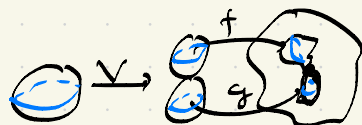


Simplicial Sets and Simplicial Homotopy:

Topological Spaces!



$$\text{Top}, \text{Top}_* \quad \pi_1(X, \cdot) = \frac{(S^1, \cdot) \longrightarrow (X, \cdot)}{\sim}$$

define π_n

$$F(Y) \{ (Y, \cdot) \longrightarrow (Z, \cdot) \}$$

$$X \wedge Y = \frac{X \times Y}{X \vee Y} \quad \text{in } \text{Top}_*$$

$$\text{Hom}_{\text{Top}_*}(X \wedge Y, Z) \cong \text{Hom}_{\text{Top}_*}(X, F(Y, Z))$$

$$X \xrightarrow{\sim} Y$$

$$\pi_i(X) \cong \pi_i(Y) \quad \forall i$$

Now for CGWH-spaces

$$\boxed{F(X \wedge Y, Z) \cong F(X, F(Y, Z))}$$

$$* = S^1$$

$$F(\Sigma X, Z) \cong F(X, \Omega Z)$$

$$\pi_0(F(\Sigma X, Z)) \cong \pi_0 F(X, \Omega Z)$$

$$[S^0, F(\Sigma X, Z)] \cong [S^0, F(X, \Omega Z)]$$

$$[S^1 \wedge \Sigma X, Z] \cong [S^1 \wedge X, \Omega Z]$$

$$\boxed{[\Sigma X, Z] \cong [X, \Omega Z]}$$

group

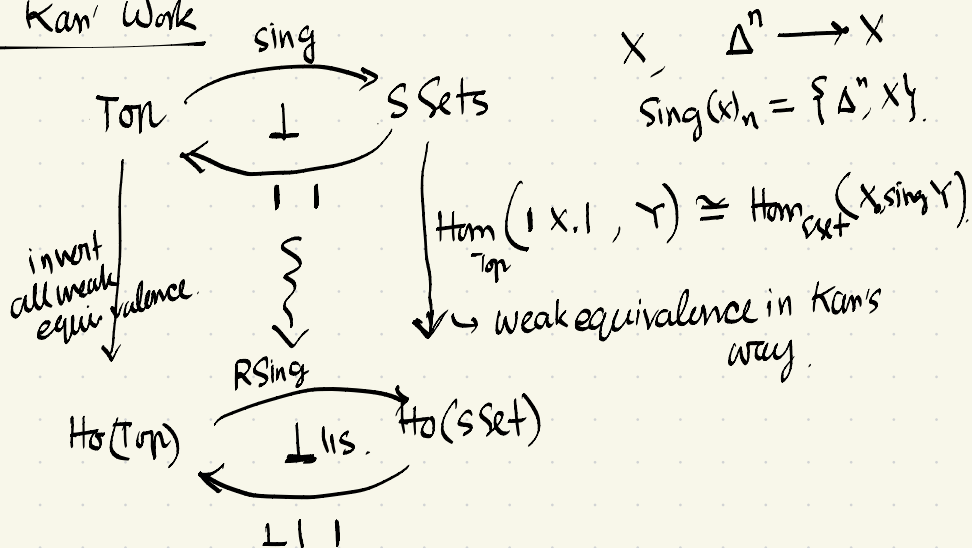
group

famous (w.p. suspension adjunction)

