Define $V_n = \bigcup_{k \geq n}^{\infty} V_n(\mathbb{R}^k)$

GRASSMANIAN MANIFOLDS

Last time:

Defined Stiefel Manifolds

$$V_n(\mathbb{R}^k) = \left\{ (v_1, v_2, \dots, v_n) \mid v_i \in \mathbb{R}^k,\ v_i \perp v_j,\ |v_i| = 1 \right\} \subseteq (S^{k-1})^n$$

Now, $\mathbb{R}^k \subseteq \mathbb{R}^{k+1}$

by $(x_1, x_2, \dots, x_k) \longrightarrow (x_1, x_2, \dots, x_k, 0)$

$\Rightarrow V_n(\mathbb{R}^n) \subseteq V_n(\mathbb{R}^{n+1})$

Define $\tau_v : V_n(\mathbb{R}^k) \longrightarrow V_{n+1}(\mathbb{R}^{k+1})$

by $\tau_v((v_1, v_2, \dots, v_n)) = (v_1, v_2, \dots, v_n, v)$

$v = (0, 0, \dots, 0, 1)$

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~~Def~~ Grassmanian manifold: $G_n(\mathbb{R}^k) = V_n(\mathbb{R}^k)/\Sigma_n$

Now, think of any n-dim subsp. of $\mathbb{R}^k$. By Gram-Schmidt process, $\exists$ an orthonormal basis of this subsp. So, to study n-dim subspaces of $\mathbb{R}^k$, studying $V_n(\mathbb{R}^k)$ is enough.

But 2 tuples in $V_n(\mathbb{R}^k)$ may generate the same subsp.

$\therefore$ Define an equivalence relation on $V_n(\mathbb{R}^k)$ $(v_1, v_2, \dots, v_n) \sim (w_1, w_2, \dots, w_n)$ if they generate the same subsp.

Then the Grassmanian manifold

$$G_n(\mathbb{R}^k) = V_n(\mathbb{R}^k)/\sim$$

- A pt in $G_n(\mathbb{R}^k)$ generates a unique n dim subsp. ω of $\mathbb{R}^k$. By abuse of notation we write $\omega \in G_n(\mathbb{R}^k)$
- $G_n(\mathbb{R}^k)$ is "space of n-planes in $\mathbb{R}^k$"
- $G_n(\mathbb{R}^k)$ is compact.
- If $n > k$, $V_n(\mathbb{R}^k) = \phi \Rightarrow G_n(\mathbb{R}^k) = \phi$.
- $G_n(\mathbb{R}^n) = \{\mathbb{R}^n\}$
- $G_1(\mathbb{R}^k) = \mathbb{R}P^{k-1}$
- As $V_n(\mathbb{R}^k) \subseteq V_n(\mathbb{R}^{k+1})$

  $\Rightarrow G_n(\mathbb{R}^k) \subseteq G_n(\mathbb{R}^{k+1})$

  So, we can define $G_n = \bigcup_{k=1}^{\infty} G_n(\mathbb{R}^k)$

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- $G_n(\mathbb{R}^k)$ is T_2
- $G_n(\mathbb{R}^k)$ is a manifold with atlas:
  For each $v \in G_n(\mathbb{R}^k)$ define
  $$U_v = \{ w \in G_n(\mathbb{R}^k) \mid w \cap v^{\perp} = 0 \}$$
  $\dim G_n(\mathbb{R}^k) = n(k-n)$

~~Grassm~~.

Grassmanian Manifold carries a vector bundle (called Universal Bundle)

Define $\gamma_{nk} = \{ (w, v) \mid w \in G_n(\mathbb{R}^k),\ v \in w \subseteq \mathbb{R}^k \}$.

So, $\gamma_{nk} \subseteq G_n(\mathbb{R}^k) \times \mathbb{R}^k$

Recall a vector bundle of rank n is a cont. map $p : E \longrightarrow B$ s.t.

