§4.1, #19

The inversions of the permutation \((n, (n - 1), (n - 2), \ldots, 2, 1)\) are given by:

\[
\begin{align*}
(n, (n - 1)), (n, (n - 2)), & \ldots, (n, 1) \\
((n - 1), (n - 2)), ((n - 1), (n - 3)), & \ldots, ((n - 1), 1) \\
\hdotsfor[6]{2} \\
(3, 2), (3, 1) \\
(2, 1).
\end{align*}
\]

Note that there are \(n - 1\) inversions in the first row, \(n - 2\) in the second row, and finally 1 in the last row. Therefore, the total number of inversions equals to

\[
(n - 1) + (n - 2) + \cdots + 1 = \frac{n(n - 1)}{2}.
\]

Thus the sign attached to the term

\[
a_1 a_{n-1} a_3 \cdots a_n
\]

is \((-1)^{n(n-1)}/2\).

§4.2, #21

The determinant of the coefficient matrix is given by

\[
\begin{vmatrix}
1 & 2 & k \\
2 & -k & 1 \\
3 & 6 & 1
\end{vmatrix} = -k + 6 + 12k + 3k^2 - 4 - 6 = 3k^2 + 11k - 4 = (3k - 1)(k + 4).
\]

Note that the given system has an infinite number of solutions if and only if the above determinant is zero, which yields the values of \(k = 1/3\) and \(k = 4\).

§4.2, #27

Bring out \(-1\) from the first row and \(1/2\) from the second row to get

\[
\begin{vmatrix}
1 & -3 & 1 \\
2 & -1 & 7 \\
3 & 1 & 13
\end{vmatrix} = -\frac{1}{2} \begin{vmatrix}
-1 & 3 & -1 \\
4 & -2 & 14 \\
3 & 1 & 13
\end{vmatrix}.
\]

Then add the second row to the first row:

\[
\begin{vmatrix}
-1 & 3 & -1 \\
4 & -2 & 14 \\
3 & 1 & 13
\end{vmatrix} = \begin{vmatrix}
3 & 1 & 13 \\
4 & -2 & 14 \\
3 & 1 & 13
\end{vmatrix}.
\]

Since the first and third rows are identical, the determinant equals to 0.
§4.2, #29
First note that det(A) is a polynomial of degree 3. Hence the equation det(A) = 0 has three roots. Then observe the following:
(a) When \( x = 0 \), the third column is zero and so det(A) = 0.
(b) When \( x = -1 \), the second and third columns are identical and so det(A) = 0.
(c) When \( x = 2 \), the first and third columns are proportional and so det(A) = 0.
Therefore, we conclude that all values of \( x \) for which det(A) = 0 are 0, -1 and 2.

§4.2, #39
First note that the determinant is a polynomial in \( x,y,z \) of degree 3. Then observe the following:
(a) When we put \( x = y \), the determinant is 0 because the first and second rows are identical. This means that the determinant has a factor \( x - y \).
(b) When we put \( x = z \), the determinant is 0 because the first and third rows are identical. This means that the determinant has a factor \( x - z \).
(c) When we put \( y = z \), the determinant is 0 because the second and third rows are identical. This means that the determinant has a factor \( y - z \).
From the above facts we conclude that
\[
\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = c(x - y)(x - z)(y - z), \tag{1}
\]
where \( c \) is a constant. To determine the constant \( c \), note that the determinant has a term \( yz^2 \). On the other hand, the right hand side of Equation (1) has a term \(-cyz^2\). Hence \( c = -1 \). Finally, put \( c = -1 \) into Equation (1) to get
\[
\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -(x - y)(x - z)(y - z) = (x - y)(y - z)(z - x).
\]