$$Ff \longrightarrow A \xrightarrow{f} B \leadsto \text{maps in homotopy}$$

We are limited to compute them

Now Kan Work



$$X, \Delta^n \rightarrow X$$

$$\text{Sing}(X)_n = \{\Delta^n, X\}$$

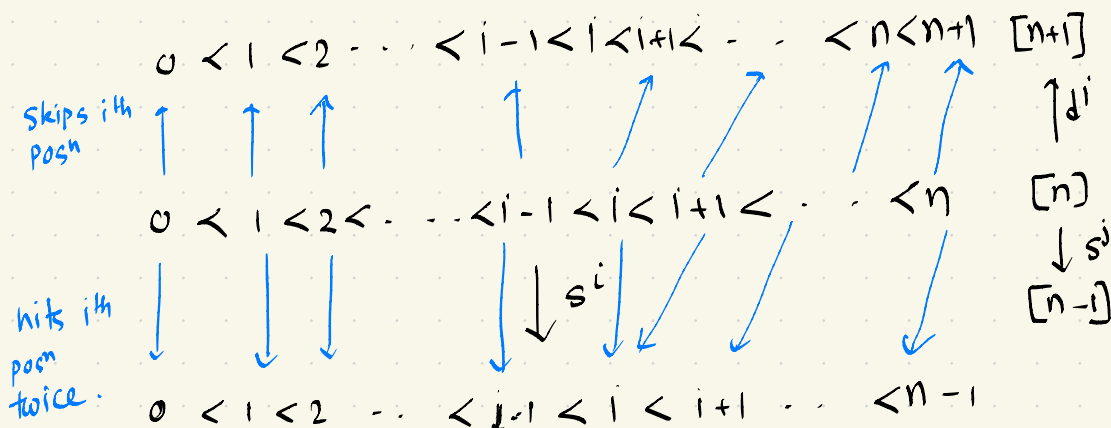
S Sets:

Δ = category of ordered sets,

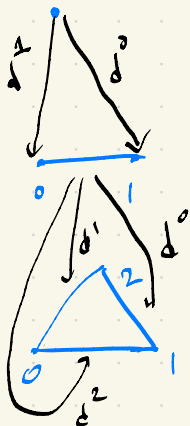
$$[n] = \{0 < 1 < \dots < n\}$$

$[n] \xrightarrow{f} [m]$ order preserving morphism.

$$a < b \Rightarrow f(a) \leq f(b)$$



[0]



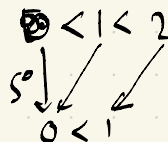
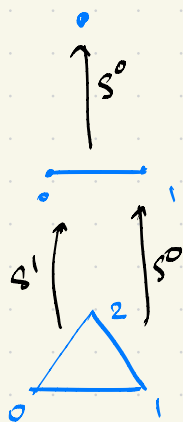
[1]

[2]

[0]

[1]

[2]



Check they have the following relationship,

$$d^j d^i = d^i d^{j-1} \text{ for } i < j$$

$$s^j s^i = s^{i-1} s^{j-1} \text{ for } i > j$$

$$s^i d^i = \begin{cases} d^i s^{i-1} & \text{for } i < j \\ \text{id} & \text{for } i = j \\ d^{i-1} s^j & \text{for } i > j+1 \end{cases}$$

Thm: any map $f: [n] \rightarrow [m]$ can be written as a composition of these.

Def:

A simplicial set is a functor $X: \Delta^{\text{op}} \rightarrow \text{sets}$.

$$[n] \rightarrow X[n] = X_n$$

$$[n+1]$$

$$\uparrow d^i$$

$$[n]$$

$$\downarrow s^j$$

$$[n-1]$$

$$\xrightarrow{X}$$

$$X_{n+1}$$

$$X d^i = d_i$$

$$X_n$$

$$X s^j = s_j$$

$$X_{n-1}$$

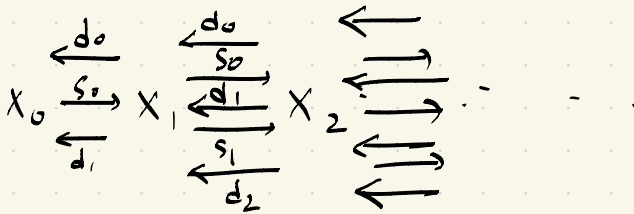
{

$$d_i d_j = d_{j-1} d_i \quad \text{for } i < j$$

$$s_i s_j = s_{j-1} s_{i-1} \quad \text{for } i > j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ \text{id} & \text{for } i = j, j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases}$$

This is another way of describing simplicial sets.



