**Def 3.6:** $p: E \to B$ fiber bundle w/ fiber $F$
$f: B' \to B$ any map

The <u>pullback of $p$ along $f$</u> is the fiber bundle
$$p': B' \times_B E \to B'$$
given by the pullback square

$$\begin{tikzcd}
B' \times_B E \arrow[r] \arrow[d, "p'"'] & E \arrow[d, "p"] \\
B' \arrow[r, "f"'] & B
\end{tikzcd}$$

We write this bundle as $f^*p : f^*E \to B'$.
It has fiber $F$.

**Lemma 3.7:** $f^*p$ is well-defined.

**Pf:**

$$\begin{tikzcd}
f^{-1}(U) \times F \arrow[rr, dashed] \arrow[dr, "f \times id_F"] \arrow[dd] & & f^*(E) \arrow[dr, "f'"] \arrow[dd] & \\
& U \times F \arrow[rr] \arrow[dd] & & E \arrow[dd, "p"] \\
f^{-1}(U) \arrow[rr] \arrow[dr, "f"'] & & B' \arrow[dr] & \\
& U \arrow[rr] & & B
\end{tikzcd}$$

Let $\mathcal{U}$ be the trivialization cover of $p$.

Let $\mathcal{U}$ be the trivialization cover of $p$.
WTS: $\exists$ an open cover s.t. $\forall U \in \mathcal{U}$,
there is a pullback square

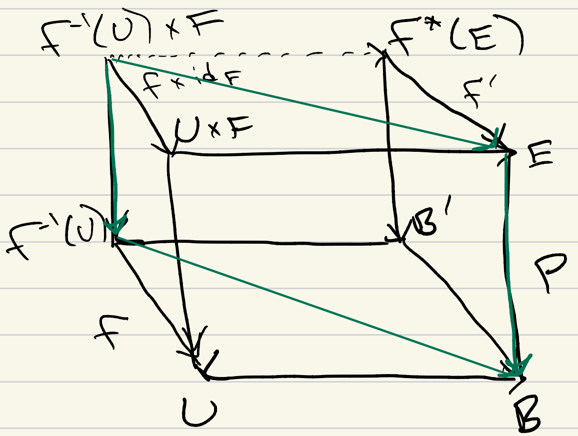
$$\begin{tikzcd} U \times F \arrow[r] \arrow[d, "\text{proj}_1"'] & E \arrow[d, "p"] \\ U \arrow[r, hook] & B \end{tikzcd}$$

"$p$ is trivial over $U$"

Claim: $\{f^{-1}(U) : U \in \mathcal{U}\}$ is an appropriate open cover.
This is well-defined, since $f$ is continuous.

For each $U \in \mathcal{U}$, the solid black diagram commutes.

$p$ is a fiber bundle $\Rightarrow$ front face is a pullback square.
The left face is also a pullback square.




So the green rectangle is a pullback square.

Since the right face is a pullback square by definition of $f^*E$, the partial inverse property gives us that the back face is a p.b. square.

$\blacksquare$

## Ex 3.8:

## Def 3.9:

For $i = 1, 2$, let $p_i : E_i \to B$ be a fiber bundle with fiber $F_i$.

The <u>product bundle</u> $E_1 \times_B E_2 \to B$ is the diagonal map in the p.b. square

$$\begin{tikzcd}
E_1 \times_B E_2 \arrow[r] \arrow[d] & E_1 \arrow[d, "p_2"] \\
E_2 \arrow[r, "p_1"'] & B
\end{tikzcd}$$

This is a fiber bundle with fiber $F_1 \times F_2$, denoted $p_1 \times p_2$.

When $p_1$ and $p_2$ are both vector bundles, the prod. bundle is also a vector bundle w/ fiber $F_1 \oplus F_2$. We call $p_1 \oplus p_2$ the Whitney sum.

**Lemma 3.10:** If $p_1$ and $p_2$ are isomorphic bundles over $B$ and $f: B' \to B$ is any map,

then

$f^* p_1$ and $f^* p_2$ are isomorphic bundles over $B'$.

Also

if $p$ is any bundle over $B$

then

$p \times p_1$ and $p \times p_2$ are isomorphic.

**Lemma 3.11:** Let $p: E \to B$ and $p': E' \to B$ be vector bundles and let $f: B' \to B$ be a map.

Then

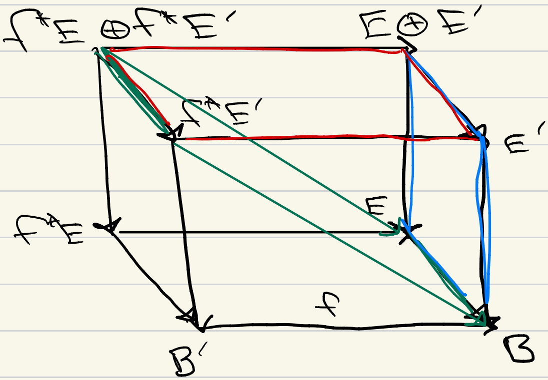
$$f^*(E \oplus E') \cong f^*(E) \oplus f^*(E').$$

