$(\Rightarrow)$

$$\boxed{A \xrightarrow{\sim} GX \iff FA \xrightarrow{\sim} X.}$$

$\downarrow \text{Cof}$ $\downarrow \text{Fib}$ $\downarrow \text{Cof}$ $\downarrow \text{Fib}$

$A, \sim$ FA cofibrant
 $\downarrow$
 GFA

$A \xrightarrow{\sim} G \underbrace{FA}_B$ B is cofibrant

$A \xrightarrow{\sim} GB \iff FA \xrightarrow[\sim]{\text{id}} FA$

$FA \xrightarrow{\sim} FA$
 $\downarrow$
 $A \xrightarrow{\sim} GFA$

$(\Leftarrow)$ $A \xrightarrow{\sim} GFA$ | $FGX \xrightarrow{\sim} X$

$\text{Cof } A \xrightarrow{\sim} GX$ fibrant X to be fibrant.
 $\downarrow \text{Cof}$
 $FA \xrightarrow{\sim} FGX$
 $\downarrow \text{S}$
 $\xrightarrow{\sim} X$