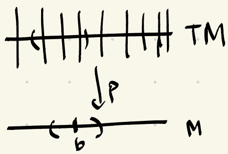
Vector Bundles:

$p^{-1}(b)$ is $T_bM \cong \mathbb{R}^n$



M is a manifold

locally it looks like $p^{-1}(U) \cong \coprod_{p\in U} T_pM$

$\cong U \times T_pM$

$= U \times \mathbb{R}^n$

- ~~Vector~~ fiber Bundle: $\left(\begin{array}{c} E \\ \downarrow p \\ B \end{array}\right)$ ← this information is called vector bundle.

  (E ← total space, B ← base, p ← projection map)

  - $p^{-1}(b)$ is a vector space. ← relax this get fiber bundle.

  - $\forall b \in B$, $\exists\, U \ni b$ open, $k \in \mathbb{Z}_{\geq 0}$ rank, $\varphi_b : p^{-1}(U) \xrightarrow{\cong} U \times \underline{\underline{\mathbb{R}^k}}_{F}$:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong", "\varphi_b"'] \arrow[dr, "p"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ & B & \end{tikzcd}$$

← local trivilization. ($\mathbb{R}^k$ = F)

- **Morphism of fiber bundle:**

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & \circlearrowleft & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

$f : \left(\begin{array}{c} E \\ \downarrow p \\ B \end{array}\right) \to \left(\begin{array}{c} E' \\ \downarrow p' \\ B \end{array}\right)$

$f_b : p^{-1}(b) \longrightarrow p'^{-1}(b)$ morphism

(vector bundle, ~~vect~~ linear morphism)

Examples:

- trivial bundle $\begin{array}{c} B \times F \\ \downarrow \\ B \end{array}$

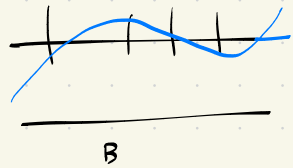
- tangent bundle. TM, sphere bundle $S(TM) = \{(x,v) \mid |v| = 1\}$, Disk bundle $D(TM) = \{(x,v) \mid |v| \leq 1\}$.

- normal bundle M^n, $M \hookrightarrow \mathbb{R}^N$ (whitney embedding).

$$\nu = \{(x,v) \mid x \in M,\ v \text{ is normal to } M \text{ at } x\}.$$

Def^n (Section):

$$\begin{tikzcd} E \arrow[d, "p"'] \\ B \arrow[u, bend right, "s"'] \end{tikzcd} : \quad p \circ s = id_B .$$



$S(TS^2)$ does not have a section.

ee why?

Poincaré Hopf thm: For a smooth closed manifold, M $S(TM)$ has a section iff Euler character 0.

A smooth manifold whose tangent bundle is trivial is called parallelizable.

Thm (Hopf invariant 1): Only parallelizable spheres are S^0, S^1, S^3, S^7 which are unit spheres in $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ - octonian
real, cmplx, hermitian

The multiplication structures in these structures, help to construct trivilization of these tangent bundles.

Formal Constructions:

for fibre: $f : U \times F \longrightarrow U \times F' \longleftrightarrow f' : U \longrightarrow Map(F, F')$ ← compact open topology

$u \longrightarrow Pr_2 \circ f|_{u \times F}$

$Pr_1 \circ f = Pr_1$

vect bundle: f fiberwise linear $\longrightarrow$ $u \longmapsto Pr_2 \circ f|_{u \times F}$ $(u \xrightarrow{f'} Lin(F, F'))$

$$\begin{tikzcd} U \times F \arrow[rr, "f"] \arrow[dr, "Pr_1"'] & \arrow[d, phantom, "\circlearrowleft"] & U \times F' \arrow[dl, "Pr_1"] \\ & U & \end{tikzcd}$$

$U \times F \to F' \rightsquigarrow U \to Map(F, F')$

$(u, x) \mapsto (u, \underline{\tilde{f}(u, x)})$

Using this lemma we can build bundles, patching together trivial bundles.

