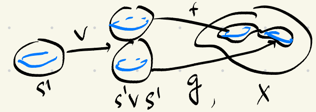
Introduction to Spectra:

$[X,Y] \leftarrow$ homotopy classes of maps.

$[(X,*),(Y,*)] \leftarrow$ " " based maps.

$\pi_n(X) = [(S^n,*),(X,*)] := [S^n,X]_*$

$X \longrightarrow Y$ weak equivalence, if $\pi_*(X) \xrightarrow{\cong} \pi_*(Y)$



Exercise: Show $\pi_n(X)$ is abelian for $n \geq 2$

Eckmann-Hilton: $(G,+,0)$, $(G,\cdot,1)$ two monoids,

$(a+b)\cdot(c+d) = (a\cdot c)+(b\cdot d)$ $\forall a,b,c,d$

then $(G,+,0) = (G,\cdot,1)$ abelian group.

Top_* closed, symmetric monoidal under '$\wedge$' product,

$$Hom_{Top_*}(X\wedge Y, Z) \cong Hom_{Top_*}(X, F(Y,Z))$$

For homotopy theory we restrict to CGWH / "spaces"

$$F(X\wedge Y, Z) \cong F(X, F(Y,Z)) \qquad F(X,Y) = \{(X,*) \longrightarrow (Y,*)\}$$

compact open top.

choose $Y = S^1$,

$$F(X\wedge S^1, Z) \cong F(X, F(S^1,Z))$$

$$F(\Sigma X, Z) \cong F(X, \Omega Z)$$

$$\pi_0(F(X,Y)) = [S^0, F(X,Y)] \cong [S^0\wedge X, Y] = [X,Y]$$

$$\pi_0(F(\Sigma X, Z)) \cong \pi_0(F(X,\Omega Z))$$

$$[\Sigma X, Z] \cong [X, \Omega Z] \leftarrow \text{loop-suspension adjunction!}$$

Computing the homotopy groups are hard!

To understand this lets see what tools we use for computing homology. (Co)

h-pushout: (1973)

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\ C \arrow[r] \arrow[drr, bend right] & D \arrow[ul, phantom, "\ulcorner"] \arrow[dr, dashed, "\exists!"] & \\ & & X \end{tikzcd}$$

$$D = B \cup_A C = \frac{B \sqcup C}{b \sim c \text{ if } \exists a,\ b = f(a),\ c = g(a)}$$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, hook] & & B \\ & M_f \arrow[ur, "\sim"] & \end{tikzcd}$$

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \\ D^2 \arrow[r] & S^2 \end{tikzcd}, \qquad \begin{tikzcd} S^1 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd},$$

$H_2(S^2) = \mathbb{Z}$,
$H_2(*) = 0$.

cofiber replacement:

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & M* \arrow[d] \\ M* \arrow[r] & S^2 \arrow[ul, phantom, "h\ulcorner"] \end{tikzcd},$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & D \arrow[ul, phantom, "h\ulcorner"] \end{tikzcd} := \begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & M_f \arrow[d] \\ M_g \arrow[r] & D \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \qquad M_f := \frac{A \times I \cup B}{(a,1) \sim b,\ b = f(a)}$$

$\simeq$

$$\cdots \to H_n(A) \to H_n(B) \oplus H_n(C) \to H_n(D) \to H_{n-1}(A) \to \cdots$$

(co)-homology Myer-Victoris sequence ← (crucial player)

h-pullback:

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} P_f \times_B P_g \arrow[r] \arrow[d] & P_g \arrow[d, two heads] \\ P_f \arrow[r] & B \end{tikzcd} \qquad P_f = \{(x, \gamma) \in A \times B^I,\ \gamma(0) = f(x)\}$$

$$\begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "\to h"] & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

$$\to \pi_n(D) \to \pi_n(A) \oplus \pi_n(C) \to \pi_n(B) \to \pi_{n-1}(D) \to \cdots$$

$$D = \{(x, y, \gamma) \mid f(x) = \gamma(0),\ g(y) = \gamma(1)\}.$$

Looks weird!

cofiber sequence:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ * \arrow[r] & C_f \arrow[ul, phantom, "h\ulcorner"] \end{tikzcd}$$

$A \to B \to C_f \rightsquigarrow$ LES in homology
h-cofiber seq

$$\begin{tikzcd} F_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "f"] \\ * \arrow[r] & B \end{tikzcd}$$

$F_f \to A \xrightarrow{f} B \rightsquigarrow$ LES in homotopy
homotopy fiber sequence

$$D = \{(x, y, \gamma) \mid f(x) = \gamma(0),\ g(y) = \gamma(1)\}.$$

Looks weird!

That means we really want a pushout version! ← we want h.pushout and pullback to coincide.

