Intro to Model Cats

Top	sSet
· Homotopy groups	· All the objects are not fibrant.
· All the objects are fibrant.	· Kan Ex^{∞}
	1950's.
geometric.	· combinatorial

Quillen (1967):

Model Cat: $(\mathcal{M}, \overset{\text{cofib}}{\rightarrowtail}, \overset{\text{fib}}{\twoheadrightarrow}, \overset{\text{weak equiv.}}{\xrightarrow{\simeq}})$

Axioms:

- $\mathcal{M}$ has all small limits and colimits.
- 2 out of 3 property for w.e.
$$\begin{tikzcd} \cdot \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & & \cdot \\ & \cdot \arrow[ur, "\simeq"'] & \end{tikzcd}$$
- Factorization

Motivation in top

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, tail] & & B \\ & Mf \arrow[ur, "\simeq"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr] \arrow[dr, "\simeq"'] & & B \\ & Pf \arrow[ur, two heads] & \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, tail] & & B \\ & C \arrow[ur, two heads, "\simeq"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, tail, "\simeq"'] & & B \\ & D \arrow[ur, two heads] & \end{tikzcd}$$

(functorial)

Necess

- lifting

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & B \arrow[d, two heads, "\simeq"] \\ C \arrow[r] \arrow[ur, dashed] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, tail, "\simeq"'] & B \arrow[d, two heads] \\ C \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

- retract (f is called retract of g)

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}"] & B \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ A' \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B' \arrow[r] & A' \end{tikzcd}$$

if $g \in \mathcal{C}, \mathcal{F}, \mathcal{W}$ so does f.

Cor: it comes with initial and terminal object.

A few Key Observations:

- $$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "l"'] & & Y \\ & Z \arrow[ur, "r"'] & \end{tikzcd}$$ $f = r \circ l$ if f has <u>left lifting prop.</u> w.r.t <u>r</u>, the f is <u>retract</u> of l.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Z \arrow[d, "r"] \\ Y \arrow[r, "="'] \arrow[ur] & Y \end{tikzcd} \rightsquigarrow \begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "l"] & X \arrow[d, "f"] \\ Y \arrow[r] \arrow[rr, bend right, equal] & Z \arrow[r, "r"] & Y \end{tikzcd}$$ f is retract of l.

$\overset{\sim}{\hookrightarrow}$. $\twoheadrightarrow_s$

- A map $f \in \mathcal{C}$ or ($\mathcal{C} \cap W$) $\Leftrightarrow$ f has <u>l.l.p</u> w.r.t maps in <u>$\mathcal{F} \cap W$</u> or ($\mathcal{F}$).

  A map $f \in \mathcal{F}$ or ($\mathcal{F} \cap W$) $\Leftrightarrow$ f has <u>r.l.p.</u> w.r.t maps in <u>$\mathcal{C} \cap W$</u> or ($\mathcal{C}$).

- Any two of $\mathcal{C}$, $\mathcal{F}$, W will determine the third. (signature feature)

- <u>pushout</u> of a <u>cofibration</u> (trivial) is a <u>cofibration</u> (trivial).

  <u>pullback</u> of a <u>fibration</u> (trivial) is a <u>fibration</u> (trivial).

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & P \arrow[d, "s"] \\ C \arrow[r] \arrow[urr] & D \arrow[r] \arrow[ur] & Q \end{tikzcd}$$

cofib. sequential <u>colimits</u>.

fibration. sequential <u>limits</u>.

cofibrant object: $\phi \rightarrowtail X$ Fibrant object: $X \twoheadrightarrow 1$

Q: When does a functor $F: M \to N$ (between Model cats) preserve weak equivalences?

<u>Ken Brown's Lemma</u>: (we see how all axioms take place)

$f: A \to B$ (A, B are cofibrant) then $\exists$ C_f and $j_f: B \xrightarrow[\sim]{} C$ s.t.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, tail] & B \arrow[dr, equal] & \\
& C_f \arrow[u, "s"] & B \arrow[l, "\sim"']
\end{tikzcd}$$

<u>Pf</u>:

$$\begin{tikzcd}
\phi \arrow[r, tail] \arrow[d, tail] & A \arrow[d, tail] \arrow[ddr, bend left, "f"] & \\
B \arrow[r, tail] \arrow[drr, bend right, equal] & A \amalg B \arrow[dr, "f \amalg id_B"] & \\
& & B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[d, tail] \arrow[dr, "j_f"] \arrow[drr, bend left, "id_B \sim"] & & \\
A \amalg B \arrow[r, tail] \arrow[rr, bend right, "f \amalg id_B"'] & C_f \arrow[r, "\sim"] & B \\
A \arrow[u, tail] \arrow[r, tail] \arrow[urr, bend right, "f"'] & C_f' \arrow[ur, "s"'] &
\end{tikzcd}$$

<u>Dually</u>:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, tail, "\sim"] & B \\
A \arrow[r, two heads, "\sim"'] & P_f \arrow[u]
\end{tikzcd}$$

<u>Cor</u>: $F: M \to N$ preserve <u>trivial cofib</u>, then it sends <u>$\sim$</u> between <u>cofibrant objects</u> to <u>$\sim$</u>.