E.g.

$$\Delta(-, [n]) : \Delta^{\text{op}} \longrightarrow \text{Sets}$$

$$[q] \longmapsto \Delta([q], [n])$$

$\uparrow$

Standard simplicial n-simplex denoted by $\Delta[n]$

$$\Delta(-, [0]) \left\{ \begin{array}{l} \Delta([0], [0]) = \{ * \} \\ \Delta([1], [0]) = \{ * \} \\ \Delta([k], [0]) = \{ * \} \end{array} \right. \quad \begin{array}{l} \text{in moral sense} \\ \text{its a point} \end{array}$$

$$\Delta(-, [1])$$

$$\Delta^1[1]$$

$$\Delta([0], [1]) = \left\{ \begin{array}{c} [0] \\ [0 < 1] \end{array} \right\} = \{\text{two maps}\}$$

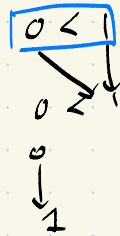
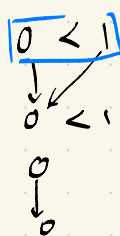
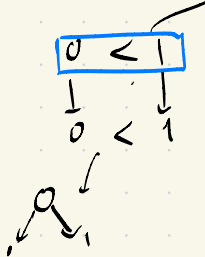
$$\Delta([1], [1]) = \left\{ \begin{array}{c} 0 < 1 \\ 0 < 1 \end{array} \right\} = \{3\text{-maps}\}$$

$$\Delta([2], [1]) = \left\{ \begin{array}{c} 0 < 1 < 2 \\ 0 < 1 \end{array} \right\} = \{5\text{ maps}\}$$

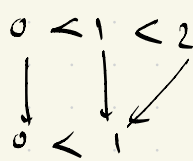
$$\Delta([k], [1]) = \{k+2 \text{ elements}\}$$

$$|\Delta[1]| = \Delta[0, 1]$$

nondegenerate maps



$$\begin{array}{cc} 0 & 1 \\ 0 \leftrightarrow 0, & 0 \rightarrow 1. \\ \hline = \Delta^1 = \mathbb{I} \end{array}$$



always get
degenerate
~~simp~~ maps.

$$\Delta[2] = \Delta(-, [2])$$

↓

$$\Delta([0], [2]) = \begin{matrix} \circ \\ 0 < 1 < 2 \end{matrix} = \{3 \text{ pts}\}$$

$$\Delta([1], [2]) = \left\{ \begin{matrix} 0 < 1 \\ 0 < 1 < 2 \end{matrix} \right\}$$

$$\Delta([2], [2]) = \begin{matrix} 0 \rightarrow 1 \rightarrow 2 \\ 0 \rightarrow 1 \rightarrow 2 \end{matrix}$$



$|\Delta[2]|$

$$|\Delta[n]| = \Delta^n$$

$$\begin{array}{ccccc} -[n] & \longrightarrow & \Delta^n & \longrightarrow & |\Delta^n| \\ \downarrow f & & \downarrow f & & \downarrow f \\ [m] & \longrightarrow & \Delta^m & \longrightarrow & |\Delta^m| \end{array}$$



$$\partial \Delta[1] = \{0, 1\} = \left\{ \begin{matrix} 0 \\ \downarrow \\ 0 \end{matrix}, \begin{matrix} 0 \\ \downarrow \\ 1 \end{matrix} \right\} = \Delta[0] \rightrightarrows \Delta[1]$$

$$= \left\{ \begin{matrix} 0 \\ \downarrow \\ 0 < 1 \end{matrix}, \begin{matrix} 0 \\ \downarrow \\ 0 < 1 \end{matrix} \right\} \quad \begin{matrix} [0] \\ \downarrow d^0, d^1 \\ [1] \end{matrix} \quad \begin{matrix} [0] \\ \downarrow d^0, d^1 \\ [1] \end{matrix} \quad \begin{matrix} [0] \\ \downarrow d^0, d^1 \\ [1] \end{matrix}$$

$$= \left\{ \begin{matrix} [0] \\ \downarrow d^0, d^1 \\ [1] \end{matrix}, \begin{matrix} [0] \\ \downarrow d^0, d^1 \\ [1] \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} \Delta[0] \\ \downarrow d^0, d^1 \\ \Delta[1] \end{matrix} \right\} = \text{imd}^0 \cup \text{imd}^1$$

$$|\partial \Delta[1]| = \partial \Delta^1$$

$$\text{simplicial circle } S^1 = \frac{\Delta[1]}{\partial \Delta[1]}$$

$$\Delta^{op} \longrightarrow \text{Set}$$

$\Delta : [n] \longmapsto \Delta[n]$ is called Yoneda embedding.

