**Solutions to some Math 2090-3 homework problems**

**§4.1, #19**

The inversions of the permutation $(n,(n-1),(n-2),\cdots,2,1)$ are given by:
$(n,(n-1)),(n,(n-2)),\ldots,(n,1)$
$((n-1),(n-2)),((n-1),(n-3)),\ldots,((n-1),1)$
$\ldots\ldots\ldots\ldots\ldots$
$(3,2),(3,1)$
$(2,1)$.
Note that there are $n-1$ inversions in the first row, $n-2$ in the second row, and finally 1 in the last row. Therefore, the total number of inversions equals to

$$(n-1)+(n-2)+\cdots+1=\frac{n(n-1)}{2}.$$

Thus the sign attached to the term

$$a_{1\,n}a_{2\,n-1}a_{3\,n-2}\cdots a_{n\,1}$$

is $(-1)^{n(n-1)/2}$.

**§4.2, #21**

The determinant of the coefficient matrix is given by

$$\begin{vmatrix}1 & 2 & k\\ 2 & -k & 1\\ 3 & 6 & 1\end{vmatrix}=-k+6+12k+3k^2-4-6=3k^2+11k-4=(3k-1)(k+4).$$

Note that the given system has an infinite number of solutions if and only if the above determinant is zero, which yields the values of $k=1/3$ and $k=4$.

**§4.2, #27**

Bring out $-1$ from the first row and $1/2$ from the second row to get

$$\begin{vmatrix}1 & -3 & 1\\ 2 & -1 & 7\\ 3 & 1 & 13\end{vmatrix}=-\frac{1}{2}\begin{vmatrix}-1 & 3 & -1\\ 4 & -2 & 14\\ 3 & 1 & 13\end{vmatrix}.$$

Then add the second row to the first row:

$$\begin{vmatrix}-1 & 3 & -1\\ 4 & -2 & 14\\ 3 & 1 & 13\end{vmatrix}=\begin{vmatrix}3 & 1 & 13\\ 4 & -2 & 14\\ 3 & 1 & 13\end{vmatrix}.$$

Since the first and third rows are identical, the determinant equals to 0.

§**4.2, #29**

First note that $\det(A)$ is a polynomial of degree 3. Hence the equation $\det(A) = 0$ has three roots. Then observe the following:

(a) Whne $x = 0$, the third column is zero and so $\det(A) = 0$.
(b) When $x = -1$, the second and third columns are identical and so $\det(A) = 0$.
(c) When $x = 2$, the first and third columns are proportional and so $\det(A) = 0$.

Therefore, we conclude that all values of $x$ for which $\det(A) = 0$ are $0, -1$ and $2$.

§**4.2, #39**

First note that the determinant is a polynomial in $x, y, z$ of degree 3. Then observe the following:

(a) When we put $x = y$, the determinant is 0 because the first and second rows are identical. This means that the determinant has a factor $x - y$.
(b) When we put $x = z$, the determinant is 0 because the first and third rows are identical. This means that the determinant has a factor $x - z$.
(c) When we put $y = z$, the determinant is 0 because the second and third rows are identical. This means that the determinant has a factor $y - z$.

From the above facts we conclude that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = c(x-y)(x-z)(y-z), \tag{1}$$

where $c$ is a constant. To determine the constant $c$, note that the determinant has a term $yz^2$. On the other hand, the right hand side of Equation (1) has a term $-cyz^2$. Hence $c = -1$. Finally, put $c = -1$ into Equation (1) to get

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -(x-y)(x-z)(y-z) = (x-y)(y-z)(z-x).$$

§**4.3, #20**

Use the first row cofactor exapnsion to get

$$\begin{vmatrix} 0 & x & y & z \\ -x & 0 & 1 & -1 \\ -y & -1 & 0 & 1 \\ -z & 1 & -1 & 0 \end{vmatrix} = -x \begin{vmatrix} -x & 1 & -1 \\ -y & 0 & 1 \\ -z & -1 & 0 \end{vmatrix} + y \begin{vmatrix} -x & 0 & -1 \\ -y & -1 & 1 \\ -z & 1 & 0 \end{vmatrix} - z \begin{vmatrix} -x & 0 & 1 \\ -y & -1 & 0 \\ -z & 1 & -1 \end{vmatrix}$$
$$= -x(-z-y-x) + y(y+z+x) - z(-x-y-z)$$
$$= x(x+y+z) + y(x+y+z) + z(x+y+z)$$
$$= (x+y+z)(x+y+z)$$
$$= (x+y+z)^2.$$