**Pf:** Consider the commutative cube

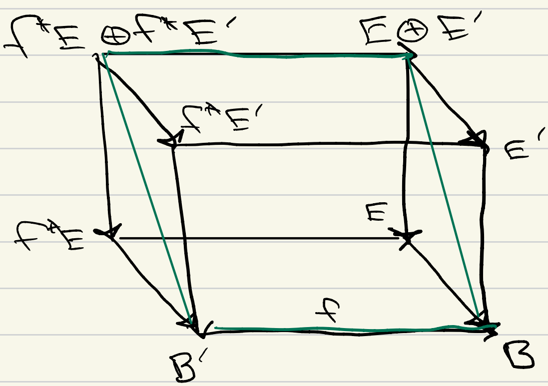
$$\begin{tikzcd}
f^*E \oplus f^*E' \arrow[rr] \arrow[dr] \arrow[dd] & & E \oplus E' \arrow[dr] \arrow[dd] & \\
& f^*E' \arrow[rr] \arrow[dd] & & E' \arrow[dd] \\
f^*E \arrow[rr] \arrow[dr] & & E \arrow[dr] & \\
& B' \arrow[rr, "f"] & & B
\end{tikzcd}$$

The front, bottom, left, and right are pullbacks by definition.

Left + bottom p.b. $\Rightarrow$ composition p.b.




Right p.b. $\Rightarrow$ top p.b. by inverse prop.




Top + front p.b. $\Rightarrow$ composition p.b.

This composition looks like

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & E \oplus E' \arrow[d] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

We know $f^*(E \oplus E')$ fits in the top left corner, by definition of $f^*$.

We just showed $f^*E \oplus f^*E'$ also works. So by the universal prop. of pullback squares,

$$f^*(E \oplus E') \cong f^*(E) \oplus f^*(E')$$

Ex 3.12: The tangent bundle to any smooth manifold is a pullback of a map from the manifold into a Grassmannian.

Let $M$ be a smooth $n$-manifold and let $f : M \hookrightarrow \mathbb{R}^N$ be a smooth embedding.

Define $g : M \to G_n(\mathbb{R}^N)$

$$x \longmapsto \left\{ \begin{array}{l} \text{linear subspace of } \mathbb{R}^N \text{ parallel to} \\ \text{the tangent plane to } M \text{ at } x \end{array} \right\}$$

Now

$$g^*\gamma_{nN} = \{(x, w, v) \in M \times \gamma_{nN} : v \text{ is in the tangent space to } M \text{ at } f(x)\}.$$

$\gamma_{nk} = \{(w,v) : w \in G_n(\mathbb{R}^k),\ v \in w \subseteq \mathbb{R}^k\}$

$\gamma_n := \gamma_{n\infty}$

~~$f^* E = B' \times_B E$~~

$\Rightarrow g^* \gamma_{nN} \cong TM.$

<u>Thm 3.13</u>: (Classification thm. for vector bundles over compact surfaces)

Sps $B$ is compact. Let $\mathrm{Vect}_n(B)$ be the set of isomorphism classes of $n$-dim'l vector bundles over $B$. Write $[B, G_n]$ for the set of homotopy classes of maps $B \to G_n$.
Then
the map $\rho : [B, G_n] \to \mathrm{Vect}_n(B)$
$f \longmapsto f^* \gamma_n$

is a bijection.

<u>Cor 3.14</u>: If $X$ and $Y$ are homotopy equivalent finite CW complexes
then
$\mathrm{Vect}_n(X)$ and $\mathrm{Vect}_n(Y)$ are in bijection

In particular, if $X$ is contractible then
all bundles over $X$ are trivial.

Def 3.15: We say that a homotopy class $\alpha \in [B, G_n]$ classifies $p: E \to B$

if for any $f \in \alpha$, $E \cong f^* \gamma_n$.

Any such choice of $f$ is a classifying map for $E$.

Ex 3.16: Consider $\oplus : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ induced by

$$(a_1, a_2, \dots) \oplus (b_1, b_2, \dots) = (a_1, b_1, a_2, b_2 \dots)$$

A point $(V, V') \in G_m \times G_n$ is a pair of subspaces of $\mathbb{R}^\infty$.

$\Rightarrow V \times V'$ is a subspace of $\mathbb{R}^\infty \times \mathbb{R}^\infty$.

Write $\oplus(V \times V') = V \oplus V'$.

This is an $n+m$-plane in $\mathbb{R}^\infty$, so it lives in $G_{m+n}$.

This induces $\oplus : G_m \times G_n \to G_{m+n}$

Let $f: B \to G_m$ and $f': B \to G_n$ be classifying maps for $E$ and $E'$, resp.

Consider

$$f_\oplus : B \xrightarrow{\text{diag}} B \times B \xrightarrow{f \times f'} G_m \times G_n \xrightarrow{\oplus} G_{m+n}.$$

Then $f_\oplus^* \gamma_n \cong E \oplus E'$.

$\oplus$ on Grassmannians classifies Whitney sums.