let X be a simplicial set.

$$\boxed{\{\Delta[n] \xrightarrow{\eta} X\} \leftrightarrow \{\eta \in X_n\}}$$

Yoneda Lemma

• $X \in \text{Top}$. Singular Complex.

$$\sum_{S \in \text{Set}} \text{Hom}_{\text{Top}}(\Delta^n, X) = S X_n.$$

$$d^i : \Delta^n \rightarrow \Delta^{n+1} \quad d^i(x_0, \dots, x_n) = (x_0, \dots, x_{i-1}, 0, x_i, x_{i+1}, \dots, x_n)$$

$$s^i : \Delta^{n+1} \rightarrow \Delta^n \quad s^i(x_0, \dots, x_{n+1}) = (x_0, \dots, x_{i-1}, x_i + x_{i+1}, \dots, x_{n+1})$$

$$|X| = \frac{\coprod X_n \times \Delta^n}{\sim} \quad \leftarrow \text{geometric realization.}$$

$$(x, s^j y) \sim (s^j x, y).$$

$$(x, d^i y) \sim (d^i x, y).$$

• Nerve of a category.

$$N C_0 = \text{ob}$$

$$N C_1 = \text{arrows}$$

$$N C_2 = \text{compositions}$$

$$\sim \mid N C = B C.$$

$$N C_{n+1} C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_{i-1} \rightarrow C_i \rightarrow \dots \rightarrow C_n$$

$$s^i \uparrow$$

$$N C_n \quad C_0 \rightarrow C_1 \rightarrow C_2 \xrightarrow{s^i} C_i \xrightarrow{s^i} C_{i+1} \rightarrow C_n$$

$$d^i \downarrow$$

$$N C_{n+1} \quad C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_{i-1} \rightarrow C_{i+1} \rightarrow C_n$$

3 - some constructions in $\mathbf{sSet}$

$\mathbf{sSet}$ has no categorical surprises, limits and colimits are calculated degree-wise.

$$\bullet (X \amalg Y)_q = X_q \amalg Y_q \bullet (X \times Y)_q = X_q \times Y_q$$

$$\mathbf{sSet}(X \times Y, Z) = \mathbf{sSet}(X, \text{Map}(Y, Z)) \quad \text{Map}(Y, Z) \in \mathbf{sSet}$$

How to define $\text{Map}(Y, Z)$ choose $X = \Delta[q]$

$$\begin{aligned} \mathbf{sSet}(\Delta[q] \times Y, Z) &= \mathbf{sSet}(\Delta[q], \text{Map}(Y, Z)) \\ &= \text{Map}(Y, Z)_q \quad (\text{Yoneda}) \end{aligned}$$

$$\text{Map}(Y, Z)_q = \mathbf{sSet}(\Delta[q] \times Y, Z).$$

For any closed monoidal cat, $\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))$

Composition: $\text{Map}(Y, Z) \times \text{Map}(X, Y) \longrightarrow \text{Map}(X, Z)$

$g \qquad f \qquad g \circ f$

$$X \times \Delta[q] \xrightarrow{\text{id} \times \Delta} X \times \Delta[q] \times \Delta[q] \xrightarrow{f \times \text{id}} Y \times \Delta[q] \xrightarrow{g} Z$$

$g \circ f$

Homotopy in terms of simplicial sets

in topological spaces, $x \begin{smallmatrix} \xrightarrow{f_0} \\ \xleftarrow{f_1} \end{smallmatrix} Y$

$$\begin{array}{ccc} X \times I & \xrightarrow{H} & Y \\ \uparrow & \searrow (G, f_1) & \\ X \times \partial I & & \end{array}$$

in simplicial sets.