On the other hand, this problem can also be solved in an indirect way as follows. First add the third and fourth columns to the second column to get

$$\begin{vmatrix} 0 & x & y & z \\ -x & 0 & 1 & -1 \\ -y & -1 & 0 & 1 \\ -z & 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & x+y+z & y & z \\ -x & 0 & 1 & -1 \\ -y & 0 & 0 & 1 \\ -z & 0 & -1 & 0 \end{vmatrix} = -(x+y+z)\begin{vmatrix} -x & 1 & -1 \\ -y & 0 & 1 \\ -z & -1 & 0 \end{vmatrix}.$$

Then in the last determinant add the second and third rows to the first row to obtain

$$\begin{vmatrix} -x & 1 & -1 \\ -y & 0 & 1 \\ -z & -1 & 0 \end{vmatrix} = \begin{vmatrix} -x-y-z & 0 & 0 \\ -y & 0 & 1 \\ -z & -1 & 0 \end{vmatrix} = (-x-y-z)\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = -(x+y+z).$$

Therefore, we have

$$\begin{vmatrix} 0 & x & y & z \\ -x & 0 & 1 & -1 \\ -y & -1 & 0 & 1 \\ -z & 1 & -1 & 0 \end{vmatrix} = -(x+y+z)\begin{vmatrix} -x & 1 & -1 \\ -y & 0 & 1 \\ -z & -1 & 0 \end{vmatrix} = (x+y+z)^2.$$

### §5.2, #11

The answer in the book is wrong. The set $\mathbf{R}^2$ with the addition and scalar multiplication as defined in this problem is not a vector space.

**First proof.** The set $\mathbf{R}^2$ is not closed under scalar multiplication. For example

$$\frac{1}{2}(2,-1) = (1,(-1)^{1/2}) = (1,i),$$

which is not an element in $\mathbf{R}^2$.

**Second proof.** It is easy to see that the element satisfying Axiom 3 is given by $(0,1)$. But then

$$(x_1,0)+(a,b) = (x_1+a,0),$$

which is impossible to be equal to $(0,1)$. This means that $(x_1,0)$ has no additive inverse for any $x_1$. Thus Axiom 4 is not satisfied.

### §5.7, #9

We use the given four vectors in $\mathbf{R}^4$ to form the following matrix:

$$\begin{bmatrix} 1 & 4 & 1 & 3 \\ 2 & 8 & 3 & 5 \\ 1 & 4 & 0 & 4 \\ 2 & 8 & 2 & 6 \end{bmatrix}$$

What we need to find is a basis for the rowspace of this matrix. Apply the row operations: (1) multiply the first row by $-2$ and then add it to the second and fourth rows, (2) multiply the first row by $-1$ and then add it to the third row. This results the following matrix:

$$\begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Next, add the second row to the third row to get

$$\begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the first and second row vectors form a basis for the row space and so the vectors $(1, 4, 1, 3)$ and $(0, 0, 1, -1)$ form a basis for the space spanned by the given four vectors.

**§6.1, #19(b)**

First we need to write $(x_1, x_2)$ as a linear combination of $\vec{v}_1 = (1, 1)$ and $\vec{v}_2 = (1, -1)$ and so let

$$(x_1, x_2) = c_1(1, 1) + c_2(1, -1).$$

Comparing the components, we see that $c_1 + c_2 = x_1$ and $c_1 - c_2 = x_2$. Solve for $c_1$ and $c_2$ to get

$$c_1 = \frac{x_1 + x_2}{2}, \qquad c_2 = \frac{x_1 - x_2}{2}.$$

Then use the linearity of $T$ and the given information on $T(\vec{v}_1)$ and $T(\vec{v}_2)$ to get

$$\begin{aligned} T(x_1, x_2) &= \frac{x_1 + x_2}{2} T(\vec{v}_1) + \frac{x_1 - x_2}{2} T(\vec{v}_2) \\ &= \frac{x_1 + x_2}{2}(2, 3) + \frac{x_1 - x_2}{2}(-1, 1) \\ &= \Big( \frac{x_1 + 3x_2}{2}, 2x_1 - x_2 \Big). \end{aligned}$$

In particular, we have $T(4, -2) = (-1, 10)$.