Blakers Massey Excision :

$$\begin{tikzcd}
Q \arrow[rr, "m - \mathrm{Conn}"] \arrow[dr, "m+n-1\ \mathrm{conn}"] \arrow[dd, "n\ \mathrm{Conn}"'] & & A \arrow[dd] \\
& R \arrow[ur] \arrow[dl] & \\
B \arrow[rr] & & P
\end{tikzcd}$$

$A \xrightarrow{m} B$

$\pi_i(A) \cong \pi_i(B) \quad i \leq m-1.$

$\pi_i(A) \twoheadrightarrow \pi_i(B) \quad i = m.$

so you can replace Q by R upto $m+n-2$ level.

$$\begin{tikzcd}
S^1 \arrow[rr] \arrow[dd] \arrow[dr, "X^2"] & & * \arrow[dd] \\
& \Omega S^2 \arrow[ur] \arrow[dl] & \\
* \arrow[rr] & & S^2
\end{tikzcd}$$

$A \xrightarrow{m} B$

$\pi_i(A) \cong \pi_i(B) \quad i \leq m-1.$

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so you can replace Q by R upto $m+n-2$ level.

We want to get rid of this connectivity!

$A \to B$ n connected $n-1$

loop space $\longrightarrow \Omega A \longrightarrow \Omega B$. $(n-1)$ connected $\qquad A \longrightarrow \Omega A$

$[S^{n-2}, \Omega A] \to [S^{n-2}, \Omega B]$.

$[S^{n-1}, A] \to [S^{n-1}, B]$.

$\Omega^2(A) \to \Omega^2(B) \quad (n-2)$ connected.

$A^2 \longrightarrow B^2 \quad n$ connected.

if $\exists\ A'$: $\Omega A' = A^2$, $= \Omega B^2$.

$A^1 \to B^1 \quad (n+1)$ connected. ? right:

$\pi_n(A) \to \pi_n(B) \qquad [S^n, A'] \to [S^n, B']$

$[\Sigma S^{n-1}, A'] \to [\Sigma S^{n-1}, B']$

$\to [S^{n-1}, \Omega A'] \cong [S^{n-1}, \Omega B']$

delooping increases connectivity

$A \to \Omega A \to \Omega^2 A \to \cdots$

$X := X_0, X_1, X_2, X_3, \cdots$

$X_1 \simeq \Omega X_2, \quad X_2 \simeq \Omega X_3.$

$X_n \xrightarrow{\simeq} \Omega X_{n+1}$

Ω-spectrum ! ←

Model them

- 1972 (May, Operads)
- 1973 (Boardman-Vogt)
- 1974 (Segal Γ-spaces).
- 1978 (May, Thomason) showed they are equivalent in homotopy cat.

$X_0 \simeq \Omega X_1$

$\Sigma X_0 \to X_1$

Now define spectrum:

$(X_\cdot)$

$$\begin{tikzcd} & \Sigma X_0 \arrow[dr, "\sigma_0"] & & \Sigma X_1 \arrow[dr, "\sigma_1"] & \\ X_0 \arrow[ur] \arrow[rr, "\sigma_0"'] & & X_1 \arrow[ur] \arrow[rr, "\sigma_1"'] & & X_2 \cdots \end{tikzcd}$$

$\mathbb{S},$

$$\begin{tikzcd} & \Sigma S^0 \arrow[dr, equal] & & \Sigma S^1 \arrow[dr, equal] & \\ S^0 \arrow[ur] \arrow[rr, "\sigma_0"'] & & S^1 \arrow[ur] \arrow[rr, "\sigma_1"'] & & S^2 \arrow[r, "\sigma_2"] & \cdots \end{tikzcd}$$

$\sigma_0 = \mathbb{Z}$, $\sigma_1 = \mathbb{Z}$

$\Omega^\infty X,$

map between two spec

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] \arrow[dr, phantom, "\circlearrowleft"] & X_1 \arrow[r] \arrow[d, "f_1"] \arrow[dr, phantom, "\circlearrowleft"] & \cdots \\ Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \end{tikzcd}$$

- homotopy pushout and pullback coincide.

$$\begin{tikzcd} X_\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y_\cdot \arrow[d] \\ Z_\cdot \arrow[r] & W_\cdot \end{tikzcd} := \begin{tikzcd} X_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y_n \arrow[d] \arrow[r, bend left, "\sigma_n^Y"] & Y_{n+1} \arrow[d] \\ Z_n \arrow[r] \arrow[dr, bend right, "\sigma_n^Z"'] & W_n \arrow[r, "\sigma_n"] & W_{n+1} \\ & Z_{n+1} \arrow[ur, bend right] & \end{tikzcd}$$

$$\begin{tikzcd} X_{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y_{n+1} \arrow[d] \\ Z_{n+1} \arrow[r] & W_{n+1} \end{tikzcd}$$

- coprduct and product coincide. $X_\cdot \vee Y \simeq \cancel{X_\cdot} X_\cdot \times Y_\cdot$

Eilenberg - Maclane Spectra : introduce it.