E.g.:

- $p: E \longrightarrow B$ fiber bundle, with fiber F
- $\{U_\alpha\}_{\alpha \in A}$ trivilization cover for B, $\varphi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times F$ (trivilization homeo)
- for $\alpha, \beta (\neq \alpha) \in A$, we have,

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times F \arrow[r, "\varphi_\alpha^{-1}"] \arrow[dr, "\pi_1"'] & p^{-1}(U_\alpha \cap U_\beta) \arrow[r, "\varphi_\beta"] \arrow[d, "p"] & (U_\alpha \cap U_\beta) \times F \arrow[dl, "\pi_1"] \\
& B &
\end{tikzcd}$$

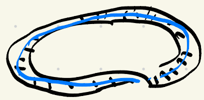
$U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} \mathrm{Map}(F,F)$:

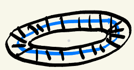
- $g_{\alpha\alpha} = id$
- $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$
- $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = id$

So if we have such g's, satisfying those conditions,
we can retrieve E by $E = \coprod_\alpha U_\alpha \times F / \sim \quad (x,v) \sim (x, g_{\alpha\beta}(x) \cdot v)$

E.g. Mobius Bundle: $U_1 = S^1 \setminus \{\text{North pole}\}$, $U_2 = S^1 \setminus \{\text{south pole}\}$

$$g_{12} : U_1 \cap U_2 \longrightarrow \mathrm{Map}(\mathbb{R}, \mathbb{R}) \supseteq Gl_1(\mathbb{R})$$
$$(x,y) \longrightarrow -1$$
$$(x,y) \longrightarrow 1,$$





$(x,v) \sim (x,v)$, U_1
$(x,v) \sim (x,-v)$, U_2

Lem: TS^1 is not iso to Mobius Bundle.

$TS^1 \setminus s_0(S^1) = S^1 \times \mathbb{R}^1 \setminus \{0\}$ not connected.

Mobius$\setminus s_0(S^1)$, still connected

By construction, we can give intrinsic example to tangent bundle.

M^n smooth, $\{(U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{R}^n)\}$

$f_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\beta \varphi_\alpha^{-1}} \varphi_\beta(U_\alpha \cap U_\beta)$

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{R}) \subseteq Map(\mathbb{R}^n, \mathbb{R}^n)$$

$$g_{\alpha\beta}(x) := df_{\varphi_\alpha(x)}$$

$g_{\alpha\alpha}(x) = dId = Id,$

$g_{\alpha\beta} = g_{\beta\alpha}^{-1}$

$df_{\varphi_\alpha(x)}$

$f_{\beta\alpha} : \varphi_\beta(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\alpha \varphi_\beta^{-1}} \varphi_\alpha(U_\alpha \cap U_\beta)$

$\uparrow \varphi_\beta$

$g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{R})$

$g_{\beta\alpha}^{-1} = (df_{\varphi_\beta(x)})^{-1} = df_{\varphi_\beta \varphi_\beta^{-1} \varphi_\alpha(x)}$

$= df_{\varphi_\alpha(x)}$

$d(f^{-1})_{f(x)} = (df_x)^{-1}$

$d(f_{\beta\alpha}^{-1})_{f\varphi_\beta(x)} = d(f_{\alpha\beta})_{f_{\alpha\beta}\varphi_\beta\varphi_\beta^{-1}\varphi_\alpha(x)}$

$= \underline{d(f)_{\alpha(x)}}$

<u>Next goal classify bundle up to iso:</u>

- $\begin{matrix} F \\ \downarrow \\ E \\ \downarrow p \\ B \end{matrix}$, $\begin{matrix} F' \\ \downarrow \\ E' \\ \downarrow p' \\ B \end{matrix}$, $f : \left(F \to \begin{matrix} E \\ \downarrow \\ B \end{matrix}\right) \xrightarrow{\cong} \left(F' \to \begin{matrix} E' \\ \downarrow \\ B \end{matrix}\right)$, then $\forall b \in B$, f_b is homeo.

conversely, F, F' are compact Hausdroff spaces, if $\forall b \in B$, f_b is homeo then f is iso.

$$\begin{tikzcd} E \arrow[rr, "\cong", "f"'] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