**§6.1, #21**

Use the linearity of $T$ to rewrite the given information as

$$2T(\vec{v}_1) + 3T(\vec{v}_2) = v_1 + v_2, \qquad (1)$$
$$T(\vec{v}_1) + T(\vec{v}_2) = v_1 + v_2. \qquad (2)$$

Apply $3 \times$ Eq. (2) $-$ Eq. (1) to get

$$T(\vec{v}_1) = 8\vec{v}_1 - 4\vec{v}_2.$$

Then apply Eq. (1) $+ (-2) \times$ Eq. (2) to get

$$T(\vec{v}_2) = -5\vec{v}_1 + 3\vec{v}_2.$$

**§6.3, #7**

The kernel of $T$ consists of all functions $y$ such that $T(y) = 0$, namely $y'' - y = 0$. This differential equation has auxilliary equation $r^2 - 1 = 0$, which has two roots $r = 1, -1$. Hence the differential equation $y'' - y = 0$ has two linearly independent solutions $e^x$ and $e^{-x}$. It follows that a basis for the kernel of $T$ is given by $\{e^x, e^{-x}\}$.

**§6.3, #17**

First we need to find what $T(\vec{v})$ equals to for any $\vec{v} \in V$. Note that any $\vec{v} \in V$ can be expressed as

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3.$$

Use the linearity of $T$ and the given information to get

$$\begin{aligned} T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) &= c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) \\ &= c_1(2\vec{w}_1 - \vec{w}_2) + c_2(\vec{w}_1 - \vec{w}_2) + c_3(\vec{w}_1 + 2\vec{w}_2) \\ &= (2c_1 + c_2 + c_3)\vec{w}_1 + (-c_1 - c_2 + 2c_3)\vec{w}_2. \end{aligned}$$

Therefore, the action of $T$ on $V$ is given by

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = (2c_1 + c_2 + c_3)\vec{w}_1 + (-c_1 - c_2 + 2c_3)\vec{w}_2. \qquad (1)$$

Now, let us find Ker($T$). By Equation (1) we need to find all vectors $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ in $V$ such that

$$(2c_1 + c_2 + c_3)\vec{w}_1 + (-c_1 - c_2 + 2c_3)\vec{w}_2 = \vec{0}.$$

But $\{\vec{w}_1, \vec{w}_2\}$ is a basis for $W$ and so

$$\begin{aligned} 2c_1 + c_2 + c_3 &= 0, \\ -c_1 - c_2 + 2c_3 &= 0. \end{aligned}$$

Use the Gauss-Jordan elimination to get the solution:

$$c_1 = -3t, \quad c_2 = 5t, \quad c_3 = t.$$

Thus the kernel of $T$ consists of all vectors $\vec{v}$ of the form

$$\begin{aligned} \vec{v} &= -3t\vec{v}_1 + 5t\vec{v}_2 + t\vec{v}_3 \\ &= t(-3\vec{v}_1 + 5\vec{v}_2 + \vec{v}_3). \end{aligned}$$

This means that the kernel of $T$ is given by

$$\mathrm{Ker}(T) = \{t(-3\vec{v}_1 + 5\vec{v}_2 + \vec{v}_3) \mid t \in \mathbf{R}\}.$$

Thus $\mathrm{Ker}(T)$ is spanned by the vector $-3\vec{v}_1 + 5\vec{v}_2 + \vec{v}_3$ and so $\dim[\mathrm{Ker}(T)] = 1$.

Next, we find the range of $T$. For this purpose we need to find out what vector $\vec{w} = a\vec{w}_1 + b\vec{w}_2$ in $W$ is the image of some vector $\vec{v}$ in $V$. By Equation (1) we need to solve the equation:

$$\begin{aligned} 2c_1 + c_2 + c_3 &= a, \\ -c_1 - c_2 + 2c_3 &= b. \end{aligned}$$

Use the Gauss-Jordan elimination to solve this system of linear equations:

$$c_1 = -3t + a + b, \quad c_2 = 5t - a - 2b, \quad c_3 = t. \tag{2}$$

Note that for any $a$ and $b$ we can always find $c_1, c_2, c_3$ as given by Equation (2) such that

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = (2c_1 + c_2 + c_3)\vec{w}_1 + (-c_1 - c_2 + 2c_3)\vec{w}_2 = a\vec{w}_1 + b\vec{w}_2.$$

This means that for any vector $\vec{w}$ in $W$, we can always find a vector $\vec{v}$ in $V$ such that $T(\vec{v}) = \vec{w}$. Hence $\mathrm{Range}(T) = W$ and so $\dim[\mathrm{Range}(T)] = 2$.