(i) $\forall\, b \in B$, ~~$\exists$ open U st $p^{-1}(U)$~~ $p^{-1}(b)$ is a vector space,

(ii) $\forall\, b \in B$, $\exists$ open U s.t. $b \in U$, $p^{-1}(U) \cong U \times \mathbb{R}^n$, and

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U \times \mathbb{R}^n \arrow[dl, "\pi_1"] \\ & B & \end{tikzcd}$$

Here $E = \gamma_{nk}$, $B = G_n(\mathbb{R}^k)$
and p is the first projection map.

Then for any $\omega \in G_n(\mathbb{R}^k)$,

$$p^{-1}(\omega) = \{\omega\} \times \omega \overset{\text{iso}}{\cong} \omega, \text{ a } n\text{-linear subsp. of } \mathbb{R}^k$$

$\forall \omega \in G_n(\mathbb{R}^k)$, take any open set U containing ω, then,

$$p^{-1}(U) = \{(\eta, \xi) \mid p(\eta, \xi) \in U\}$$

$$= \{(\eta, \xi) \mid \eta \in U, \xi \in \eta\}$$

$$\cong \bigcup_{\eta \in U} \{\eta\} \times \eta$$

~~Also, $p^{-1}(U)$~~

$$\overset{\varphi}{\cong} \bigcup_{\eta \in U} \{\eta\} \times \mathbb{R}^n$$

$$= U \times \mathbb{R}^n$$

Also,

$$\begin{tikzcd}
p^{-1}(U) \arrow[r, "\varphi"] \arrow[dr, "p"'] & U \times \mathbb{R}^n \arrow[d, "\pi_1"] \\
 & B
\end{tikzcd}$$

$$\pi_1 \circ \varphi((\eta, \xi)) = \eta = p((\eta, \xi))$$

So, $\boxed{p : \gamma_{nk} \longrightarrow G_n(\mathbb{R}^k) \text{ is a vector bundle}}$

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Lemma: For any $k > 1$, (including $k = \infty$)
γ_{1k} is not trivializable.

~~Define a left action~~

Another def^n of $G_n(\mathbb{R}^k)$

Consider $V_n(\mathbb{R}^k)$. Any element in ~~The group O(n),~~ $V_n(\mathbb{R}^k)$ is a $n \times k$ matrix.
The group $O(n)$ acts on $V_n(\mathbb{R}^k)$ by multiplication on left. ~~Denote this~~

Now, consider the map

$$\tau_V^{nk} : V_n(\mathbb{R}^k) \longrightarrow V_{n+1}(\mathbb{R}^{k+1})$$

$$\tau_V((x_1, x_2, \dots, x_n)) = (x_1, x_2, \dots, x_n, x)$$

$x = (0, 0, \dots, 1)$

~~Then~~ ~~Note that~~ ~~the~~

$$\cancel{\tau_V^{nk}(g \cdot \gamma) = g;}$$

Also note there is a ~~similar~~ natural map

$$i : O(n) \longrightarrow O(n+1)$$

$$i(M) = \begin{pmatrix} M & \\ & 1 \end{pmatrix}.$$

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$\therefore$ We have the diagram,

$$\begin{tikzcd} V_n(\mathbb{R}^k) \arrow[r, "M"] \arrow[d, "\tau_V"'] & V_n(\mathbb{R}^k) \arrow[d, "\tau_V"] \\ V_{n+1}(\mathbb{R}^{k+1}) \arrow[r, "i(M)"] & V_{n+1}(\mathbb{R}^{k+1}) \end{tikzcd}$$

and this diagram commutes.

So, τ_V is $O(n)$-equivariant $\forall k$

Proceeding as $k \longrightarrow \infty$, we get a map

$$\tau : G_n \longrightarrow G_{n+1}$$

~~Also, not~~ This map can be directly obtained from the notion of classifying spaces.