$\Rightarrow$ $f|_b = f|_{p^{-1}(b)} : p^{-1}(b) \xrightarrow{\sim} p'^{-1}(b)$

$(\Leftarrow)$ $\forall b \in B, \exists U :$ $p^{-1}(U) \cong U \times F$

$p'^{-1}(U) \cong U \times F'$

<u>we define</u> $\tilde{f} : U \times F \xrightarrow{\cong} U \times F'$

$\tilde{f}(b, x) = (b, f_b(x))$ f_b is <u>homeo.</u>

$\tilde{f}$ is <u>homeo.</u>

$\tilde{g}(b,y) = (b, f_b^{-1}(y))$ $\quad \hat{f} : U \to \underset{}{\overset{\text{top group.}}{\mathrm{Homeo}(F, F')}}$

$$\underbrace{U \xrightarrow[]{\hat{\tilde{f}}\ \text{cts}} \mathrm{Homeo}(F, F') \xrightarrow{\text{cts.}} \mathrm{Homeo}(F', F)}_{\hat{\tilde{g}}\ \text{cts.}}$$

$$\begin{tikzcd}
U \times F \arrow[r, bend left, "\tilde{f}"] & U \times F' \arrow[l, bend left, "\tilde{g}"] \\
p^{-1}(U) \arrow[u, "\Phi", "\cong"'] \arrow[r, bend left, "f"] \arrow[dr, "p"'] & p'^{-1}(U) \arrow[u, "\cong"'] \arrow[l, bend left, "g"] \arrow[d, "p'"] \\
& U
\end{tikzcd}$$

<u>Stiefel Manifolds:</u>

$F \to E \xrightarrow{p} B \quad \rightsquigarrow \quad \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$

we will use this to compute homotopy groups of stiefel manifolds.

$V_n(\mathbb{R}^k)$ = the set of orthogonal n frames in $\mathbb{R}^k$.

$= \{(v_1, \ldots v_n) \in (\mathbb{R}^k)^n \mid v_i \perp v_j,\ |v_i| = 1\} \subseteq (S^{k-1})^n$

compact.

$V_n(\mathbb{R}^n) = O(n), \qquad V_1(\mathbb{R}^k) = S^{k-1}$.

<u>Lemma:</u> $p : V_n(\mathbb{R}^k) \to S^{k-1}$ is a fiber bundle with fiber $V_{n-1}(\mathbb{R}^{k-1})$.

$(v_1, \ldots v_n) \mapsto v_n$

$p^{-1}(x) = (v_1, \ldots v_{n-1}, x) \qquad v_i \in x^{\perp}$.

$p^{-1}(U_i) \to U_i \times V_{n-1}(\mathbb{R}^{k-1})$

$(v_1 \ldots v_n) \mapsto (v_n, GS(\underbrace{\text{--}\ \text{--}}_{\text{linearly independent}}))$

<u>Propn:</u> For $1 < m < k-2$, $\pi_m V_{n-1}(\mathbb{R}^{k-1}) \cong \pi_m V_n(\mathbb{R}^k)$

$$V_{n-1}(\mathbb{R}^{k-1}) \longrightarrow V_n(\mathbb{R}^k) \longrightarrow S^{k-1}$$

$$\cdots \longrightarrow \underset{\substack{\| \\ 0}}{\pi_{m+1}(S^{k-1})} \to \pi_m V_{n-1}(\mathbb{R}^{k-1}) \xrightarrow[\cong]{} \pi_m V_n(\mathbb{R}^k) \longrightarrow \underset{\substack{\| \\ 0}}{\pi_m(S^{k-1})} \longrightarrow \cdots$$

π_m depends on $k-n$ ← to increase this, better to look at colim.

$$\mathbb{R}^n \subseteq \mathbb{R}^{n+1} \subseteq \cdots \qquad \operatorname{Colim} \mathbb{R}^n = \mathbb{R}^\infty$$

$$\{V_n$$

$$V_n(\mathbb{R}^n) \subseteq V_n(\mathbb{R}^{n+1}) \subseteq \cdots \qquad \operatorname{Colim}_k V_n(\mathbb{R}^k) = V_n(\operatorname{Colim}_k \mathbb{R}^k) = V_n(\mathbb{R}^\infty)$$

$$V_n := \operatorname{Colim} V_n(\mathbb{R}^k)$$

<u>Claim</u>: V_n is weakly contractible.

$$\pi_m V_n = [S^m, V_n] \qquad S^m \longrightarrow V_n = \bigvee_{k=n}^{\infty} V_n(\mathbb{R}^k)$$

$V_n(\mathbb{R}^k)$ for some k by compact object argument.

for $\underline{k > m+n}$

$$\pi_m(V_n \mathbb{R}^k) = 0.$$

$$\pi_m(V_n) = 0$$

Vector Bundles:

Already talking about it.