§4.3, #20
Use the first row cofactor expansion to get
\[
\begin{vmatrix} 0 & x & y & z \\ -x & 0 & 1 & -1 \\ -y & -1 & 0 & 1 \\ -z & 1 & -1 & 0 \end{vmatrix} = -x \begin{vmatrix} -x & 1 & -1 \\ -y & 0 & 1 \\ -z & 1 & 0 \end{vmatrix} + y \begin{vmatrix} -x & 0 & -1 \\ -y & 1 & -1 \\ -z & 0 & 1 \end{vmatrix} + z \begin{vmatrix} -x & 0 & 1 \\ -y & -1 & 0 \\ -z & 1 & -1 \end{vmatrix}
\]
\[
= -x(-z - y - x) + y(y + z + x) - z(-x - y - z)
= x(x + y + z) + y(x + y + z) + z(x + y + z)
= (x + y + z)(x + y + z)
= (x + y + z)^2.
\]
On the other hand, this problem can also be solved in an indirect way as follows. First add the third and fourth columns to the second column to get
\[
\begin{vmatrix}
0 & x & y & z \\
-x & 0 & 1 & -1 \\
-y & -1 & 0 & 1 \\
-z & 1 & -1 & 0 \\
\end{vmatrix}
= \begin{vmatrix}
0 & x + y + z & y & z \\
-x & 0 & 1 & -1 \\
-y & 0 & 0 & 1 \\
-z & 0 & -1 & 0 \\
\end{vmatrix}
= -(x + y + z) \begin{vmatrix}
x & 1 & -1 \\
y & 0 & 1 \\
z & -1 & 0 \\
\end{vmatrix}.
\]

Then in the last determinant add the second and third rows to the first row to obtain
\[
\begin{vmatrix}
-x & 1 & -1 \\
-y & 0 & 1 \\
-z & -1 & 0 \\
\end{vmatrix}
= \begin{vmatrix}
0 & x + y - z & 0 & 0 \\
-y & 0 & 1 \\
-z & -1 & 0 \\
\end{vmatrix}
= (-x - y - z) \begin{vmatrix}
0 & 1 \\
-1 & 0 \\
\end{vmatrix}
= -(x + y + z).
\]

Therefore, we have
\[
\begin{vmatrix}
0 & x & y & z \\
-x & 0 & 1 & -1 \\
-y & -1 & 0 & 1 \\
-z & 1 & -1 & 0 \\
\end{vmatrix}
= -(x + y + z) \begin{vmatrix}
x & 1 & -1 \\
y & 0 & 1 \\
z & -1 & 0 \\
\end{vmatrix}
= (x + y + z)^2.
\]

§5.2, #11
The answer in the book is wrong. The set \( \mathbb{R}^2 \) with the addition and scalar multiplication as defined in this problem is not a vector space.

**First proof.** The set \( \mathbb{R}^2 \) is not closed under scalar multiplication. For example
\[
\frac{1}{2} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -(1)^{1/2} \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix},
\]
which is not an element in \( \mathbb{R}^2 \).

**Second proof.** It is easy to see that the element satisfying Axiom 3 is given by \((0, 1)\). But then
\[
(x_1, 0) + (a, b) = (x_1 + a, 0),
\]
which is impossible to be equal to \((0, 1)\). This means that \((x_1, 0)\) has no additive inverse for any \(x_1\). Thus Axiom 4 is not satisfied.

§5.7, #9
We use the given four vectors in \( \mathbb{R}^4 \) to form the following matrix:
\[
\begin{pmatrix}
1 & 4 & 1 & 3 \\
2 & 8 & 3 & 5 \\
1 & 4 & 0 & 4 \\
2 & 8 & 2 & 6 \\
\end{pmatrix}
\]
What we need to find is a basis for the rowspace of this matrix. Apply the row operations:
(1) multiply the first row by $-2$ and then add it to the second and fourth rows, (2) multiply the first row by $-1$ and then add it to the third row. This results the following matrix:

$$
\begin{bmatrix}
1 & 4 & 1 & 3 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Next, add the second row to the third row to get

$$
\begin{bmatrix}
1 & 4 & 1 & 3 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Therefore, the first and second row vectors form a basis for the row space and so the vectors $(1, 4, 1, 3)$ and $(0, 0, 1, -1)$ form a basis for the space spanned by the given four vectors.

§6.1, #19(b)

First we need to write $(x_1, x_2)$ as a linear combination of $\vec{v}_1 = (1, 1)$ and $\vec{v}_2 = (1, -1)$ and so let

$$(x_1, x_2) = c_1(1, 1) + c_2(1, -1).$$

Comparing the components, we see that $c_1 + c_2 = x_1$ and $c_1 - c_2 = x_2$. Solve for $c_1$ and $c_2$ to get

$$c_1 = \frac{x_1 + x_2}{2}, \quad c_2 = \frac{x_1 - x_2}{2}.$$

Then use the linearity of $T$ and the given information on $T(\vec{v}_1)$ and $T(\vec{v}_2)$ to get

$$T(x_1, x_2) = \frac{x_1 + x_2}{2}T(\vec{v}_1) + \frac{x_1 - x_2}{2}T(\vec{v}_2)$$

$$= \frac{x_1 + x_2}{2}(2, 3) + \frac{x_1 - x_2}{2}(-1, 1)$$

$$= \left(\frac{x_1 + 3x_2}{2}, 2x_1 - x_2\right).$$

In particular, we have $T(4, -2) = (-1, 10)$.