Classifying space: For a topological group G let EG be a weakly contractible space. Then classifying space

$$BG = EG/G.$$

So if $G = O(n)$ and $EG = V_n$

Then $BO(n) = V_n/O(n)$.

But $V_n/O(n) \cong G_n \Rightarrow G_n \cong BO(n)$

PULLBACKS :-

Let $\mathcal{C}$ be a category

Let $X \xrightarrow{f} Z \xleftarrow{g} Y$

be a diagram in $\mathcal{C}$ (sometimes they are called cospans)

Pullback of this cospan is the commutative square

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "f"] \\ X \arrow[r, "g"'] & Z \end{tikzcd}$$

s.t. if we have any ~~comm.~~ comm. square

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "G"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

then $\exists$ a unique morphism

$\phi : A \to P$ so that

$$\begin{tikzcd} A \arrow[dr, "\phi"] \arrow[drr, bend left, "F"] \arrow[ddr, bend right, "G"'] & & \\ & P \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

commutes i.e, $F = f' \circ \phi$

$G = g' \circ \phi$

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"Pullbacks are unique up to isomorphism ϕ".

<u>Example</u>: Consider the category of topological spaces and the cospan

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

Define $P = X \times_Z Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$

Check if this is a pullback:

We have
$$\begin{tikzcd} P \arrow[r, "f' = \pi_2"] \arrow[d, "g' = \pi_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Obviously this diagram commutes.

Now, consider any A s.t.
$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "G"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Define $\phi : A \longrightarrow P$

by $\phi(a) = (G(a), F(a))$

Now, this map is well-defined because
$f(G(a)) = g(F(a))$ [As the 2nd diagram commutes].

G, F are cont. $\Rightarrow$ ϕ is cont.

By construction, ϕ is unique

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...etry Speaker Feedback

...ongly Agree

(*) $\cancel{U \times_F E}$ For the cospan,

$$U \xrightarrow{i} B \xleftarrow{p} E,$$

we have pullback $U \times_B E = \{(u,e) \mid u = p(e)\}$

$$= \cancel{\{(u,e) \mid e \in p^{-1}(U)\}}$$

Moreover, $= U \times p^{-1}(U) \cong p^{-1}(U)$

$$f' \circ \phi = \pi_2 \circ \phi = F$$
$$g' \circ \phi = \pi_1 \circ \phi = G$$

$\therefore$ P is a pullback

(*)

Using this as inspiration, we redefine fiber bundle:

A map $p : E \longrightarrow B$ is a fiber bundle with fiber F if $\exists$ an open cover $\mathcal{U}$ of B s.t. $\forall\, U \in \mathcal{U}$ $\exists$ a pullback square

$$\begin{tikzcd}
U \times F \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E \arrow[d, "p"] \\
U \arrow[r, hook, "i"] & B
\end{tikzcd}$$

~~Call this new defn D2 and the prev. def D1 Condition~~

Call this new defn D2 and the old one D1. We imply they are equivalent.

~~D1 ⟹ D2~~ Why? For the cospan

$$U \xrightarrow{i} B \xleftarrow{p} E$$

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we already have the pullback

$$U \times_B E = \{(u,e) \in U \times E \mid i(u) = p(e)\}$$

$$= \{(u,e) \in U \times E \mid u = p(e)\}$$

$$= U \times p^{-1}(U)$$

$$\cong p^{-1}(U)$$

If p is a fiber bundle,

$$p^{-1}(U) \cong U \times F \quad (\text{so } D1 \Rightarrow D2)$$

If $D2$ holds then $\exists$ a unique isomorphism bet^n $p^{-1}(U)$ and $U \times F$ so $D2 \Rightarrow D1$.

Lemma:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

$\leftarrow$ Let this be a pullback square of spaces. Take any $c \in C$. Then $f'^{-1}(c)$ is cannonical isomorphic to $f^{-1}(g(c))$. In particular,