Prop^n 1 $P: E \longrightarrow B$ be an n-dim'l vector bundle $\exists$ n sections $B \xrightarrow{S_i} E$: $\forall x \in B$, $s_1(x), \cdots s_n(x)$ L.I $\iff$ E is trivilizable.

($\Rightarrow$) Suppose $E \cong B\times\mathbb{R}^n$, $s_i : B \to E$ $s_i(b) = (b, e_i)$, e_i's basis for $\mathbb{R}^n$.

$\Rightarrow$ s_i's are L.I.

($\Leftarrow$) suppose L.I $s_1, \dots, s_n$ exist.

Then define $P : B\times\mathbb{R}^n \longrightarrow E$, define

$$P(b, (a_1 \dots a_n)) := (b, \Sigma a_i s_i(b))$$

Cor. Mobius strip has no everywhere non-zero section.

Grassmannians: $G_n(\mathbb{R}^k) = V_n(\mathbb{R}^k)/O(n)$ = space of n planes in $\mathbb{R}^k$ (unordered)

$G_n(\mathbb{R}^n) = V_n(\mathbb{R}^n)/O(n) \cong *$, $G_1(\mathbb{R}^k) = S^{n-1}/O(1) \cong \mathbb{R}P^{n-1}$.

$$G_n(\mathbb{R}^k) \hookrightarrow G_n(\mathbb{R}^{k+1}) \hookrightarrow \cdots$$

$$G_n = \operatorname{colim}_k G_n(\mathbb{R}^k)$$

. $G_n(\mathbb{R}^k)$ is a Manifold of dim^n $n(k-n)$.

$\nu \in G(\mathbb{R}^k)$, a nplane in $\mathbb{R}^k$. $\nu^\perp$ $(k-n)$ dim^l.

$U_\nu := \{ \omega \in G_n(\mathbb{R}^k) \mid \omega \cap \nu^\perp = 0 \} \cong (\mathbb{R}^{k-n})^n$.

Ψ ν $\quad \tilde{\omega} : \underset{n}{\nu} \longrightarrow \underset{(k-n)}{\nu^\perp}$ $\quad (a, \tilde{\omega}(a)) = \omega(a)$, $\underline{\tilde{\omega}(a) \perp a}$.

$(k-n) \times n$ matrix.

$\boxed{G_n \cong BO(n)}$ thm

Def^n: G - Top group. $EG \underset{\text{weakly}}{\simeq} *$, with a free G action

$$BG = EG/G \qquad \Rightarrow G_n \cong BO(n)$$

$$G_n = V_n/O(n)$$

$$T_v : V_n(\mathbb{R}^k) \longrightarrow V_{n+1}(\mathbb{R}^{k+1}) \quad \text{is } O(n) \text{ equivariant}$$

$$(a_1, \dots a_n) \longmapsto (a_1, \dots a_n, 1)$$

$$\begin{tikzcd}
V_n(\mathbb{R}^k) \arrow[r] \arrow[d, "O(n)"'] \arrow[dr, phantom, "\circlearrowleft"] & V_{n+1}(\mathbb{R}^{k+1}) \arrow[d, "O(n) \hookrightarrow O(n+1)"] \\
V_n(\mathbb{R}^k) \arrow[r] & V_{n+1}(\mathbb{R}^{k+1})
\end{tikzcd}$$

equivariance diagram

$$M \mapsto \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array}\right]$$

$\Downarrow$

by taking Colimit $\quad G_n \xrightarrow{T} G_{n+1}$

$\Downarrow$

$BO(n) \longrightarrow BO(n+1)$ ← this will play a key role in development in K-theory later.

Next Universal bundle, classification of rank n bundle by grassmannians. Chp 3.