$K(A,n)$, $\pi_i(K(A,n)) = A \quad i=n$
$\qquad\qquad\qquad = 0 \quad$ otherwise.

$$\pi_i(\Omega K(A,n)) = \pi_{i+1}(K(A,n))$$
$$= \begin{cases} A & i+1=n \\ 0 & i \neq n-1 \end{cases}$$
$$= \pi_i K(A, n-1).$$

$$\Omega K(A,n) \simeq K(A,n-1).$$

$$\underset{\text{Ab}}{\overset{\cap}{A}} \rightsquigarrow \underset{\text{Sp}}{\overset{\cap}{HA}} \leftarrow \text{eilenberg-maclane spectra.}$$

singular Homlogy, $\tilde{H}^n(X,A) \underset{\uparrow}{\cong} [X, K(A,n)]$
brown representability.

Reduced Cohomology theories:

A reduced cohomology is a sequence of functors,

$$h^i : \text{Top}_*^{op} \longrightarrow \text{Ab} \quad \forall i \in \mathbb{Z},$$

together with natural suspension isos, $\sigma^i : h^i(X) \xrightarrow{\cong} h^{i+1}(\Sigma X)$,

$$\begin{tikzcd} h^i(X) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & h^{i+1}(\Sigma X) \arrow[d] \\ h^i(Y) \arrow[r, "\cong"] & h^{i+1}(\Sigma Y) \end{tikzcd}$$

with the following axioms.

① Homotopy invariance: $\begin{tikzcd} X \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\simeq"] & Y \end{tikzcd} \rightsquigarrow \begin{tikzcd} h^i(X) \arrow[r, bend left, "h^i(f_1)"] \arrow[r, bend right, "h^i(f_2)"'] \arrow[r, phantom, "\|"] & h^i(Y) \end{tikzcd}$

② Exact: $A \xrightarrow{f} B \longrightarrow C_f$ homotopy cofiber sequence

$$h^i(C_f) \longrightarrow h^i(B) \longrightarrow h^i(A) \quad \text{is exact,}$$

③ Additivity: $\{X_j\}$ be collection of pointed spaces,

$$h^i(\vee_j X_j) \cong \prod_j h^i(X_j)$$

Remark: There is another axiom, dimension axiom.

$$h^i(S^0) = \begin{cases} \mathbb{Z} & i=0. \\ 0 & \text{otherwise}, \end{cases}$$

we don't include it, because $\exists!$ cohomology theory satisfying all of them namely singular cohomology.

long exact sequence :

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d, "g"] \arrow[dl, phantom, "\ulcorner" near start, "h"] & * \arrow[d] \\
* \arrow[r] & C_f \arrow[r, "\ell"] \arrow[d] \arrow[ur, phantom, "h \ulcorner"] & C_g \cong \Sigma A \arrow[d] \\
& * \arrow[r] & \Sigma B \arrow[ul, phantom, "\ulcorner" near start]
\end{tikzcd}$$

← puppe sequence

$$\cdots \to h^i(\Sigma B) \to h^i(\Sigma A) \to h^i(C_f) \to h^i(B) \to h^i(A) \to \cdots$$

$$h^{i-1}(B) \to h^{i-1}(A) \to h^i(C_f) \to h^i(B) \to h^i(A) \to \cdots$$

Myer Vietoris Sequence :

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d, "\sim"] \\
C \arrow[r] \arrow[ur, phantom, "h \ulcorner"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f \vee g"] \arrow[d] & B \vee C \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] \arrow[ur, phantom, "h \ulcorner"] & D \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma A
\end{tikzcd}$$

$D = B \cup^h_A C$

$\underline{B \vee C \cup x}$

$f \vee g\,(a) \sim *$

$= f(a) \sim * \sim g(a) \quad = B \vee^h_A C$

~~[illegible]~~

$$\cdots \to h^i(\Sigma A) \to h^i(D) \to h^i(B) \oplus h^i(C) \to h^i(A) \to \cdots$$

$h^i(\Sigma A) \cong h^{i-1}(A)$

Thm : X be an Ω-spectrum.

$$h^n(Y) = \begin{cases} [Y, X_n] & n \geq 0 \\ [Y, \Omega^{-n} X_0] & n < 0 \end{cases}$$

is a reduced cohomology theory. (its called generalized cohomology theory represented by X).

converse of the theorem is also true and called Brown's Representability.