§6.1, #21

Use the linearity of $T$ to rewrite the given information as

$$2T(\vec{v}_1) + 3T(\vec{v}_2) = v_1 + v_2, \quad (1)$$

$$T(\vec{v}_1) + T(\vec{v}_2) = v_1 + v_2. \quad (2)$$
Apply $3 \times \text{Eq. (2)} - \text{Eq. (1)}$ to get
\[ T(\vec{v}_1) = 8\vec{v}_1 - 4\vec{v}_2. \]
Then apply $\text{Eq. (1)} + (-2) \times \text{Eq. (2)}$ to get
\[ T(\vec{v}_2) = -5\vec{v}_1 + 3\vec{v}_2. \]

§6.3, #7
The kernel of $T$ consists of all functions $y$ such that $T(y) = 0$, namely $y'' - y = 0$. This differential equation has auxiliary equation $r^2 - 1 = 0$, which has two roots $r = 1, -1$. Hence the differential equation $y'' - y = 0$ has two linearly independent solutions $e^x$ and $e^{-x}$. It follows that a basis for the kernel of $T$ is given by \{e^x, e^{-x}\}.

§6.3, #17
First we need to find what $T(\vec{v})$ equals to for any $\vec{v} \in V$. Note that any $\vec{v} \in V$ can be expressed as
\[ \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3. \]
Use the linearity of $T$ and the given information to get
\[
T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) \\
= c_1(2\vec{w}_1 - \vec{w}_2) + c_2(\vec{w}_1 - \vec{w}_2) + c_3(\vec{w}_1 + 2\vec{w}_2) \\
= (2c_1 + c_2 + c_3)\vec{w}_1 + (-c_1 - c_2 + 2c_3)\vec{w}_2.
\]
Therefore, the action of $T$ on $V$ is given by
\[ T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = (2c_1 + c_2 + c_3)\vec{w}_1 + (-c_1 - c_2 + 2c_3)\vec{w}_2. \quad (1) \]
Now, let us find $\text{Ker}(T)$. By Equation (1) we need to find all vectors $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ in $V$ such that
\[(2c_1 + c_2 + c_3)\vec{w}_1 + (-c_1 - c_2 + 2c_3)\vec{w}_2 = \vec{0}.\]
But \{\vec{w}_1, \vec{w}_2\} is a basis for $W$ and so
\[
2c_1 + c_2 + c_3 = 0, \\
-c_1 - c_2 + 2c_3 = 0.
\]
Use the Gauss-Jordan elimination to get the solution:
\[
c_1 = -3t, \quad c_2 = 5t, \quad c_3 = t.
\]
Thus the kernel of $T$ consists of all vectors $\vec{v}$ of the form
\[
\vec{v} = -3t\vec{v}_1 + 5t\vec{v}_2 + t\vec{v}_3 \\
= t(-3\vec{v}_1 + 5\vec{v}_2 + \vec{v}_3).
\]
This means that the kernel of $T$ is given by

$$\text{Ker}(T) = \{ t(-3\vec{v}_1 + 5\vec{v}_2 + \vec{v}_3) \mid t \in \mathbb{R} \}.$$ 

Thus Ker$(T)$ is spanned by the vector $-3\vec{v}_1 + 5\vec{v}_2 + \vec{v}_3$ and so dim[Ker$(T)$] = 1.

Next, we find the range of $T$. For this purpose we need to find out what vector $\vec{w} = a\vec{w}_1 + b\vec{w}_2$ in $W$ is the image of some vector $\vec{v}$ in $V$. By Equation (1) we need to solve the equation:

\begin{align*}
2c_1 + c_2 + c_3 &= a, \\
-c_1 - c_2 + 2c_3 &= b.
\end{align*}

Use the Gauss-Jordan elimination to solve this system of linear equations:

\begin{align*}
c_1 &= -3t + a + b, \\
c_2 &= 5t - a - 2b, \\
c_3 &= t.
\end{align*} (2)

Note that for any $a$ and $b$ we can always find $c_1, c_2, c_3$ as given by Equation (2) such that

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = (2c_1 + c_2 + c_3)\vec{w}_1 + (-c_1 - c_2 + 2c_3)\vec{w}_2 = a\vec{w}_1 + b\vec{w}_2.$$ 

This means that for any vector $\vec{w}$ in $W$, we can always find a vector $\vec{v}$ in $V$ such that $T(\vec{v}) = \vec{w}$. Hence Range$(T) = W$ and so dim[Range$(T)$] = 2.

