MATH 2090-3 **Review for Exam (2)** Spring 2001

1. Define the invertibility of a square matrix.
2. **Theorem**: An $n \times n$ matrix A is invertible if and only rank$(A) = n$.
3. How to check when a matrix A is invertible and, if so, how to find its inverse? (Apply the Gauss-Jordan elimination to the matrix $[A \,|\, I\,]$)
4. If A is an invertible matrix, then the solution of $A\vec{x} = \vec{b}$ is given by $\vec{x} = A^{-1}\vec{b}$.
5. State the definition of the determinant of a matrix $A = [a_{ij}]$. Use the definition to find the determinant of a 2×2 matrix.
6. If A is a triangular matrix, what is the value of its determinant?
7. How to use the row operations to evaluate the determinant?
8. **Theorem**: A square matrix A is invertible if and only if $\det(A) \neq 0$.
9. **Theorem**: $\det(AB) = \det(A)\det(B)$.
10. If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$
11. Define minors and cofactors of a matrix.
12. State the cofactor expansion (by a row and by a column)
13. Define the cofactor matrix and adjoint of a matrix.
14. **Theorem**: If A is invertible, then its inverse is given by $A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)$.
15. State Cramer's formula and use it to solve a system of linear equations.
16. State the definition of a vector space.
17. Define the following standard vector spaces: $\mathbb{R}^n$, M_{mn}, M_n, P_n, P, $C[a, b]$.
18. What is a subspace of a vector space?
19. **Theorem**: A nonempty subset of a vector space is a subspace if and only if it is closed under addition and scalar multiplication.
20. A linear combination of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a vector of the form
    $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$.
21. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a subspace of V. It is called the span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.
22. Define linear independence and linear dependence.
23. **Theorem**: The vectors $\vec{v}_1$, $\vec{v}_2, \ldots$ $\vec{v}_n$ with $n \geq 2$ are linearly dependent if and only if at lease one of them can be written as a linear combination of others.
24. Define a basis for a vector space.
25. Give standard bases for the vector spaces in Item 17 above.
26. Define the dimension of a vector space and find the dimensions of the six standard vector spaces.
27. Define the null space of a matrix.
28. How to find a basis for and the dimension of the null space nullsp(A) of a matrix A?

MATH 2090-3 **Review for Exam (3)** Spring 2001

1. Define the row space and column space.
2. How to find a basis for the row space?
3. How to find a basis for the column space of a matrix A?
   Method 1: Reduce A to REF or RREF. The column vectors of A corresponding to those column vectors containing the leading ones in the REF or RREF form a basis for the column space of A.
   Method 2: Use the fact that the column space of A is the row space of A^T and work on A^T.
4. **Theorem**: $\dim[\text{rowsp}(A)] = \dim[\text{colsp}(A)] = \text{rank}(A)$.
5. Define the null space and nullity of a matrix.
6. **Rank-Nullity Theorem**: Let A be an $m \times n$ matrix. Then $\text{rank}(A) + \text{nullity}(A) = n$.
7. State the definition of a linear transformation T from V into W.
8. Let $T : V \to W$ be a linear transformation. Show that $T(\vec{0}) = \vec{0}$ and $T(-\vec{v}) = -T(\vec{v})$ for all $\vec{v} \in V$.
9. **Theorem**: A mapping $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation if and only if it is given by $T(\vec{x}) = A\vec{x}$, where A is an $m \times n$ matrix. In fact, $A = [T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)]$, where $\{\vec{e}_1, \vec{v}_2, \ldots, \vec{e}_n\}$ is the standard basis for $\mathbf{R}^n$.
10. Define the kernel and range of a linear transformation T from V into W.
11. Show that $\text{Ker}(T)$ is a subspace of V and $\text{Range}(T)$ is a subspace of W.
12. If $T : \mathbf{R}^n \to \mathbf{R}^m$ is given by an $m \times n$ matrix A, then
    $$\text{Ker}(T) = \text{nullspace}(A), \quad \text{Range}(T) = \text{colsp}(A).$$
13. How to find the kernel and range of a linear transformation?
14. **Rank-Nullity Theorem** (general case): If $T : V \to W$ is a linear transformation, then $\dim[\text{Ker}(T)] + \dim[\text{Range}(T)] = \dim(V)$.
15. Define eigenvalues, eigenvectors, characteristic polynomial, and characteristic equation.
16. Define eigenspace E_λ for an eigenvalue λ. **Fact**: $\dim(E_\lambda) \leq$ multiplicity of λ.
17. **Theorem**: Eigenvectors corresponding to distinct eigenvalues are linearly independent.
18. State the definition of a matrix being nondefective.
19. **Theorem**: A matrix A is nondefective if and only if $\dim(E_\lambda) =$ multiplicity of λ for all eigenvalues λ of A.
20. State the definition that a matrix is diagonalizable. **Fact**: A matrix is diagonalizable if and only if it is nondefective.
21. If A is nondefective, how to find a matrix S such that $S^{-1}AS$ is a diagonal matrix?
22. Homogeneous vector differential equations and its fundamental matrix.
23. State the general solution of a nonhomogeneous VDE.