$A \overset{\sim}{\rightarrowtail} B \xrightarrow{F} F(A) \overset{\sim}{\rightarrowtail} F(B)$

$$\begin{tikzcd}
A \arrow[r, "\sim"] \arrow[dr, tail, "\sim"'] & B \arrow[d, equal] \\
& B \\
C_f \arrow[uur, "s"] & B \arrow[l, "\sim"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \arrow[r, "\sim"] \arrow[d, tail, "\sim"'] & F(B) \arrow[d, equal, "s"] \\
F(C_f) \arrow[ur, "\sim"] & F(B) \arrow[l, "\sim"]
\end{tikzcd}$$

Homotopy Category

A cofibrant, B fibrant: on $\mathrm{Hom}(A, B)$ left and right homotopy coincide.

$f, g : A \rightrightarrows B$, $f \sim g$. ← (they have a notion of homotopy)

$$\mathrm{Ho}\mathcal{M} := \mathcal{M}(RQA, RQB)/\sim,$$

$$\begin{tikzcd} & QA \arrow[r, "\sim"] & A \\ & \phi \arrow[u, tail] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, tail, "\sim"] \arrow[dr] & RA \arrow[d] \\ & 1 \end{tikzcd}$$

$Q \Rightarrow id_{\mathcal{M}}$, $id_{\mathcal{M}} \Rightarrow R$

$h\mathcal{M}_{cf} :=$ objects bifibrant objects.

$$[A, B] = \frac{\mathcal{M}(A, B)}{\sim}$$

we can look them upto weak equivalences.

$\mathcal{M}[\mathcal{W}^{-1}]$

$$\bullet \xrightarrow{\sim} \bullet \xleftarrow{\sim} \bullet \xrightarrow{s} \bullet \xleftarrow{\sim} \bullet$$

$\mathcal{M}$, $\mathcal{W}$, $\boxed{\mathcal{M}[\mathcal{W}^{-1}]} = \mathrm{Ho}\mathcal{M}$.

example.

$$\begin{tikzcd} X\times\{1\} \arrow[r, "h'"] \arrow[d] & X \arrow[d, two heads] \\ Y\times I \arrow[r, "h"] \arrow[ur] & * \end{tikzcd}$$

All objects are fibrant

Top / Top$_*$

- $\simeq$:= weak homotopy equivalences.
- $\twoheadrightarrow$:= serre fibration

$$\begin{tikzcd} X_0 \arrow[r, "\tilde{h}_0"] \arrow[d, hook] & E \arrow[d, two heads] \\ X\times I \arrow[r, "h"] \arrow[ur, "h"] & B \end{tikzcd}$$

HLP.

HLP w.r.t all cw complexes.

$$\begin{tikzcd} D^n\times\{0\} \arrow[r] \arrow[d, "\mathrm{s}\wr"] & E \arrow[d, two heads] \\ D^n\times\{I\} \arrow[r] \arrow[ur] & B \end{tikzcd}, \quad \begin{tikzcd} \partial D^n \arrow[r] \arrow[d] & E \arrow[d, "s"] \\ D^n \arrow[r] \arrow[ur] & B \end{tikzcd}$$

- cofibrations :=

$J = \{D^n\times\{0\} \xrightarrow{\sim} D^n\times\{I\}\}$

$$\begin{tikzcd} \sqcup \partial D^n \arrow[r] \arrow[d] & X \arrow[d] \\ D^n \arrow[r] & Y \end{tikzcd}$$

I := $\{\partial D^n \rightarrowtail D^n\}$ generation set of cofib.

$$\begin{tikzcd} \sqcup \partial D^n \arrow[r] \arrow[d] & X_i \arrow[d] \\ \sqcup D^n \arrow[r] & X_{i+1} \end{tikzcd}$$

All objects are cofibrant

sSet / sSet$_*$

- $\simeq$:= $X \xrightarrow{\simeq} Y$ if $|X| \xrightarrow{\simeq} |Y|$.
- $\twoheadrightarrow$:= Kan fibrations.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, "s"] & X \arrow[d, two heads] \\ \Delta[n] \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

$\Lambda \simeq \Delta^n$.

cofibrantly generated model cats

- $\rightarrowtail$:= $K \subset L$,

$I = K \subseteq K(0) \subseteq K(1) - \cdots \subseteq \varinjlim(K(i)) = L$.