\section*{8.9 Extra problem}

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find $e^A$.

The characteristic polynomial is easily found to be $p(\lambda) = \lambda^2 - 5\lambda - 2$. Hence the eigenvalues are given by

$$\lambda_1 = \frac{5 + \sqrt{33}}{2}, \quad \lambda_2 = \frac{5 - \sqrt{33}}{2}. $$

We can derive the corresponding eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 3 + \sqrt{33} \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 3 - \sqrt{33} \end{bmatrix}. $$

Let $S$ be the matrix $S = [\vec{v}_1, \vec{v}_2]$, namely,

$$S = \begin{bmatrix} 4 & 4 \\ 3 + \sqrt{33} & 3 - \sqrt{33} \end{bmatrix}. $$ (1)

Then we have the following identity

$$S^{-1}AS = \begin{bmatrix} \frac{5 + \sqrt{33}}{2} & 0 \\ 0 & \frac{5 - \sqrt{33}}{2} \end{bmatrix}. $$

Multiply $S$ from the left and $S^{-1}$ from the right to get

$$A = S \begin{bmatrix} \frac{5 + \sqrt{33}}{2} & 0 \\ 0 & \frac{5 - \sqrt{33}}{2} \end{bmatrix} S^{-1}. $$
Therefore, $e^A$ is given by

$$e^A = S \begin{bmatrix} e^{\frac{5+\sqrt{33}}{2}} & 0 \\ 0 & e^{\frac{5-\sqrt{33}}{2}} \end{bmatrix} S^{-1}. \quad (2)$$

Now, we can use row operations to find $S^{-1}$:

$$S^{-1} = \begin{bmatrix} \frac{-3+\sqrt{33}}{2\sqrt{33}} & \frac{1}{2\sqrt{33}} \\ \frac{3+\sqrt{33}}{2\sqrt{33}} & -\frac{1}{2\sqrt{33}} \end{bmatrix}. \quad (3)$$

Finally, put Equations (1) and (3) into Equation (2) and carry out matrix multiplication to get

$$e^A = \begin{bmatrix} \frac{-3+\sqrt{33}}{2\sqrt{33}} e^{\frac{5+\sqrt{33}}{2}} + \frac{3+\sqrt{33}}{2\sqrt{33}} e^{\frac{5-\sqrt{33}}{2}} & 2\left(e^{\frac{5+\sqrt{33}}{2}} e^{\frac{5-\sqrt{33}}{2}}\right) \sqrt{33} \\ 3\left(\frac{5+\sqrt{33}}{2\sqrt{33}} - e^{\frac{5-\sqrt{33}}{2}}\right) & \frac{3+\sqrt{33}}{2\sqrt{33}} e^{\frac{5+\sqrt{33}}{2}} + (-3+\sqrt{33})e^{\frac{5-\sqrt{33}}{2}} \end{bmatrix}. $$

§9.5, #53

Let $X_1$ and $X_2$ be the Laplace transforms of $x_1$ and $x_2$, respectively. Take the Laplace transform of the given differential equations with the initial conditions to get

$$sX_1 - 1 = 2X_1 - X_2, \quad (1)$$
$$sX_2 = X_1 + 2X_2. \quad (2)$$

From Equation (2), we solve for $X_2$

$$X_2 = \frac{1}{s-2}X_1. \quad (3)$$

Put this $X_2$ into Equation (1):

$$sX_1 - 1 = 2X_1 - \frac{1}{s-2}X_1,$$

which can be rewritten as

$$(s-2)X_1 + \frac{1}{s-2}X_1 = 1,$$

or

$$\frac{(s-2)^2 + 1}{s-2}X_1 = 1.$$ 

Thus we find $X_1$ to be

$$X_1 = \frac{s-2}{(s-2)^2 + 1}. \quad (4)$$
Use the First Shifting Theorem to take the inverse Laplace transform to get

\[ x_1 = e^{2t} \cos t. \]

Next, use Equations (3) and (4) to find \( X_2 \)

\[ X_2 = \frac{1}{(s - 2)^2 + 1}. \]

Again use the First Shifting Theorem to take the inverse Laplace transform to get

\[ x_2 = e^{2t} \sin t. \]