§**8.9** Extra problem *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. *Find* $e^A$.

The characteristic polynomia is easily found to be $p(\lambda) = \lambda^2 - 5\lambda - 2$. Hence the eigenvalues are given by

$$\lambda_1 = \frac{5 + \sqrt{33}}{2}, \quad \lambda_2 = \frac{5 - \sqrt{33}}{2}.$$

We can derive the corresponding eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 3 + \sqrt{33} \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 3 - \sqrt{33} \end{bmatrix}.$$

Let $S$ be the matrix $S = [\vec{v}_1, \vec{v}_2]$, namely,

$$S = \begin{bmatrix} 4 & 4 \\ 3 + \sqrt{33} & 3 - \sqrt{33} \end{bmatrix}. \tag{1}$$

Then we have the following identity

$$S^{-1}AS = \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{bmatrix}.$$

Multiply $S$ from the left and $S^{-1}$ from the right to get

$$A = S \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{bmatrix} S^{-1}.$$

Therefore, $e^A$ is given by

$$e^A = S \begin{bmatrix} e^{\frac{5+\sqrt{33}}{2}} & 0 \\ 0 & e^{\frac{5-\sqrt{33}}{2}} \end{bmatrix} S^{-1}. \tag{2}$$

Now, we can use row operations to find $S^{-1}$:

$$S^{-1} = \begin{bmatrix} \frac{-3+\sqrt{33}}{8\sqrt{33}} & \frac{1}{2\sqrt{33}} \\ \frac{3+\sqrt{33}}{8\sqrt{33}} & -\frac{1}{2\sqrt{33}} \end{bmatrix}. \tag{3}$$

Finally, put Equations (1) and (3) into Equation (2) and carry out matrix multiplication to get

$$e^A = \begin{bmatrix} \frac{\left(-3+\sqrt{33}\right)e^{\frac{5+\sqrt{33}}{2}}+\left(3+\sqrt{33}\right)e^{\frac{5-\sqrt{33}}{2}}}{2\sqrt{33}} & \frac{2\left(e^{\frac{5+\sqrt{33}}{2}}-e^{\frac{5-\sqrt{33}}{2}}\right)}{\sqrt{33}} \\ \frac{3\left(e^{\frac{5+\sqrt{33}}{2}}-e^{\frac{5-\sqrt{33}}{2}}\right)}{\sqrt{33}} & \frac{\left(3+\sqrt{33}\right)e^{\frac{5+\sqrt{33}}{2}}+\left(-3+\sqrt{33}\right)e^{\frac{5-\sqrt{33}}{2}}}{2\sqrt{33}} \end{bmatrix}.$$

### §9.5, #53

Let $X_1$ and $X_2$ be the Laplace transforms of $x_1$ and $x_2$, respectively. Take the Laplace transform of the given differential equations with the initial conditions to get

$$sX_1 - 1 = 2X_1 - X_2, \tag{1}$$
$$sX_2 = X_1 + 2X_2. \tag{2}$$

From Equation (2), we solve for $X_2$

$$X_2 = \frac{1}{s-2}X_1. \tag{3}$$

Put this $X_2$ into Equation (1):

$$sX_1 - 1 = 2X_1 - \frac{1}{s-2}X_1,$$

which can be rewritten as

$$(s-2)X_1 + \frac{1}{s-2}X_1 = 1,$$

or

$$\frac{(s-2)^2+1}{s-2}X_1 = 1.$$

Thus we find $X_1$ to be

$$X_1 = \frac{s-2}{(s-2)^2+1}. \tag{4}$$

Use the First Shifting Theorem to take the inverse Laplace transform to get

$$x_1 = e^{2t} \cos t.$$

Next, use Equations (3) and (4) to find $X_2$

$$X_2 = \frac{1}{(s-2)^2 + 1}.$$

Again use the First Shifting Theorem to take the inverse Laplace transform to get

$$x_2 = e^{2t} \sin t.$$