1. Define the invertibility of a square matrix.

2. **Theorem**: An $n \times n$ matrix $A$ is invertible if and only if $\text{rank}(A) = n$.

3. How to check when a matrix $A$ is invertible and, if so, how to find its inverse?
   (Apply the Gauss-Jordan elimination to the matrix $[A | I]$)

4. If $A$ is an invertible matrix, then the solution of $A\vec{x} = \vec{b}$ is given by $\vec{x} = A^{-1}\vec{b}$.

5. State the definition of the determinant of a matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$. Use the definition to find the determinant of a $2 \times 2$ matrix.

6. If $A$ is a triangular matrix, what is the value of its determinant?

7. How to use the row operations to evaluate the determinant?

8. **Theorem**: A square matrix $A$ is invertible if and only if $\det(A) \neq 0$.

9. **Theorem**: $\det(AB) = \det(A)\det(B)$.

10. If $A$ is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

11. Define minors and cofactors of a matrix.

12. State the cofactor expansion (by a row and by a column)

13. Define the cofactor matrix and adjoint of a matrix.

14. **Theorem**: If $A$ is invertible, then its inverse is given by $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$.

15. State Cramer’s formula and use it to solve a system of linear equations.

16. State the definition of a vector space.

17. Define the following standard vector spaces: $\mathbb{R}^n$, $M_{mn}$, $M_n$, $P_n$, $P$, $C[a, b]$.

18. What is a subspace of a vector space?

19. **Theorem**: A nonempty subset of a vector space is a subspace if and only if it is closed under addition and scalar multiplication.

20. A linear combination of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a vector of the form

   $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$.

21. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a subspace of $V$. It is called the span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

22. Define linear independence and linear dependence.

23. **Theorem**: The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with $n \geq 2$ are linearly dependent if and only if at least one of them can be written as a linear combination of others.

24. Define a basis for a vector space.

25. Give standard bases for the vector spaces in Item 17 above.

26. Define the dimension of a vector space and find the dimensions of the six standard vector spaces.

27. Define the null space of a matrix.

28. How to find a basis for and the dimension of the null space nullsp($A$) of a matrix $A$?
1. Define the row space and column space.

2. How to find a basis for the row space?

3. How to find a basis for the column space of a matrix \( A \)?
   - **Method 1:** Reduce \( A \) to REF or RREF. The column vectors of \( A \) corresponding to those column vectors containing the leading ones in the REF or RREF form a basis for the column space of \( A \).
   - **Method 2:** Use the fact that the column space of \( A \) is the row space of \( A^T \) and work on \( A^T \).

4. **Theorem:** \( \dim[\text{rowsp}(A)] = \dim[\text{colsp}(A)] = \text{rank}(A) \).

5. Define the null space and nullity of a matrix.

6. **Rank-Nullity Theorem:** Let \( A \) be an \( m \times n \) matrix. Then \( \text{rank}(A) + \text{nullity}(A) = n \).

7. State the definition of a linear transformation \( T \) from \( V \) into \( W \).

8. Let \( T : V \rightarrow W \) be a linear transformation. Show that \( T(\vec{0}) = \vec{0} \) and \( T(-\vec{v}) = -T(\vec{v}) \) for all \( \vec{v} \in V \).

9. **Theorem:** A mapping \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a linear transformation if and only if it is given by \( T(\vec{x}) = A\vec{x} \), where \( A \) is an \( m \times n \) matrix. In fact, \( A = [T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)] \), where \( \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\} \) is the standard basis for \( \mathbb{R}^n \).

10. Define the kernel and range of a linear transformation \( T \) from \( V \) into \( W \).

11. Show that \( \text{Ker}(T) \) is a subspace of \( V \) and \( \text{Range}(T) \) is a subspace of \( W \).

12. If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is given by an \( m \times n \) matrix \( A \), then \( \text{Ker}(T) = \text{nullspace}(A), \ \text{Range}(T) = \text{colsp}(A) \).

13. How to find the kernel and range of a linear transformation?

14. **Rank-Nullity Theorem** (general case): If \( T : V \rightarrow W \) is a linear transformation, then \( \dim[\text{Ker}(T)] + \dim[\text{Range}(T)] = \dim(V) \).

15. Define eigenvalues, eigenvectors, characteristic polynomial, and characteristic equation.

16. Define eigenspace \( E_\lambda \) for an eigenvalue \( \lambda \). **Fact:** \( \dim(E_\lambda) \leq \text{multiplicity of } \lambda \).

17. **Theorem:** Eigenvectors corresponding to distinct eigenvalues are linearly independent.

18. State the definition of a matrix being nondefective.

19. **Theorem:** A matrix \( A \) is nondefective if and only if \( \dim(E_\lambda) = \text{multiplicity of } \lambda \) for all eigenvalues \( \lambda \) of \( A \).

20. State the definition that a matrix is diagonalizable. **Fact:** A matrix is diagonalizable if and only if it is nondefective.

21. If \( A \) is nondefective, how to find a matrix \( S \) such that \( S^{-1}AS \) is a diagonal matrix?


23. State the general solution of a nonhomogeneous VDE.