$$\begin{array}{ccc} |X| \times |\Delta[1]| & \xrightarrow{h} & Y \\ \uparrow & \searrow \scriptstyle (f_0, f_1) & \\ |X| \times |\Delta[1]| & & \end{array}$$

this way we
define it.

$$\begin{array}{ccc} X \times \Delta[1] & \xrightarrow{h} & Y \\ \uparrow & \searrow \scriptstyle (f_0, f_1) & \\ X \times \Delta[1] & & \end{array}$$

$X \xrightarrow{h} Y$ is homotopy equivalence if $\exists g: Y \rightarrow X$ s.t.
 $g \circ f \simeq \text{id}$ and $f \circ g \simeq \text{id}$.

Simplicial Abelian Groups.

Ab = category of abelian groups. $\mathcal{A} = \mathbf{sAb}$.

$$\begin{array}{ccc} \text{Ab} & \begin{array}{c} \xleftarrow{\mathbb{Z}[\]} \\ \xrightarrow{\quad} \end{array} & \text{Sets} \\ \downarrow & \cup & \downarrow \\ \mathbf{sAb} & \begin{array}{c} \xleftarrow{\mathbb{Z}[\]} \\ \xrightarrow{\quad} \end{array} & \mathbf{sSet} \end{array}$$

$X \mapsto \mathbb{Z}[X]$ free abelian group on set X .

$\mathbf{sSet}$ defined degreewise.

$$\text{Hom}_{\mathbf{sAb}}(M \otimes P, N) = \text{Hom}_{\mathbf{sAb}}(M, \text{Map}(P, N))$$

$$\begin{aligned} M &= \mathbb{Z}[\Delta[q]] \quad , \quad \text{Hom}_{\mathbf{sAb}}(P \otimes \mathbb{Z}[\Delta[q]], N) = \text{Hom}_{\mathbf{sAb}}(\mathbb{Z}[\Delta[q]], \text{Map}(P, N)) \\ &= \text{Map}(P, N)_q \end{aligned}$$

Note: $\underline{A}(M, N)_q = A(M \otimes \mathbb{Z}[\Delta(-[q])], N)$

and we can define homotopy naturally. (like before)

$$C_* = \{ \dots \leftarrow C_{q-1} \leftarrow C_q \leftarrow C_{q+1} \leftarrow \dots \}$$

$f_* : C_* \rightarrow D_*$ such that levelwise diagrams commute.

$Ch :=$ category of chain complexes $Ch^{\geq 0}$ non negative graded chain complexes

defined cycle, boundaries and homologies.

$$M \in SAb, \leadsto C_*(M) \in Ch_{\geq 0}$$

$$M_0 \xleftarrow{d_0 - d_1} M_1 \xleftarrow{d_0 - d_1 + d_2} M_2 \xleftarrow{d_0 - d_1 + d_2 - d_3} \dots$$

singular homology of a topological space Y ,

$$H_*(Y) := H_*(C_*, \mathbb{Z}[\text{sing } Y])$$

more generally, for a simplicial set X , we define

$$H_*(X) = H_*(C_*, \mathbb{Z}[X])$$

$$\text{if } M \in SAb, \boxed{H_*(C_* M) \cong \pi_*(|UM|)} \quad \underline{\text{Kan}}$$

Normalized Chain Complex:

$$M \in SAb \leadsto C_*^n(M) \in Ch^{\geq 0}$$

$$C_q^n(M) = \bigcap_{i=0}^{q-1} \ker \{d_i : M_q \rightarrow M_{q-1}\} \xrightarrow{d_q} C_{q-1}^n(M)$$

Dold Kan: $A \xrightarrow{Ch^{nm}} Ch_{\geq 0}$ induces equivalence of cats,

$$H_*(A) \cong H_*(Ch_{\geq 0})$$

Pointed Case:

S_* for pointed simplicial sets.