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & K(0) \arrow[d] \\ \Delta[n] \arrow[r] & K(1) \end{tikzcd}$$

$\{\partial\Delta[n] \rightarrowtail \Delta[n]\}$ generating set for cofibration.

Proper Model Cat :

$\mathcal{M}$ is called **left** proper if

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[d, tail] & B \arrow[d, tail] \\ C \arrow[r, "\sim"'] \arrow[ur, phantom, "\ulcorner", very near end] & D \end{tikzcd}$$

right proper.

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "\sim"] \\ C \arrow[r, "\sim"'] & D \end{tikzcd}$$

$\mathcal{M}$ is both left and right proper then its called proper.

e.g. SSet, sKel, Top, Top_*, Spectra.

Quillen Equivalence :

We want a functor between model cats preserving the structure.

A left Quillen functor $F: M \to N$,

- F has right adjoint, G
- F preserves cofibrations and trivial cofibrations

is called Quillen equivalence if

$\forall\ A \in \mathcal{C}_M,\ X \in \mathcal{F}_N,$

$$M(A, GX) \cong N(FA, X)$$ re strict fu. $\underbrace{wM(A, GX) \simeq wN(FA, X)}$.

$$\Updownarrow\quad M[W_M^{-1}] \simeq M[W_N^{-1}]$$

$$\quad \wr\ \uparrow$$

$$hM_{cf} \simeq hN_{cf}.$$

equivalently define right Quillen Functor.

G preserves. fibrations and trivial fibrations.

$FA \longrightarrow X \quad \rightsquigarrow \quad A \longrightarrow GX$

$$\begin{tikzcd} A \arrow[d] \\ B \end{tikzcd} \quad \begin{tikzcd} X \arrow[r] \arrow[d, "s"'] & GA \arrow[d] \\ Y \arrow[r] \arrow[ur] & GB \end{tikzcd} \rightsquigarrow \begin{tikzcd} FX \arrow[r] \arrow[d, "s"] & A \arrow[d] \\ FY \arrow[r] \arrow[ur] & B \end{tikzcd}$$

e.g.

- $sSet \xrightarrow{|\ |} Top$ Quillen equivalence
- $sSet \xrightarrow{(\ )_+} cSet.$ } Quillen Functor.
- $sSet. \xrightarrow{\mathbb{Z}} Ab.$

Remark: if both F and G preserve weak equivalence, F is a Quillen Equivalence

$\iff$ $\forall$ cofibrant objects $A \in \mathcal{M}$, $X \in \mathcal{N}$,

$$A \xrightarrow{\sim} GFA \qquad FGX \xrightarrow{\sim} X.$$

Start with F is Quillen equivalence.

$$\begin{tikzcd} A \arrow[r, "\sim"] & GX \end{tikzcd}$$

$X = FA$.

$\{ F$

$$\begin{tikzcd} FA \arrow[r, "\sim"] \arrow[dr, "\sim"'] & FGX \arrow[d] \\ & X \end{tikzcd}$$

analogous is G.

$$\begin{tikzcd} FA \arrow[r, "\sim"] & X \end{tikzcd}$$

$\{ G$

$$\begin{tikzcd} GFA \arrow[r, "\sim"] & GX \\ & A \arrow[ul] \arrow[u, "\sim"'] \end{tikzcd}$$

F, G Quillen functor that preserve weak equivalences.

F is Quillen Equiv $\Rightarrow$ $\forall$ ~~bifibrant~~, A, X, $A \xrightarrow{\sim} GFA$, $FGA \xrightarrow{\sim} A$

Cof $\downarrow$ fib

$(\Rightarrow)$ $\boxed{A \xrightarrow{\sim} GX \iff FA \xrightarrow{\sim} X}$. Cof, fi; cof, fib

$A, \sim FA$ cofibrant

$\wr$

GFA

$A \xrightarrow{\sim} G\underbrace{FA}_{B}$ B is cofibr

$A \xrightarrow{\sim} GB \xleftarrow{\sim} FA \xrightarrow[\sim]{id} FA$

$FA \xrightarrow{\sim} FA$

$\wr$

$A \xrightarrow{\sim} GFA$

$(\Leftarrow)$ $A \xrightarrow{\sim} GFA$, $FGX \xrightarrow{\sim} X$

cof $A \xrightarrow{\sim} GX$, $\boxed{\text{fibrant}}$ X to be fibrant.

$$\begin{array}{ccc} & A \xrightarrow{\sim} GX & \\ & \wr & \\ FA & \xrightarrow{\sim} & FGX \\ & \searrow\sim & \downarrow s \\ & & X \end{array}$$