$$X, Y \in \text{Sets}_*, \quad X \vee Y, \quad X \wedge Y := \frac{X \times Y}{X \vee Y}$$

$$X \in S_* \quad \Sigma X = S' \wedge X, \quad S' = \frac{\Delta[1]}{\partial \Delta[1]} \leftarrow \text{simplicial circle.}$$

$$[0] \rightarrow [1] \sim \Delta(-, [0]) \xrightarrow{i_1} \Delta(-, [1])$$

$$i_1 \downarrow 0 \quad \downarrow 0 \quad i_2 \quad \partial \Delta[1] = \text{im } i_1 \cup \text{im } i_2$$

$$S^n := S' \wedge S^{n-1} \quad S^0 = \partial \Delta(-, [1]) = \{0, 1\}$$

$$+ : \text{sets} \longrightarrow \text{sets}_* \quad (\text{adds a basepoint})$$

$$\text{Ab} \xleftarrow{\tilde{Z}[1]} \text{sets}_* \xleftarrow{+} \text{sets} \quad \tilde{Z}[X] = \frac{\mathbb{Z}[X]}{\mathbb{Z}[*]}$$

1. $(Z \times S)_+ \cong Z_+ \wedge S_+$,
2. $X \wedge (Y_1 \vee Y_2) \cong (X \wedge Y_1) \vee (X \wedge Y_2)$,
3. $S^0 \wedge X \cong X$,
4. $\tilde{Z}[X \vee Y] \cong \tilde{Z}[X] \oplus \tilde{Z}[Y]$,
5. $\tilde{Z}[X \wedge Y] \cong \tilde{Z}[X] \otimes \tilde{Z}[Y]$
6. if $A \subseteq X$, then $0 \rightarrow \tilde{Z}[A] \rightarrow \tilde{Z}[X] \rightarrow \tilde{Z}[X/A] \rightarrow 0$ is exact.

These relations are satisfied!

perform them degree-wise and we get corresponding constructions for pointed simplicial sets.

$$\pi_n(|\tilde{Z}[X]|) \cong \tilde{H}_n(*, \tilde{Z}[X]) \quad (\text{By dold-Kan})$$

Defⁿ: n^{th} Integral Eilenberg-MacLane space is $\tilde{Z}[S^n]$

$$\pi_q \tilde{Z}[S^n] = \begin{cases} 0 & \text{if } n \neq q \\ \mathbb{Z} & \text{if } n = q \end{cases} \quad \boxed{|\tilde{Z}[S^n]| = K(\mathbb{Z}, n)}$$

Mapping Spaces and Homotopies

$X, Y \in \mathbf{SSet}_*$, $\underline{S}_*(X, Y)$ pointed simplicial set,

$$\underline{S}_*(X, Y)_q = \{ \text{pointed simplicial maps } X \wedge \Delta[q]_+ \rightarrow Y \}$$

and we have similar properties like $\underline{S}(X, Y)$,

$$\underline{S}_*(Y, Z) \wedge \underline{S}_*(X, Y) \rightarrow \underline{S}_*(X, Z)$$

$$\underline{S}_*(X \wedge Y, Z) \cong \underline{S}_*(Y, \underline{S}_*(X, Z))$$

$f_0, f_1 \in \underline{S}_*(X, Y)$ pointed homotopy between them is a map
 $X \wedge \Delta[1]_+ \rightarrow Y$.

Loop Spaces and cohomology:

$$\text{sing-}\Omega |X| \cong \underline{S}_*(S^1, \text{sing}|X|)$$

$$\Omega X := \underline{S}_*(S^1, \text{sing}|X|) \cong \text{sing-}\Omega |X|.$$

loop and suspension are not adjoint,

reduced cohomology of $X \in \mathbf{SSet}$, is defined by

$$\tilde{H}^n(X) = \pi_0 \underline{S}_*(X, \tilde{Z}[S^n])$$

$$Y \rightarrow \text{sing-}\Omega |X|.$$

$$S^1 \wedge X \rightarrow Y \rightarrow \text{sing}|Y|.$$

$$X \rightarrow \underline{S}_*(S^1, \text{sing}|X|) = \underline{\Omega} Y$$

$$S^1 \wedge X \rightarrow \text{sing}|Y| \xrightarrow{?} Y$$